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# RECURSIVELY DIVISIBLE NUMBERS

Abstract. Divisible numbers are useful whenever a whole needs to be divided into equal parts. But sometimes we need to divide the parts into subparts, and subparts into sub-subparts, and so on, in a recursive way. To understand numbers that are recursively divisible, I introduce the recursive divisor function: a recursive analog of the usual divisor function. I study the number and sum of recursive divisors and give a geometric interpretation of recursive divisibility. I show that the number of recursive divisors is twice the number of ordered factorizations, a problem much studied in its own right. By computing those numbers which are more recursively divisible than all of their predecessors, I recover many of the numbers prevalent in design and technology, and suggest new ones which have yet to be adopted. These are useful for recursively modular systems which operate across multiple organizational length scales.

## 1. Introduction

1.1. **Divisor function.** The usual divisor function is

$$\sigma_x(n) = \sum_{m|n} m^x.$$

It sums the divisors of $n$ raised to some integer power $x$.

When $x = 0$, the divisor function counts the number of divisors of $n$ and is written $d(n)$. It is well known that for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j}$, with $p_1, p_2, \dots, p_j$ prime,

$$d(n) = (1+\alpha_1)(1+\alpha_2)\dots(1+\alpha_j). \tag{1}$$

Numbers $n$ for which $d(n)$ is larger than that of all of the predecessors of $n$ are called highly composite numbers and have been extensively studied [3, 4].

When $x = 1$, the divisor function sums the divisors of $n$ and is written $\sigma(n)$. It is well known that

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\,\frac{p_2^{\alpha_2+1}-1}{p_2-1}\dots\frac{p_j^{\alpha_j+1}-1}{p_j-1}. \tag{2}$$

Numbers $n$ for which $\sigma(n)/n$ is larger than that of all of the predecessors of $n$ are called super-abundant numbers and have also been studied [4].

1.2. **Recursive divisor function.** In contrast with the usual divisor function, in this paper I am concerned not only with the divisors of a number $n$ but also the divisors of the resultant quotients, and the divisors of those resultant quotients, and so on. I introduce and study the recursive divisor function,

$$\kappa_x(n) = n^x + \sum_{m \lfloor n} \kappa_x(m), \tag{3}$$

where the sum is over the proper divisors of $n$. When $x = 0$, I call this the number of recursive divisors $a(n)$. When $x = 1$, I call this the sum of recursive divisors $b(n)$.

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**Definition 1.** *The number of recursive divisors is* $a(1) = 1$ *and*

$$a(n) = 1 + \sum_{m \lfloor n} a(m),$$

*where* $m \lfloor n$ *means* $m$ *is a proper divisor of* $n$.

For example, $a(10) = 1 + a(1) + a(2) + a(5) = 6$, illustrated in Fig. 1.

**Definition 2.** *The sum of recursive divisors is* $b(1) = 1$ *and*

$$b(n) = n + \sum_{m \lfloor n} b(m).$$

For example, $b(10) = 10 + b(1) + b(2) + b(5) = 20$, illustrated in Fig. 1. Note that $a(n)$ depends only on the set of exponents in the prime factorization of $n$, but $b(n)$ depends on both the set of exponents and the set of primes.

1.3. **General remarks.** My path to the recursive divisor function began with a discussion with colleagues about numbers whose divisors have many divisors, and so on. To my surprise, when I wrote down (3), I found that, so far as I could tell, it had not been investigated. As I discuss below, the related (for $\kappa = 1$) number of ordered factorizations has been studied for some time. However, casting the problem more generally in terms of the recursive divisor function suggests a deeper perspective, not only in terms of $\kappa$, but by motivating other properties. While here I study when $a(n)$ and $b(n)$ are high, more recently I considered when $a(n) = n$ and $a(n) > n$ [1], which are the recursive counterparts of the perfect and abundant numbers. Parallels between theorems concerning the divisor function and its recursive analog suggest a deeper connection between the two, worthy of further investigation.

I believe this work may stimulate significant new research activity, for three

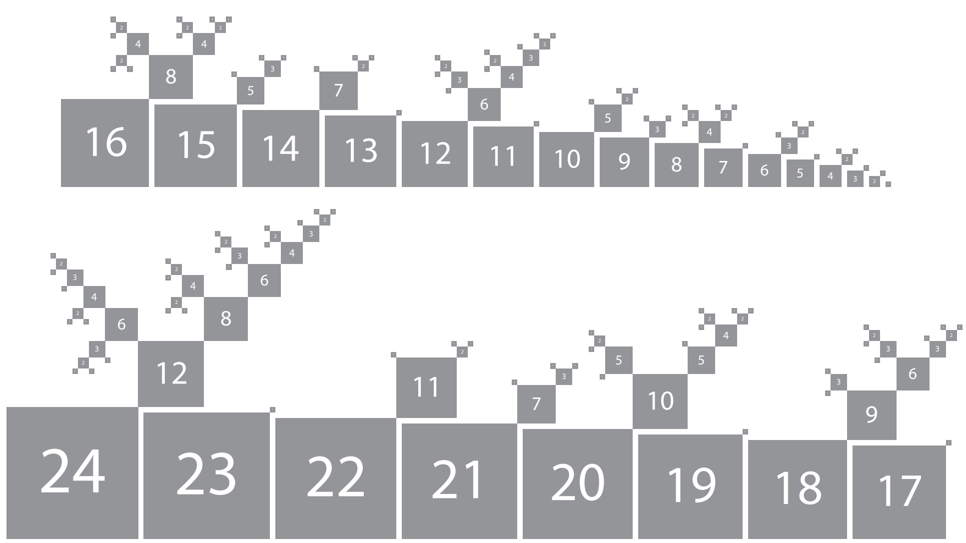


FIGURE 1. Divisor trees for 1 to 24. The number of recursive divisors $a(n)$ counts the number of squares in each tree and the sum of recursive divisors $b(n)$ adds up the side length of the squares in each tree. Divisor trees can be generated for any $n$ at lims.ac.uk/recursively-divisible-numbers.

reasons. First, the novelty of the recursive divisor function suggests there is low-hanging fruit. Second, I have mainly relied on elementary methods, so this paper is accessible to a wide audience. Third, I list a number of specific open questions at the end. As well as prompting new research, effective papers should, in the words of Thurston, "advance human understanding of mathematics" [2]. To this end, I have included a geometric interpretation of recursively divisible numbers and tables and plots of how they grow.

1.4. **Example.** Consider one of the earliest references to a number that can be divided into equal parts in many ways. Plato writes in his *Laws* that the ideal population of a city is 5040, since this number has more divisors than any number less than it. He observes that 5040 is divisible by 60 numbers, including one to 10. A highly divisible population is useful for dividing the city into equal-sized sectors for administrative, social and military purposes.

This conception of divisibility can be extended. Once the city is divided into equal parts, it is often necessary to divide a part into equal subparts. For example, if 5040 is divided into 15 parts of 336, each part can in turn be divided into subparts in 20 ways, since 336 has 20 divisors. But if 5040 is divided into 16 parts of 315, each part can be divided into subparts in only 12 ways, since 315 has 12 divisors. Thus the division of the whole into 15 parts offers more optionality for further subdivisions than the division into 16 parts. Similar reasoning can be applied to the divisibility of the subparts into sub-subparts, and so on, in a recursive way.

The goal of this paper is to quantify the notion of recursive divisibility and understand the properties of numbers which possess it to a large degree.

1.5. **Outline of paper.** Including this introduction, this paper is divided into six parts. In part 2, I introduce divisor trees (Figs. 1 and 2), which give a geometrical interpretation of the recursive divisor function. Using this, I show that the number of recursive divisors is twice the number of ordered factorizations, a problem much studied in its own right [5, 6, 7, 8, 9]. Examining the internal structure of divisor trees yields a relation between the sum and number of recursive divisors.

In part 3, I investigate properties of the number of recursive divisors, taking advantage of their relation to the number of ordered factorizations. I give recursion relations for when $n$ is the product of distinct primes, and for when $n$ is the product of primes to a power. The latter can be solved for up to three primes.

In part 4, I investigate properties of the sum of recursive divisors. I give recursion relations for when $n$ is the product of primes to a power. These can be solved using the relation between the sum and number of recursive divisors in part 2.

In part 5, I study numbers which are recursively divisible to a high degree. I call numbers with a record number of recursive divisors recursively highly composite, and list them up to a million. These have been studied in the context of the number of ordered factorizations [9]. I call numbers with a record sum of recursive divisors, normalized by $n$, recursively super-abundant, and also list them up to a million.

In part 6, I survey applications of highly recursive numbers in design and technology. I conclude with a list of open problems.

Throughout, $m|n$ means $m$ divides $n$ and $m\lfloor n$ means $m$ is a proper divisor of $n$.

## 2. Recursive divisor function and divisor trees

2.1. **Divisor trees.** A geometric interpretation of the recursive divisor function $\kappa(n)$ can be had by drawing the divisor tree for a given value of $n$. Divisor trees for 1 to 24 are shown in Fig. 1. The number of recursive divisors $a(n)$ counts the number of squares in each tree, whereas $d(n)$ in (1) counts the number of squares in the main diagonal. The sum of recursive divisors $b(n)$ adds up the side length of the squares in each tree, whereas $\sigma(n)$ in (2) adds up the side length of the squares in the main diagonal. This can be extended to $\kappa_2(n)$, which adds up area, and so on, but in this paper I only consider $a(n) = \kappa_0(n)$ and $b(n) = \kappa_1(n)$.

A divisor tree is constructed as follows. First, draw a square of side length $n$. Let $m_1, m_2, \ldots$ be the proper divisors of $n$ in descending order. Then draw squares of side length $m_1$, $m_2, \ldots$ with each consecutive square situated to the upper right of its predecessor, kitty-corner. This forms the main arm of a divisor tree. Now, for each of the squares of side length $m_1, m_2, \ldots$, repeat the process. Let $l_1, l_2, \ldots$ be the proper divisors of $m_1$ in descending order. Then draw squares of side length $l_1$, $l_2, \ldots$, but with the sub-arm rotated 90° counter-clockwise. Do the same for each of the remaining squares in the main arm. This forms the branches off of the main arm. Now, continue repeating this process, drawing arms off of arms off of arms, and so on, until the arms are single squares of size 1.

2.2. **Properties of divisor trees.** In order to establish properties of the number and sum of recursive divisors, it helps to consider a more fine-grained description of divisor trees, namely, one that counts the number of divisors of a given size.

**Definition 3.** *The number of recursive divisors of $n$ of size $k$ is $a(n, n) = 1$ and*

$$a(n, k) = \sum_{m \lfloor n} a(m, k)$$

*for $k \lfloor n$ and $a(n, k) = 0$ otherwise.*

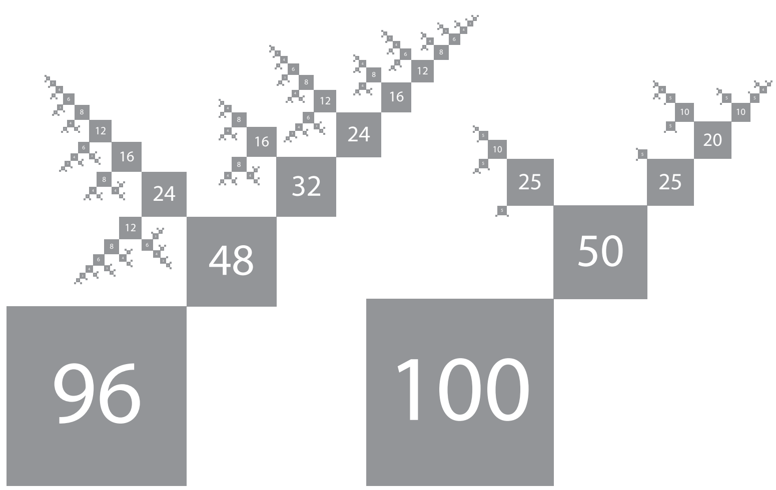


Figure 2. Divisor trees for 96 and 100. There are $a(96) = 224$ squares in the left tree and $a(100) = 52$ squares in the right. The sum of the side length of the squares, or one-fourth of the tree perimeter, is $b(96) = 768$ in the left tree and $b(100) = 340$ in the right.

**Lemma 1.** *The number of recursive divisors of size $k$ satisifes $a(k\,n, k) = a(n, 1)$.*

*Proof.* By Definition 3,

$$a(n,1) = \sum_{m \lfloor n} a(m,1) \tag{4}$$

and so

$$a(k\,n,k) = \sum_{m \lfloor k\,n} a(m,k).$$

Since $a(m, k) = 0$ if $k$ does not divide $m$, this can be rewritten as

$$a(k\,n,k) = \sum_{m \lfloor n} a(k\,m,k). \tag{5}$$

Let the prime omega function $\Omega(n)$ sum the exponents in the prime factorization of $n$, that is, for $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_j^{\alpha_j}$, $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_j$. I prove the lemma by induction on $\Omega(n)$. The base case $\Omega(n) = 0$, or $n = 1$, holds by Definition 3: $a(k{\cdot}1, k) = a(1, 1)$. I now show that if $a(k\,n, k) = a(n, 1)$ for all $n$ such that $\Omega(n) < i$, then $a(k\,n, k) = a(n, 1)$ for all $n$ such that $\Omega(n) < i+1$. To see why, observe that in (5) all of the proper divisors $m$ of $n$ must have $\Omega(m)$ at most $\Omega(n) - 1$. Therefore by assumption all of the $a(k\,m, k)$ in (5) reduce to $a(m, 1)$, and the right side of (5) takes the form of the right side of (4) and thus equals $a(n, 1)$. □

**Lemma 2.** *For $n > 1$, the number of recursive divisors of size 1 is equal to half the total number of recursive divisors, that is, $a(n, 1) = a(n)/2$.*

*Proof.* Clearly

$$a(n) = \sum_{m | n} a(n,m).$$

By Lemma 1, this becomes

$$a(n) = \sum_{m|n} a(n/m,1) = \sum_{m|n} a(m,1) = a(n,1) + \sum_{m \lfloor n} a(m,1).$$

Inserting Definition 3 with $k = 1$ into the above gives the desired result. □

2.3. **Relation to the number of ordered factorizations.** Here I show that for $n > 1$, the number of recursive divisors $a(n)$ is twice the number of ordered factorizations into integers greater than one, which I call $g(n)$. But before getting to that, I first introduce $g(n)$ and mention some of the work on it.

The number of ordered factorizations $g(n)$ satisifes $g(1) = 1$ and

$$g(n) = \sum_{m \lfloor n} g(m).$$

For example, 12 is the product of integers greater than one in eight ways: $12 = 6 \cdot 2 = 2 \cdot 6 = 4 \cdot 3 = 3 \cdot 4 = 3 \cdot 2 \cdot 2 = 2 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3$. So $g(12) = 8$.

Kalmar [5] was the first to consider $g(n)$, and it was later studied more systematically by Hille [6]. Over the last 80 years several authors have extended Hille's results [7, 8, 9], some of which we will mention later.

**Theorem 1.** *Let $g(n)$ be the number of ordered factorizations into integers greater than one and set $g(1) = 1$. Then for $n > 1$, $a(n) = 2g(n)$.*

*Proof.* The definition of $g(n)$ is identical to Definition 3 for $k = 1$, that is, identical to $a(n, 1)$. Since $g(1) = a(1, 1) = 1$, the proof follows from Lemma 2. □

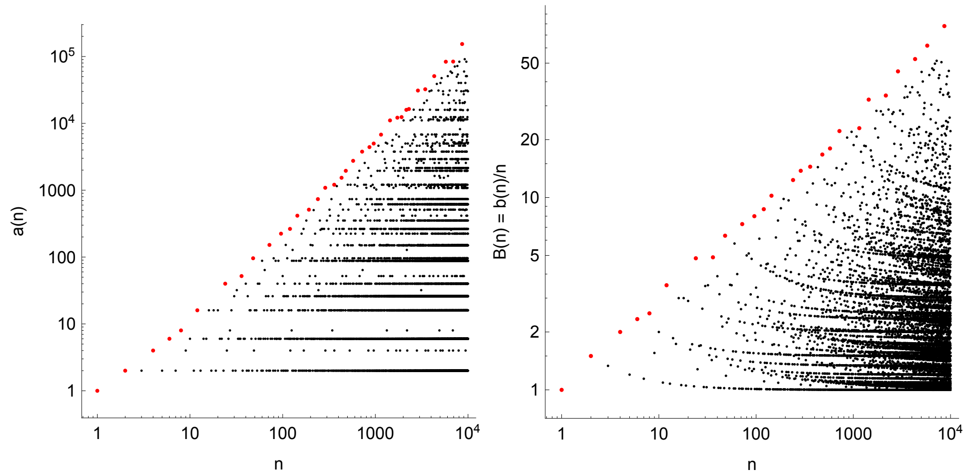


FIGURE 3. On the left are the numbers of recursive divisors $a(n)$. The recursively highly composite numbers, which satisfy $a(n) > a(m)$ for $m < n$, are the big red points. On the right are the sums of recursive divisors $b(n)$, normalized by $n$. The recursively superabundnant numbers, which satisfy $b(n)/n > b(m)/m$ for $m < n$, are the big red points.

2.4. **Relating the number and sum of recursive divisors.** The quantity $b(n)$ is more difficult to calculate than $a(n)$. Here I give an expression for $b(n)$ in terms of $a(n)$. I will use it later to explicitly determine $b(n)$ for certain values of $n$.

**Theorem 2.** *Let $B(n) = b(n)/n$ and $A(n) = a(n)/n$. Then*

$$B(n) = \frac{1}{2} + \frac{1}{2} \sum_{m|n} A(m).$$

*Proof.* We can write $b(n)$ as

$$b(n) = \sum_{m|n} m\, a(n,m).$$

By Lemma 1,

$$b(n) = \sum_{m|n} m\, a(n/m, 1) = n + \sum_{m \lfloor n} m\, a(n/m, 1).$$

By Lemma 2,

$$b(n) = n + \frac{1}{2} \sum_{m \lfloor n} m\, a(n/m) = \frac{n}{2} + \frac{1}{2} \sum_{m|n} m\, a(n/m) = \frac{n}{2} + \frac{n}{2} \sum_{m|n} a(m)/m.$$

Dividing by $n$, the theorem follows. □

## 3. NUMBER OF RECURSIVE DIVISORS

3.1. **Table and plot of values.** The number of recursive divisors $a(n)$ is harder to compute than the number of divisors $d(n)$. The first 72 values of $a(n)$ and an algorithm to generate them are given in Table 1. The first $10^4$ are plotted in Fig. 3.

3.2. **Distinct primes.** Let $n = p_1\, p_2 \dots p_k$ be the product of $k$ distinct primes. By (1), $d(p_1\, p_2 \dots p_k) = 2^k$. Here I calculate $a(p_1\, p_2 \dots p_k)$.

**Theorem 3.** *Let $n = p_1\, p_2 \dots p_k$ be the product of $k$ distinct primes. Then the exponential generating function of $a(p_1 p_2 \dots p_k)$ is*

$$\mathrm{EG}(a(p_1 p_2 \dots p_k), x) = \frac{e^x}{2 - e^x}.$$

*Proof.* This theorem is equivalent to

$$a(p_1 p_2 \dots p_k) = 1 + \sum_{i=0}^{k-1} \binom{k}{i} a(p_1 p_2 \dots p_i). \tag{6}$$

I prove it by induction. First note that $a(p_1) = 1 + \binom{1}{0} a(1) = 2$. I now show that if (6) is true for $k$, then it is true for $k+1$. I do so by adding to $a(p_1 p_2 \dots p_k)$ all of the $a$ values for divisors that include $p_{k+1}$, apart from $a(p_1 p_2 \dots p_{k+1})$, since $p_1 p_2 \dots p_{k+1}$ is not a proper divisor of itself. I also add $a(p_1 p_2 \dots p_k)$, which is left out of the expression for $a(p_1 p_2 \dots p_k)$ for the same reason. There are $\binom{k}{0}$ $a$ values that are the product of one prime that include $p_{k+1}$, $\binom{k}{1}$ that are the product of two primes that include $p_{k+1}$, and so on. Since $a$ depends on the number of primes but not their values, we are not concerned with how we label the primes. Then

$$\begin{aligned}
a(p_1 \dots p_{k+1}) &= a(p_1 \dots p_k) + \binom{k}{0} a(p_1) + \binom{k}{1} a(p_1 p_2) + \dots + \binom{k}{k-1} a(p_1 \dots p_k) \\
&\quad + a(p_1 \dots p_k) \\
&= 1 + a(p_1 \dots p_k) + \sum_{i=0}^{k-1} \binom{k}{i} a(p_1 \dots p_i) + \sum_{i=0}^{k-1} \binom{k}{i} a(p_1 \dots p_{i+1}) \\
&= 1 + (k+1) a_1(p_1 \dots p_k) + \sum_{i=0}^{k-1} \left( \binom{k}{i} + \binom{k}{i-1} \right) a(p_1 \dots p_i) \\
&= 1 + \sum_{i=0}^{k} \binom{k+1}{i} a(p_1 \dots p_i). \quad \square
\end{aligned}$$

So for the product of $k = 1, 2, \dots$ distinct primes, $a(k) = 2, 6, 26, 150, 1082, \dots$.

3.3. **Primes to a power.** When $n$ is the product of primes to powers, $a(n)$ satisfies recursion relations relating it to values of $a(n)$ for primes to lower powers.

**Theorem 4.** *Let $p, q$ and $r$ be prime. Then*

$$\begin{aligned}
a(p^c) &= 2a(p^{c-1}), \\
a(p^c q^d) &= 2\Big(a(p^{c-1} q^d) + a(p^c q^{d-1}) - a(p^{c-1} q^{d-1})\Big), \\
a(p^c q^d r^e) &= 2\Big(a(p^{c-1} q^d r^e) + a(p^c q^{d-1} r^e) + a(p^c q^d r^{e-1}) - a(p^c q^{d-1} r^{e-1}) \\
&\quad - a(p^{c-1} q^d r^{e-1}) - a(p^{c-1} q^{d-1} r^e) + a(p^{c-1} q^{d-1} r^{e-1})\Big).
\end{aligned}$$

*Analogous recursion relations apply for the product of more primes to powers.*

*Proof.* The approach is similar to, but somewhat simpler than, that used to prove Theorem 6. However, Hille [6] and Chor *et al.* [7] proved that identical recursion relations govern $g(n)$, the number of ordered factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and inserting this into Hille's and Chor's recursion relations gives the desired results. □

**Corollary 1.** *Let $\tau$ be the maximum exponent in the prime factorization of $n$. Then $2^\tau$ divides $a(n)$.*

*Proof.* All of the recursion relations in Theorem 4 have a factor of 2 on the right side. The corollary is implied by iterating the recursion relation $\tau$ times. Each time, the exponents on the right are reduced by at most 1. Iterating until the smallest exponent is reduced to 0, the exponent disappears since, for example, $a(p^c q^0) = a(p^c)$. Continuing this process ultimately gives a total of $\tau$ factors of 2. The $a(n)$ are expressed as a product of an integer and $2^\tau$ in Table 2. □

The three recursion relations shown in Theorem 4 can be solved explicitly.

**Theorem 5.** *Let $p, q$ and $r$ be prime. Then*

$$\begin{aligned} a(p^c) &= 2^c \\ a(p^c q^d) &= 2^c \sum_{i=0}^{d} \binom{d}{i} \binom{c+i}{i}. \\ a(p^c q^d r^e) &= \sum_{j=0}^{d} (-1)^j \binom{d}{j} \binom{c+d-j}{d} a(p^{c+d-j} r^e). \end{aligned}$$

*Proof.* The result for $n = p^c$ follows by inspection. For $n = p^c q^d$ and $n = p^c q^d r^e$, Chor *et al.* [7] give the analogous results for $g(n)$, the number of ordered

| $n$ | $a(n)$ | $b(n)$ | $n$ | $a(n)$ | $b(n)$ | $n$ | $a(n)$ | $b(n)$ | $n$ | $a(n)$ | $b(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 19 | 2 | 20 | 37 | 2 | 38 | 55 | 6 | 74 |
| 2 | 2 | 3 | 20 | 16 | 58 | 38 | 6 | 62 | 56 | 40 | 196 |
| 3 | 2 | 4 | 21 | 6 | 34 | 39 | 6 | 58 | 57 | 6 | 82 |
| 4 | 4 | 8 | 22 | 6 | 38 | 40 | 40 | 156 | 58 | 6 | 92 |
| 5 | 2 | 6 | 23 | 2 | 24 | 41 | 2 | 42 | 59 | 2 | 60 |
| 6 | 6 | 14 | 24 | 40 | 116 | 42 | 26 | 132 | 60 | 88 | 346 |
| 7 | 2 | 8 | 25 | 4 | 32 | 43 | 2 | 44 | 61 | 2 | 62 |
| 8 | 8 | 20 | 26 | 6 | 44 | 44 | 16 | 106 | 62 | 6 | 98 |
| 9 | 4 | 14 | 27 | 8 | 46 | 45 | 16 | 96 | 63 | 16 | 124 |
| 10 | 6 | 20 | 28 | 16 | 74 | 46 | 6 | 74 | 64 | 64 | 256 |
| 11 | 2 | 12 | 29 | 2 | 30 | 47 | 2 | 48 | 65 | 6 | 86 |
| 12 | 16 | 42 | 30 | 26 | 104 | 48 | 96 | 304 | 66 | 26 | 188 |
| 13 | 2 | 14 | 31 | 2 | 32 | 49 | 4 | 58 | 67 | 2 | 68 |
| 14 | 6 | 26 | 32 | 32 | 112 | 50 | 16 | 112 | 68 | 16 | 154 |
| 15 | 6 | 26 | 33 | 6 | 50 | 51 | 6 | 74 | 69 | 6 | 98 |
| 16 | 16 | 48 | 34 | 6 | 56 | 52 | 16 | 122 | 70 | 26 | 184 |
| 17 | 2 | 18 | 35 | 6 | 50 | 53 | 2 | 54 | 71 | 2 | 72 |
| 18 | 16 | 54 | 36 | 52 | 176 | 54 | 40 | 190 | 72 | 152 | 524 |

TABLE 1. The first 72 values of the number of recursive divisors $a(n)$ and the sum of recursive divisors $b(n)$. A Mathematica algorithm for $a(n)$ is: `n = 2; max = 72; a = {1}; While[n <= max, a = Append[a, 1 + Total[Part[a, Delete[Divisors[n], -1]]]]; n++]; a` For $b(n)$, replace the 1 before `Total` with $n$.

factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and applying this to Chor's results gives the desired recurrence relations. □

## 4. Sum of recursive divisors

I now turn to the sum of recursive divisors $b(n)$. This quantity is more intricate than $a(n)$, because it depends on the primes as well as their exponents in the prime factorization of $n$. The first 72 values and an algorithm to generate them are shown in Table 1. The first $10^4$ are plotted in Fig. 3.

4.1. **Primes to a power.** When $n$ is equal to the product of primes to powers, $b(n)$ satisfies recursion relations relating it to values of $b(n)$ for primes to lower powers. The recursion relations are similar to those for $a(n)$, but more complex.

**Theorem 6.** *Let $p, q$ and $r$ be prime. Then*

$$\begin{aligned} b(p^c) &= 2b(p^{c-1}) + (p-1)p^{c-1}, \\ b(p^c q^d) &= 2\Big(b(p^{c-1}q^d) + b(p^c q^{d-1}) - b(p^{c-1}q^{d-1})\Big) + (p-1)(q-1)p^{c-1}q^{d-1}, \\ b(p^c q^d r^e) &= 2\Big(b(p^{c-1}q^d r^e) + a(p^c q^{d-1} r^e) + a(p^c q^d r^{e-1}) \\ &- b(p^c q^{d-1} r^{e-1}) - a(p^{c-1} q^d r^{e-1}) - a(p^{c-1} q^{d-1} r^e) \\ &+ b(p^{c-1} q^{d-1} r^{e-1})\Big) + (p-1)(q-1)(r-1)p^{c-1}q^{d-1}r^{e-1}. \end{aligned}$$

*Proof.* I first prove the case of $n = p^c$. From Definition 2,

$$b(p^c) = p^c + \sum_{i=0}^{c-1} b(p^i). \tag{7}$$

Adding $b(p^c)$ to both sides and with $c \to c-1$,

$$\sum_{i=0}^{c-1} b(p^i) = 2b(p^{c-1}) - p^{c-1},$$

which, when inserted into (7), gives the desired recurrence relation.

I now prove the case of $n = p^c q^d$. From Definition 2,

$$b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d-1} b(p^c q^j). \tag{8}$$

Adding $b(p^c q^d)$ to both sides,

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d} b(p^c q^j), \tag{9}$$

which we can equally write

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c} \sum_{j=0}^{d} b(p^i q^j). \tag{10}$$

With $d \to d-1$ in (9), we find

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - p^c q^{d-1} - \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} b(p^i q^j). \tag{11}$$

With $c \to c-1$ and $d \to d-1$ in (10), and inserting the result into (11), yields

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - 2b(p^{c-1} q^{d-1}) + (1-p)p^{c-1}q^{d-1}. \tag{12}$$

With $c \to c-1$ in (10), we find

$$\sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) = 2b(p^{c-1} q^d) - p^{c-1} q^d. \tag{13}$$

Inserting (12) and (13) into (8) gives the desired recursion relation.

For $n = p^c q^d r^e$, the proof is similar to that for $n = p^c q^d$ and is omitted here. □

4.2. **Explicit values.** The recursion relations in Theorem 6 can be solved. I only give the results for $n = p^c$ and $n = p^c q^d$. For $n = p^c q^d r^e$, the solution is more intricate but can be solved in a similar way to that for $n = p^c q^d$.

**Theorem 7.** *Let $p$ and $q$ be prime, and $B(n) = b(n)/n$. Then*

$$\begin{aligned} B(p^c) &= \frac{p-1-(2/p)^c}{p-2} \quad \text{for } p \text{ odd}, \\ B(2^c) &= (c+2)/2, \\ B(p^c q^d) &= \frac{1}{2} + \frac{1}{2} \sum_{i=0}^{c} \frac{2^i}{p^i} \sum_{j=0}^{d} \frac{1}{q^j} \sum_{k=0}^{j} \binom{i+k}{k} \binom{j}{k}. \end{aligned}$$

*Proof.* I first prove the case of $n = p^c$. From Theorem 2,

$$B(p^c) = \frac{1}{2} + \frac{1}{2} \sum_{i=0}^{c} A(p^i). \tag{14}$$

From Theorem 4, $a(p^c) = 2^c$ and $A(p^i) = a(p^i)/p^i = (2/p)^i$. Inserting this into (14), we find the desired result. For $p = 2$, $B(2^c) = (c+2)/2$.

I now prove the case of $n = p^c q^d$. From Theorem 2,

$$B(p^c q^d) = \frac{1}{2} + \frac{1}{2} \sum_{i=0}^{c} \sum_{j=0}^{d} A(p^i q^j). \tag{15}$$

Theorem 5 gives $a(p^c q^d)$ explicitly. Inserting $A(p^i q^j) = a(p^i q^j)/(p^i q^j)$ into (15) yields the desired result. For $p = 2$, the result simplifies to contain just two sums. □

## 5. Numbers that are recursively divisible to a high degree

5.1. **Highly composite and super-abundant numbers.** I briefly review highly composite and super-abundant numbers [3, 4] before considering their recursive analogues. A number $n$ is highly composite if it has more divisors than any of its predecessors, that is, $d(n) > d(m)$ for all $m < n$. These are shown in the right side of Table 2. A number $n$ is super-abundant if the sum of its divisors, normalized by $n$, is greater than that of any of its predecessors, that is, $\sigma(n)/n > \sigma(m)/m$ for all $m < n$. These are the starred numbers in the right side of Table 2. For small $n$, super-abundant numbers are also highly composite, but later this ceases to be the case. The first super-abundant number that is not highly composite is 1,163,962,800 (A166735 [10]), and in fact only 449 numbers have both properties (A166981 [10]).

| $n$ | | . | $a(n)$ | $n$ | | | $d(n)$ |
|---|---|---|---|---|---|---|---|
| $*1$ | $=$ | $1$ | $1$ | $*1$ | $=$ | $1$ | $1$ |
| $*2$ | $=$ | $2$ | $1 \cdot 2$ | $*2$ | $=$ | $2$ | $2$ |
| $*4$ | $=$ | $2^2$ | $1 \cdot 2^2$ | $*4$ | $=$ | $2^2$ | $3$ |
| $*6$ | $=$ | $2 \cdot 3$ | $3 \cdot 2$ | $*6$ | $=$ | $2 \cdot 3$ | $4$ |
| $8$ | $=$ | $2^3$ | $1 \cdot 2^3$ | | | | |
| $*12$ | $=$ | $2^2 \cdot 3$ | $4 \cdot 2^2$ | $*12$ | $=$ | $2^2 \cdot 3$ | $6$ |
| $*24$ | $=$ | $2^3 \cdot 3$ | $5 \cdot 2^3$ | $*24$ | $=$ | $2^3 \cdot 3$ | $8$ |
| $*36$ | $=$ | $2^2 \cdot 3^2$ | $13 \cdot 2^2$ | $*36$ | $=$ | $2^2 \cdot 3^2$ | $9$ |
| $*48$ | $=$ | $2^4 \cdot 3$ | $6 \cdot 2^4$ | $*48$ | $=$ | $2^4 \cdot 3$ | $10$ |
| | | | | $*60$ | $=$ | $2^2 \cdot 3 \cdot 5$ | $12$ |
| $72$ | $=$ | $2^3 \cdot 3^2$ | $19 \cdot 2^3$ | | | | |
| $96$ | $=$ | $2^5 \cdot 3$ | $7 \cdot 2^5$ | | | | |
| $*120$ | $=$ | $2^3 \cdot 3 \cdot 5$ | $33 \cdot 2^3$ | $*120$ | $=$ | $2^3 \cdot 3 \cdot 5$ | $16$ |
| $144$ | $=$ | $2^4 \cdot 3^2$ | $26 \cdot 2^4$ | | | | |
| | | | | $*180$ | $=$ | $2^2 \cdot 3^2 \cdot 5$ | $18$ |
| $192$ | $=$ | $2^6 \cdot 3$ | $8 \cdot 2^6$ | | | | |
| $*240$ | $=$ | $2^4 \cdot 3 \cdot 5$ | $46 \cdot 2^4$ | $*240$ | $=$ | $2^4 \cdot 3 \cdot 5$ | $20$ |
| $288$ | $=$ | $2^5 \cdot 3^2$ | $34 \cdot 2^5$ | | | | |
| $*360$ | $=$ | $2^3 \cdot 3^2 \cdot 5$ | $151 \cdot 2^3$ | $*360$ | $=$ | $2^3 \cdot 3^2 \cdot 5$ | $24$ |
| $432$ | $=$ | $2^4 \cdot 3^3$ | $96 \cdot 2^4$ | | | | |
| $480$ | $=$ | $2^5 \cdot 3 \cdot 5$ | $61 \cdot 2^5$ | | | | |
| $576$ | $=$ | $2^6 \cdot 3^2$ | $43 \cdot 2^6$ | | | | |
| $*720$ | $=$ | $2^4 \cdot 3^2 \cdot 5$ | $236 \cdot 2^4$ | $*720$ | $=$ | $2^4 \cdot 3^2 \cdot 5$ | $30$ |
| | | | | $*840$ | $=$ | $2^3 \cdot 3 \cdot 5 \cdot 7$ | $32$ |
| $864$ | $=$ | $2^5 \cdot 3^3$ | $138 \cdot 2^5$ | | | | |
| $960$ | $=$ | $2^6 \cdot 3 \cdot 5$ | $78 \cdot 2^6$ | | | | |
| $*1152$ | $=$ | $2^7 \cdot 3^2$ | $53 \cdot 2^7$ | | | | |
| | | | | $*1260$ | $=$ | $2^2 \cdot 3^2 \cdot 5 \cdot 7$ | $36$ |
| $*1440$ | $=$ | $2^5 \cdot 3^2 \cdot 5$ | $346 \cdot 2^5$ | | | | |
| | | | | $*1680$ | $=$ | $2^4 \cdot 3 \cdot 5 \cdot 7$ | $40$ |
| $1728$ | $=$ | $2^6 \cdot 3^3$ | $190 \cdot 2^6$ | | | | |
| $1920$ | $=$ | $2^7 \cdot 3 \cdot 5$ | $97 \cdot 2^7$ | | | | |
| $*2160$ | $=$ | $2^4 \cdot 3^3 \cdot 5$ | $996 \cdot 2^4$ | | | | |
| $2304$ | $=$ | $2^8 \cdot 3^2$ | $64 \cdot 2^8$ | | | | |
| | | | | $*2520$ | $=$ | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | $48$ |
| $*2880$ | $=$ | $2^6 \cdot 3^2 \cdot 5$ | $484 \cdot 2^6$ | | | | |
| $3456$ | $=$ | $2^7 \cdot 3^3$ | $253 \cdot 2^7$ | | | | |
| $*4320$ | $=$ | $2^5 \cdot 3^3 \cdot 5$ | $1590 \cdot 2^5$ | | | | |
| | | | | $*5040$ | $=$ | $2^4 \cdot 3^2 \cdot 5 \cdot 7$ | $60$ |
| $*5760$ | $=$ | $2^7 \cdot 3^2 \cdot 5$ | $653 \cdot 2^7$ | | | | |
| $6912$ | $=$ | $2^8 \cdot 3^3$ | $328 \cdot 2^8$ | | | | |
| | | | | $7560$ | $=$ | $2^3 \cdot 3^3 \cdot 5 \cdot 7$ | $64$ |
| $*8640$ | $=$ | $2^6 \cdot 3^3 \cdot 5$ | $2402 \cdot 2^6$ | | | | |
| | | | | $*10080$ | $=$ | $2^5 \cdot 3^2 \cdot 5 \cdot 7$ | $72$ |
| $*11520$ | $=$ | $2^8 \cdot 3^2 \cdot 5$ | $856 \cdot 2^8$ | | | | |

TABLE 2. The left side shows the recursively highly composite numbers and the recursively super-abundant numbers (starred) up to a million. All of the recursively super-abundant numbers shown are also recursively highly composite, apart from one, 181,440. The right side shows the highly composite numbers and the super-abundant numbers (starred) up to a million. All of the super-abundant numbers shown are also highly composite.

| $n$ | | | $a(n)$ |
|---|---|---|---|
| ∗17280 | = | $2^7 \cdot 3^3 \cdot 5$ | $3477 \cdot 2^7$ |
| 23040 | = | $2^9 \cdot 3^2 \cdot 5$ | $1096 \cdot 2^9$ |
| ∗25920 | = | $2^6 \cdot 3^4 \cdot 5$ | $10368 \cdot 2^6$ |
| ∗30240 | = | $2^5 \cdot 3^3 \cdot 5 \cdot 7$ | $20874 \cdot 2^5$ |
| ∗34560 | = | $2^8 \cdot 3^3 \cdot 5$ | $4864 \cdot 2^8$ |
| 46080 | = | $2^{10} \cdot 3^2 \cdot 5$ | $1376 \cdot 2^{10}$ |
| ∗51840 | = | $2^7 \cdot 3^4 \cdot 5$ | $15979 \cdot 2^7$ |
| ∗60480 | = | $2^6 \cdot 3^3 \cdot 5 \cdot 7$ | $34266 \cdot 2^6$ |
| ∗69120 | = | $2^9 \cdot 3^3 \cdot 5$ | $6616 \cdot 2^9$ |
| 86400 | = | $2^7 \cdot 3^3 \cdot 5^2$ | $28481 \cdot 2^7$ |
| ∗103680 | = | $2^8 \cdot 3^4 \cdot 5$ | $23692 \cdot 2^8$ |
| ∗120960 | = | $2^7 \cdot 3^3 \cdot 5 \cdot 7$ | $53485 \cdot 2^7$ |
| 138240 | = | $2^{10} \cdot 3^3 \cdot 5$ | $8790 \cdot 2^{10}$ |
| 161280 | = | $2^9 \cdot 3^2 \cdot 5 \cdot 7$ | $17656 \cdot 2^9$ |
| ∗172800 | = | $2^8 \cdot 3^3 \cdot 5^2$ | $42520 \cdot 2^8$ |
| ∗207360 | = | $2^9 \cdot 3^4 \cdot 5$ | $34026 \cdot 2^9$ |
| ∗241920 | = | $2^8 \cdot 3^3 \cdot 5 \cdot 7$ | $80176 \cdot 2^8$ |
| 276480 | = | $2^{11} \cdot 3^3 \cdot 5$ | $11447 \cdot 2^{11}$ |
| 311040 | = | $2^8 \cdot 3^5 \cdot 5$ | $103540 \cdot 2^8$ |
| ∗345600 | = | $2^9 \cdot 3^3 \cdot 5^2$ | $61436 \cdot 2^9$ |
| ∗362880 | = | $2^7 \cdot 3^4 \cdot 5 \cdot 7$ | $267219 \cdot 2^7$ |
| ∗414720 | = | $2^{10} \cdot 3^4 \cdot 5$ | $47576 \cdot 2^{10}$ |
| ∗483840 | = | $2^9 \cdot 3^3 \cdot 5 \cdot 7$ | $116256 \cdot 2^9$ |
| 552960 | = | $2^{12} \cdot 3^3 \cdot 5$ | $14652 \cdot 2^{12}$ |
| 604800 | = | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | $480953 \cdot 2^7$ |
| 622080 | = | $2^9 \cdot 3^5 \cdot 5$ | $156278 \cdot 2^9$ |
| 691200 | = | $2^{10} \cdot 3^3 \cdot 5^2$ | $86362 \cdot 2^{10}$ |
| ∗725760 | = | $2^8 \cdot 3^4 \cdot 5 \cdot 7$ | $422932 \cdot 2^8$ |
| 829440 | = | $2^{11} \cdot 3^4 \cdot 5$ | $65018 \cdot 2^{11}$ |
| ∗967680 | = | $2^{10} \cdot 3^3 \cdot 5 \cdot 7$ | $163934 \cdot 2^{10}$ |

*Recursively super-abundant but not recursively highly composite*

∗181440 = $2^6 \cdot 3^4 \cdot 5 \cdot 7$

| $n$ | | | $d(n)$ |
|---|---|---|---|
| ∗15120 | = | $2^4 \cdot 3^3 \cdot 5 \cdot 7$ | 80 |
| 20160 | = | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 84 |
| ∗25200 | = | $2^4 \cdot 3^2 \cdot 5^2 \cdot 7$ | 90 |
| ∗27720 | = | $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 96 |
| 45360 | = | $2^4 \cdot 3^4 \cdot 5 \cdot 7$ | 100 |
| 50400 | = | $2^5 \cdot 3^2 \cdot 5^2 \cdot 7$ | 108 |
| ∗55440 | = | $2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 120 |
| 83160 | = | $2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$ | 128 |
| ∗110880 | = | $2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 144 |
| ∗166320 | = | $2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$ | 160 |
| 221760 | = | $2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 168 |
| ∗277200 | = | $2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$ | 180 |
| ∗332640 | = | $2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$ | 192 |
| 498960 | = | $2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$ | 200 |
| ∗554400 | = | $2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$ | 216 |
| ∗665280 | = | $2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$ | 224 |
| ∗720720 | = | $2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 240 |

5.2. **Recursively highly composite numbers.** By analogy with highly composite numbers, a number $n$ is recursively highly composite if it has more recursive divisors than any of its predecessors.

**Definition 4.** *A number $n$ is recursively highly composite if $a(n) > a(m)$ for all $m < n$.*

These numbers are shown in the left side of Table 2. From the third term, they correspond to the sequence records of indices of $g(n)$, the K-champion numbers [9]. Because $a(n)$ depends only on the exponents in the prime factorization of $n$, the exponents in recursively highly composite numbers must be non-increasing.

5.3. **Recursively super-abundant numbers.** By analogy with super-abundant numbers, a number $n$ is recursively super-abundant if the sum of its recursive divisors, normalized by $n$, is greater than that of any of its predecessors.

**Definition 5.** *A number $n$ is recursively super-abundant if $b(n)/n > b(m)/m$ for all $m < n$.*

These numbers are starred in the left side of Table 2. Early on, recursively super-abundant numbers are recursively highly composite. The first exception is 181,440.

## 6. Applications

6.1. **Design grids.** In graphic and digital design, the layout of graphics and text is often constrained to lie on an underlying rectangular grid [11]. The grid elements are the primitive building blocks from which bigger columns or rows can be formed. For example, grids of 24 and 96 columns are often used for books and websites, respectively [11]. Using a grid reduces the space of possible designs, making it easier to navigate. And the design elements become more interoperable, like how Lego bricks snap into place with one another, making it faster to build new designs.

What are the best grid sizes? The challenge is committing to a grid size now that provides the greatest optionality for an unknown future. Imagine, for example, that we have to cut a pie into slices, to be divided up later for an unknown number of colleagues. How many slices should we choose? The answer in this case is straightforward: the best grids are the ones with the most divisors, such as the highly composite or super-abundant numbers [3, 4].

What are the best grid sizes? The challenge is committing to a grid size now that provides the greatest optionality for an unknown future. Imagine, for example, that we have to cut a pie into slices, to be divided up later for an unknown number of colleagues. How many slices should we choose? The answer in this case is straightforward: the best grids are the ones with the most divisors, such as the highly composite or super-abundant numbers [3, 4].

But the story gets more complicated when we need to consider multiple steps into the future. For instance, imagine now that each colleague takes his share of pie home to further divide it amongst his family. In this case, not only does the whole need to be highly divisible, but the parts need to be highly divisible, too. This process can be extended in a recursive way.

Recursive modularity, in which there are multiple levels of organization, has long been a feature of graphic and digital design. For example, newspapers are divided into columns for different stories, and columns into sub-columns of text. But with the rise of digital technologies, recursive modularity is becoming the rule. Different

| $n$ | *Design and technology* | | *Display standards* | |
|---|---|---|---|---|
| *24 | $24 \times 16$ | Biotech 384-well assay | | |
| *48 | $128 \times 48$ | TRS 80 | | |
| 72 | 72 points/in | Adobe typography point | | |
| 96 | $96 \times 65$ | Nokia 1100 phone | | |
| *120 | $120 \times 160$ | Nokia 100 phone | $160 \times 120$ | QQVGA |
| 144 | $144 \times 168$ | Pebble Time watch | | |
| *240 | $240 \times 64$ | Atari Portfolio | $320 \times 240$ | Quarter VGA |
| 288 | $352 \times 288$ | Video CD | $352 \times 288$ | CIF |
| *360 | $360 \times 360$ | LG Watch Style | $640 \times 360$ | nHD |
| 480 | $320 \times 480$ | iPhone 1–3 | $640 \times 480$ | VGA |
| 576 | 576 lines | PAL analog television | $1024 \times 576$ | WSVGA |
| *720 | $720 \times 364$ | Macintosh XL, Hercules | $1280 \times 720$ | HD |
| 864 | | | $1152 \times 864$ | XGA+ |
| 960 | | Facebook website to 2019 | | |
| *1152 | | | $1152 \times 2048$ | QWXGA |
| *1440 | | 3.5" disk block size | $2560 \times 1440$ | Quad HD |
| 1920 | | | $1920 \times 1080$ | Full HD |
| *2160 | $2160 \times 1440$ | Microsoft Surface Pro 3 | $4096 \times 2160$ | 4K Ultra HD |
| 2304 | $2304 \times 1440$ | MacBook Retina | | |
| *2880 | $2880 \times 1800$ | 15" MacBook Pro Retina | $5120 \times 2880$ | 5K |
| 3456 | | Canon EOS 1100D | | |
| *4320 | | | $7680 \times 4320$ | 8K Ultra HD |
| *8640 | | | $15360 \times 8640$ | 16K Ultra HD |

TABLE 3. Recursively divisible numbers predict the numbers that frequently show up in design and technology and display standards. All of the numbers $n$ are recursively highly composite; those that are starred are also recursively super-abundant.

pages of a website are divided into different numbers of columns, each of which can be broken down into smaller design elements. Often one column from the website fills the full screen of a phone.

6.2. **Examples.** Recursively divisible numbers are especially well suited to recursive modularity. They provide maximal optionality for dividing wholes into parts in a recursive way. They are frequently used in design and technology and display standards. Examples of these are shown in Table 3.

In design and technology, these numbers are used for the screen resolutions of watches, phones, cameras and computers. They appear in typesetting, websites and experimental equipment, such as test tube microplates.

In display standards, many resolutions use these numbers in the height or width, measured in pixels. Because these standards tend to preserve certain aspect ratios, such as 16:9, usually just one of the two dimensions is highly recursively divisible.

6.3. **Open questions.** There are many open questions about the recursive divisor function and recursively divisible numbers. I list eight here.
1. For what values of $n$ does a divisor tree overlap itself?
2. For what values of $n$ do divisor trees have an (approximate) fractal dimension?
3. Is the normalized sum of the squares of the recursive divisors, $\kappa_2(n)/n^2$, bounded?
4. Theorem 2 relates $a(n) = \kappa_0(n)$ and $b(n) = \kappa_1(n)$. What about $\kappa_1(n)$ and $\kappa_2(n)$?
5. What is the recursion relation for $b(n)$ when $n$ is the product of $k$ distinct primes?

6. How many numbers are recursively highly composite? Recursively super-abundant?
7. Recursively perfect numbers satisfy $a(n) = n$. How dense are they?
8. Recursively abundant numbers satisfy $a(n) > n$. Are any odd and, if so, what is the smallest?

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