PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

RECURSIVELY DIVISIBLE NUMBERS

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Abstract. Divisible numbers are useful whenever a whole needs to be divided into equal parts. But sometimes we need to divide the parts into subparts, and subparts into sub-subparts, and so on, in a recursive way. To understand numbers that are recursively divisible, I introduce the recursive divisor function: a recursive analog of the usual divisor function. I give a geometric interpretation of recursively divisible numbers and study their general properties, in particular the number and sum of the recursive divisors. I show that the number of the recursive divisors is equal to twice the number of ordered factorizations, a problem much studied in its own right. By computing those numbers which are more recursively divisible than all of their predecessors, I recover many of the numbers used in design, technology and displays, and suggest new ones which have not been adopted but should be. These are useful for recursively modular systems which must operate across multiple organizational length scales.

1. Introduction

1.1. **Plato's ideal city.** Consider one of the earliest references to numbers which can be divided into equal parts in many ways. Plato writes in his *Laws* [1] that the ideal population of a city is 5040, since this number has more divisors than any number less than it. He observes that 5040 is divisible by 60 numbers, including one to 10. A highly divisible population is useful for dividing the city into equal-sized sectors for administrative, social and military purposes.

This conception of divisibility can be extended. Once the city is divided into equal parts, it is often necessary to divide a part into equal subparts. For example, if 5040 is divided into 15 parts of 336, each part can in turn be divided into subparts in 20 ways, since 336 has 20 divisors. But if 5040 is divided into 16 parts of 315, each part can be divided into subparts in only 12 ways, since 315 has 12 divisors. Thus the division of the whole into 15 parts offers more optionality for further subdivisions than the division into 16 parts. Similar reasoning can be applied to the divisibility of the subparts into sub-subparts, and so on, in a hierarchical way.

The goal of this paper is to quantify the notion of recursive divisibility and understand the properties of numbers which possess it to a large degree.

1.2. **Modular design.** In graphic and digital design, grid systems [2, 3] use a fixed number of columns or rows which form the primitive building blocks from which bigger columns or rows are made. For example, a grid of 24 columns is often used for books [2], and a grid of 96 columns is frequently used for websites. In exchange for giving up some freedom to choose the size of columns or rows by using a smaller grid size, the space of possible designs gets smaller, making it easier to navigate. And the design elements become more interoperable, like how Lego bricks snap into

Received by the editors March 14, 2021.

place on a discrete grid, making it faster to build new designs.

What are the best grid sizes for modular design? The challenge is committing to a grid size now that provides the greatest optionality for an unknown future. Imagine, for example, that I have to cut a pie into slices, to be divided up later for an unknown number of colleagues. How many slices should I choose? The answer in this case is relatively straightforward: the best grids are the ones with the most divisors, such as the highly composite or super-abundant numbers [4, 5].

But the story gets more complicated when it is necessary to consider multiple steps into the future. For instance, imagine now that each colleague takes his share of pie home to further divide it amongst his family. In this case, not only does the whole need to be highly divisible, but the parts need to be highly divisible, too. This process can be extended in a hierarchical way.

Design across multiple length scales has long been a feature of graphic design. For example, newspapers are divided into columns for different stories, and columns into sub-columns of text. But with the rise of digital technologies, recursive modularity is becoming the rule. Different pages of a website are divided into different numbers of columns, each of which can be broken down into smaller design elements. Often one column from the website fills the full screen of a phone.

What are the design rules for recursive modularity? The state-of-the-art is more lore than logic [2, 3], and there is little in the way of quantitative reasoning.

This paper gives a mathematical basis for choosing grids that are suitable for recursive design. It explains the preponderance of certain numbers in graphic design and display technologies, and predicts new numbers which have not been used but should be. More generally, it helps us understand recursively modular systems which must simultaneously operate across multiple organizational length scales.

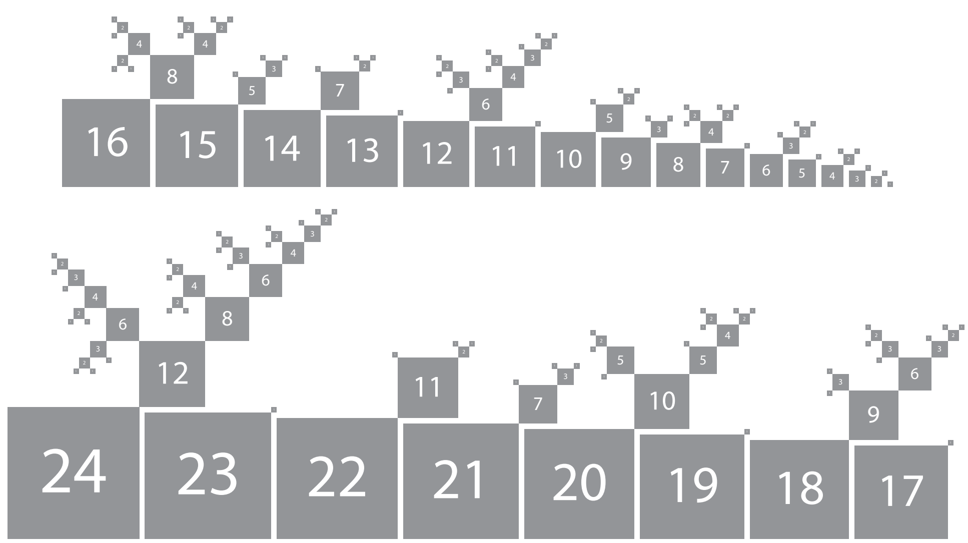

FIGURE 1. Divisor trees for 1 to 24. The number of recursive divisors $a(n)$ counts the number of squares in each tree and the sum of recursive divisors $b(n)$ adds up the side lengths of the squares in each tree. Divisor trees can be generated for any number n at lims.ac.uk/recursively-divisible-numbers.

1.3. **Outline of paper.** Including this introduction, this paper is divided into five parts. In part 2, I review the usual divisor function and define the recursive divisor function. I consider two specific instances of the latter: the number of recursive divisors and the sum of recursive divisors. I introduce divisor trees (Figures 1 and 2), which give a geometrical interpretation of the recursive divisor function. Using this, I show that the number of recursive divisors is twice the number of ordered factorizations, a problem much studied in its own right [6, 7, 8, 9, 10]. By examining the internal structure of divisor trees, I find a relation between the number and sum of recursive divisors.

In part 3, I investigate properties of the number of recursive divisors, taking advantage of their relation to the number of ordered factorizations. I give recursion relations for when n is the product of distinct primes, and for when n is the product of primes to a power. The latter can be solved for up to three primes.

In part 4, I investigate properties of the sum of recursive divisors. I give recursion relations for when n is the product of primes to a power. These can be solved using the relation between the sum and number of recursive divisors in part 2.

In part 5, I investigate numbers which are recursively divisible to a high degree. I call numbers with a record number of recursive divisors recursively highly composite, and list them up to one million. These have been studied in the context of the number of ordered factorizations [10]. I call numbers with a record sum of recursive divisors, normalized by n, recursively super-abundant, and also list them up to one million. I survey applications of highly recursive numbers in design, technology and displays, and conclude with a list of open problems.

2. Recursive divisor function and divisor trees

Throughout this paper I write $m|n$ to indicate m divides n and $m\lfloor n$ to indicate m is a proper divisor of n.

2.1. **Divisor function.** In order to write down a recursive divisor function, I first recall the usual divisor function,

$$\sigma_x(n) = \sum_{m|n} m^x.$$

This sums the divisors of n raised to some integer power x.

When $x = 0$, the divisor function counts the number divisors of n and is written $d(n)$. It is well known that for $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j}$, where $p_1, p_2, \dots, p_j$ are prime,

$$d(n) = (1 + \alpha_1)(1 + \alpha_2) \dots (1 + \alpha_j). \qquad (1)$$

Numbers n for which $d(n)$ is larger than that of all of the predecessors of n are called highly composite numbers and have been extensively studied [4, 5].

When $x = 1$, the divisor function sums the divisors of n and is written $\sigma(n)$. It is well known that

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \dots \frac{p_j^{\alpha_j+1} - 1}{p_j - 1}.$$

Numbers n for which $\sigma(n)/n$ is larger than that of all of the predecessors of n are called super-abundant numbers and have also been studied [5].

The quantity $\sigma(n)/n$ can be contrasted with $d(n)$ in the following way. Highly divisible numbers can be divided into equal-sized parts in many different ways. But for practical applications, not all divisions are equally useful. In general, divisions into fewer parts are more useful than divisions into many parts, because we are more likely to encounter the need for the former. In other words, we are more likely to need to divide a region into halves than into thirds, and into thirds than into fourths, and so on. To give preferential treatment to numbers with smaller divisors, consider that $d(n)$ effectively awards a point for each divisor of n. To give fewer points for larger divisors, let us award 1 point for numbers that can be divided into 1 part, $1/2$ point for numbers that can be divided into 2 parts, $1/3$ point for numbers that can be divided into 3 parts, and so on. This scheme gives the score of $\sigma(n)/n$.

2.2. **Recursive divisor function.** In contrast with the usual divisor function, in this paper I am concerned not only with the divisors of a number n but also the divisors of the resultant quotients, and the divisors of those resultant quotients, and so on. I introduce and study the recursive divisor function,

$$\kappa_x(n) = n^x + \sum_{m \lfloor n} \kappa_x(m),$$

where the sum is over the proper divisors of n. When $x = 0$, I call this the number of recursive divisors $a(n)$. When $x = 1$, I call this the sum of recursive divisors $b(n)$.

Definition 1. *The number of recursive divisors is* $a(1) = 1$ *and*

$$a(n) = 1 + \sum_{m \lfloor n} a(m),$$

where $m \lfloor n$ *means* m *is a proper divisor of* n.

For example, $a(10) = 1 + a(1) + a(2) + a(5) = 6$. Note that $a(n)$ depends only on the set of exponents in the prime factorization of n and not on the primes themselves.

Definition 2. *The sum of recursive divisors is* $b(1) = 1$ *and*

$$b(n) = n + \sum_{m \lfloor n} b(m).$$

For example, $b(10) = 10 + b(1) + b(2) + b(5) = 20$. Unlike $a(n)$, $b(n)$ depends on both the exponents and the primes in the prime factorization of n. Ultimately I will be interested in $b(n)/n$, analogous to $\sigma(n)/n$ described above, but for now it is more natural to define and work with $b(n)$.

2.3. **Divisor trees.** A geometric interpretation of the recursive divisor function $\kappa(n)$ can be had by drawing the divisor tree for a given value of n. Divisor trees for 1 to 24 are shown in Figure 1. The number of recursive divisors $a(n)$ counts the number of squares in each tree, whereas $d(n)$ counts the number of squares in the main diagonal. The sum of recursive divisors $b(n)$ adds up the side lengths of the squares in each tree, whereas $\sigma(n)$ adds up the side lengths of the squares in the main diagonal. This can be extended to $\kappa_2(n)$, which adds up area, and so on, but in this paper I only consider $a(n) = \kappa_0(n)$ and $b(n) = \kappa_1(n)$.

A divisor tree is constructed as follows. First, draw a square of side length n. Let $m_1, m_2, \ldots$ be the proper divisors of n in descending order. Then draw squares of side length m_1, $m_2, \ldots$ with each consecutive square situated to the upper right of its predecessor, kitty-corner. This forms the main arm of a divisor tree. Now, for each of the squares of side length $m_1, m_2, \ldots$, repeat the process. Let $l_1, l_2, \ldots$ be the proper divisors of m_1 in descending order. Then draw squares of side length l_1, $l_2, \ldots$, but with the sub-arm rotated 90° counter-clockwise. Do the same for each of the remaining squares in the main arm. This forms the branches off of the main arm. Now, continue repeating this process, drawing arms off of arms off of arms, and so on, until the arms are single squares of size 1.

2.4. **Properties of divisor trees.** In order to establish properties of the number and sum of recursive divisors, it helps to consider a more fine-grained description of divisor trees, namely, the number and sum of divisors of a given size.

Proposition 1. *Let the number of recursive divisors of size k be $a(n,k)$. Then $a(n,n) = 1$ and for $k\lfloor n$,*

$$a(n,k) = \sum_{m\lfloor n} a(m,k),$$

and $a(n,k) = 0$ otherwise.

Proof. The Proposition follows immediately on considering the divisor trees. □

Lemma 1. *The number of recursive divisors of size k satisifes $a(k\,n,k) = a(n,1)$.*

Proof. By Proposition 1,

(2)
$$a(n,1) = \sum_{m\lfloor n} a(m,1)$$

and so

$$a(k\,n,k) = \sum_{m\lfloor k\,n} a(m,k).$$

Since $a(m,k) = 0$ if k does not divide m, this can be rewritten as

(3)
$$a(k\,n,k) = \sum_{m\lfloor n} a(k\,m,k).$$

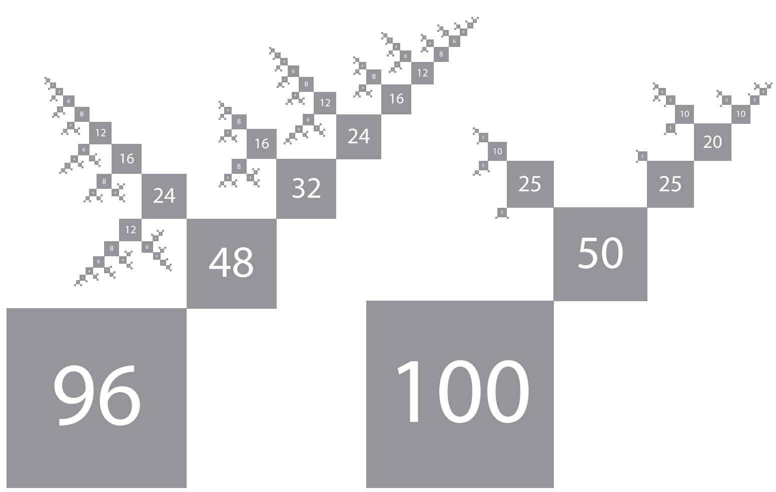

FIGURE 2. Divisor trees for 96 and 100. There are $a(96) = 224$ squares in the left tree and $a(100) = 52$ squares in the right. The sum of the side lengths of the squares, or one-fourth of the tree perimeter, is $b(96) = 768$ in the left tree and $b(100) = 340$ in the right.

Let the prime omega function $\Omega(n)$ sum the exponents in the prime factorization of n, that is, for $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_j^{\alpha_j}$, $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_j$. I prove the theorem by induction on $\Omega(n)$. The base case $\Omega(n) = 0$, or $n = 1$, holds by Proposition 1: $a(k{\cdot}1, k) = a(1, 1)$. I now show that if $a(k\,n, k) = a(n, 1)$ for all n such that $\Omega(n) < i$, then $a(k\,n, k) = a(n, 1)$ for all n such that $\Omega(n) < i+1$. To see why, observe that in (3) all of the proper divisors m of n must have $\Omega(m)$ at most $\Omega(n) - 1$. Therefore by assumption all of the $a(k\,m, k)$ in (3) reduce to $a(m, 1)$, and the right side of (3) takes the form of the right side of (2) and thus equals $a(n, 1)$. □

Lemma 2. *For $n > 1$, the number of recursive divisors of size 1 is equal to half the total number of recursive divisors, that is, $a(n, 1) = a(n)/2$.*

Proof. Clearly

$$a(n) = \sum_{m|n} a(n, m).$$

By Lemma 1, this becomes

$$a(n) = \sum_{m|n} a(n/m, 1) = \sum_{m|n} a(m, 1) = a(n, 1) + \sum_{m\lfloor n} a(m, 1).$$

Inserting Proposition 1 with $k = 1$ into the above gives the desired result. □

2.5. **Relation to the number of ordered factorizations.** Here I show that for $n > 1$, the number of recursive divisors $a(n)$ is twice the number of ordered factorizations into integers greater than one, which I call $g(n)$. But before getting to that, I first introduce $g(n)$ and mention some of the work on it.

The number of ordered factorizations $g(n)$ satisifes $g(1) = 1$ and

$$g(n) = \sum_{m\lfloor n} g(m).$$

For example, 12 is the product of integers greater than one in eight ways: $12 = 6 \cdot 2 = 2 \cdot 6 = 4 \cdot 3 = 3 \cdot 4 = 3 \cdot 2 \cdot 2 = 2 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3$. So $g(12) = 8$.

Kalmar [6] was the first to consider $g(n)$, and it was later studied more systematically by Hille [7]. Over the last 80 years several authors have extended Hille's results [8, 9, 10], some of which we will mention later.

Theorem 1. *Let $g(n)$ be the number of ordered factorizations into integers greater than one and set $g(1) = 1$. Then for $n > 1$, $a(n) = 2g(n)$.*

Proof. The definition of $g(n)$ is identical to Proposition 1 for $k = 1$, that is, identical to $a(n, 1)$. Since $g(1) = a(1, 1) = 1$, the proof follows from Lemma 2. □

2.6. **Relation between the number and sum of recursive divisors.** The $b(n)$ are more difficult to calculate than the $a(n)$, and it would be helpful to have an expression relating the $b(n)$ to the $a(n)$. Here I give just such a relation. I will use it later to explicitly determine $b(n)$ for certain values of n.

Theorem 2. *Let $B(n) = b(n)/n$ and $A(n) = a(n)/n$. Then*

$$B(n) = \frac{1}{2} + \frac{1}{2} \sum_{m|n} A(m).$$

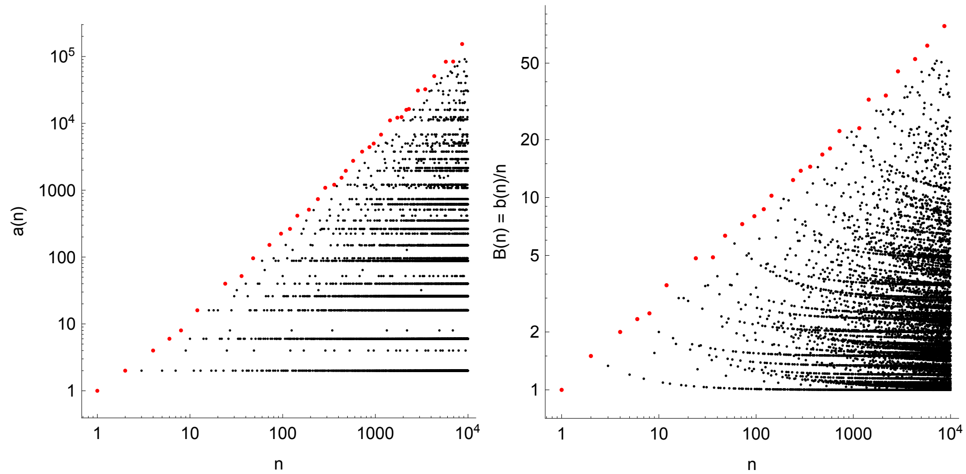

FIGURE 3. On the left are the numbers of recursive divisors $a(n)$. The recursively highly composite numbers, which satisfy $a(n) > a(m)$ for $m < n$, are the big red points. On the right are the sums of recursive divisors $b(n)$, normalized by n. The recursively super-abundant numbers, which satisfy $b(n)/n > b(m)/m$ for $m < n$, are the big red points.

Proof. Let $b(n, m)$ be the sum of the side lengths of squares of size m in the nth divisor tree. Then

$$b(n) = \sum_{m|n} b(n, m) = \sum_{m|n} m\, a(n, m).$$

By Lemma 1,

$$b(n) = \sum_{m|n} m\, a(n/m, 1) = n + \sum_{m\lfloor n} m\, a(n/m, 1).$$

By Lemma 2,

$$b(n) = n + \frac{1}{2}\sum_{m\lfloor n} m\, a(n/m) = \frac{n}{2} + \frac{1}{2}\sum_{m|n} m\, a(n/m) = \frac{n}{2} + \frac{n}{2}\sum_{m|n} a(m)/m.$$

Dividing by n, the theorem follows. □

3. Number of recursive divisors

The number of recursive divisors $a(n)$ is not as readily determined as the number of divisors $d(n)$. The first 72 values of $a(n)$ are shown in Table 1, which also gives an algorithm to generate them. The first 10^4 values are plotted in Figure 3.

3.1. **Distinct primes.** Let $n = p_1\, p_2 \ldots p_k$ be the product of k distinct primes. By (1), $d(p_1\, p_2 \ldots p_k) = 2^k$. Here I calculate the less straightforward $a(p_1\, p_2 \ldots p_k)$.

Theorem 3. *Let $n = p_1\, p_2 \ldots p_k$ be the product of k distinct primes. Then the exponential generating function of $a(p_1 p_2 \ldots p_k)$ is*

$$\mathrm{EG}(a(p_1 p_2 \ldots p_k), x) = \frac{e^x}{2 - e^x}.$$

Proof. This theorem is equivalent to

$$a(p_1p_2\dots p_k) = 1 + \sum_{i=0}^{k-1} \binom{k}{i} a(p_1p_2\dots p_i). \tag{4}$$

I prove it by induction. First note that $a(p_1) = 1 + \binom{1}{0}a(1) = 2$. I now show that if (4) is true for k, then it is true for $k+1$. I do so by adding to $a(p_1p_2\dots p_k)$ all of the a values for divisors that include p_{k+1}, apart from $a(p_1p_2\dots p_{k+1})$, since $p_1p_2\dots p_{k+1}$ is not a proper divisor of itself. I also add $a(p_1p_2\dots p_k)$, which is left out of the expression for $a(p_1p_2\dots p_k)$ for the same reason. There are $\binom{k}{0}$ a values that are the product of one prime that include p_{k+1}, $\binom{k}{1}$ that are the product of two primes that include p_{k+1}, and so on. Since a depends on the number of primes but not their values, we are not concerned with how we label the primes. Then

$$\begin{aligned} a(p_1\dots p_{k+1}) &= a(p_1\dots p_k) + \binom{k}{0}a(p_1) + \binom{k}{1}a(p_1p_2) + \dots + \binom{k}{k-1}a(p_1\dots p_k) \\ &\quad + a(p_1\dots p_k) \\ &= 1 + a(p_1\dots p_k) + \sum_{i=0}^{k-1}\binom{k}{i}a(p_1\dots p_i) + \sum_{i=0}^{k-1}\binom{k}{i}a(p_1\dots p_{i+1}) \\ &= 1 + (k+1)a_1(p_1\dots p_k) + \sum_{i=0}^{k-1}\left(\binom{k}{i} + \binom{k}{i-1}\right)a(p_1\dots p_i) \end{aligned}$$

n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$
1	1	1	19	2	20	37	2	38	55	6	74
2	2	3	20	16	58	38	6	62	56	40	196
3	2	4	21	6	34	39	6	58	57	6	82
4	4	8	22	6	38	40	40	156	58	6	92
5	2	6	23	2	24	41	2	42	59	2	60
6	6	14	24	40	116	42	26	132	60	88	346
7	2	8	25	4	32	43	2	44	61	2	62
8	8	20	26	6	44	44	16	106	62	6	98
9	4	14	27	8	46	45	16	96	63	16	124
10	6	20	28	16	74	46	6	74	64	64	256
11	2	12	29	2	30	47	2	48	65	6	86
12	16	42	30	26	104	48	96	304	66	26	188
13	2	14	31	2	32	49	4	58	67	2	68
14	6	26	32	32	112	50	16	112	68	16	154
15	6	26	33	6	50	51	6	74	69	6	98
16	16	48	34	6	56	52	16	122	70	26	184
17	2	18	35	6	50	53	2	54	71	2	72
18	16	54	36	52	176	54	40	190	72	152	524

TABLE 1. The first 72 values of the number of recursive divisors $a(n)$ and the sum of recursive divisors $b(n)$. A concise Mathematica algorithm for $a(n)$ is: `n = 2; max = 72; a = {1}; While[n <= max, a = Append[a, 1 + Total[Part[a, Delete[Divisors[n], -1]]]]; n++]; a` An algorithm for $b(n)$ is: `n = 2; max = 72; b = {1}; While[n <= max, b = Append[b, n + Total[Part[b, Delete[Divisors[n], -1]]]]; n++ ]; b`

$$= 1 + \sum_{i=0}^{k} \binom{k+1}{i} a(p_1 \dots p_i). \quad \square$$

So for the product of $k = 1, 2, \dots$ distinct primes, $a(k) = 2, 6, 26, 150, 1082, \dots$.

3.2. **Primes to a power.** When n is the product of primes to powers, $a(n)$ satisfies recursion relations relating it to values of $a(n)$ for primes to lower powers.

Theorem 4. *Let p, q and r be prime. Then*

$$\begin{aligned}
a(p^c) &= 2a(p^{c-1}), \\
a(p^c q^d) &= 2\Big(a(p^{c-1}q^d) + a(p^c q^{d-1}) - a(p^{c-1}q^{d-1})\Big), \\
a(p^c q^d r^e) &= 2\Big(a(p^{c-1}q^d r^e) + a(p^c q^{d-1} r^e) + a(p^c q^d r^{e-1}) - a(p^c q^{d-1} r^{e-1}) \\
&- a(p^{c-1} q^d r^{e-1}) - a(p^{c-1} q^{d-1} r^e) + a(p^{c-1} q^{d-1} r^{e-1})\Big).
\end{aligned}$$

Analogous recursion relations apply for the product of more primes to powers.

Proof. The approach is similar to, but somewhat simpler than, that used to prove Theorem 6. However, Hille [7] and Chor *et al.* [8] proved that identical recursion relations govern $g(n)$, the number of ordered factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and inserting this into Hille's and Chor's recursion relations gives the desired results. $\square$

Corollary 1. *Let τ be the maximum exponent in the prime factorization of n. Then 2^τ divides $a(n)$.*

Proof. All of the recursion relations in Theorem 4 have a factor of 2 on the right side. The corollary is implied by iterating the recursion relation τ times. Each time, the exponents on the right are reduced by at most 1. Iterating until the smallest exponent is reduced to 0, the exponent disappears since, for example, $a(p^c q^0) = a(p^c)$. Continuing this process ultimately gives a total of τ factors of 2. The $a(n)$ are expressed as a product of and integer and 2^τ in Table 3. $\square$

The first three recursion relations in Theorem 4 can be solved explicitly.

Theorem 5. *Let p, q and r be prime. Then*

$$\begin{aligned}
a(p^c) &= 2^c \\
a(p^c q^d) &= 2^c \sum_{i=0}^{d} \binom{d}{i} \binom{c+i}{i}. \\
a(p^c q^d r^e) &= \sum_{j=0}^{d} (-1)^j \binom{d}{j} \binom{c+d-j}{d} a(p^{c+d-j} r^e).
\end{aligned}$$

Proof. The result for $n = p^c$ follows by inspection. For $n = p^c q^d$ and $n = p^c q^d r^e$, Chor *et al.* [8] give the analogous results for $g(n)$, the number of ordered factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and applying this to Chor's results gives the desired recurrence relations. $\square$

4. Sum of recursive divisors

I now turn to the sum of recursive divisors $b(n)$. The first 72 values and an algorithm to generate them are shown in Table 1. The first 10^4 values are plotted in Figure 3.

4.1. **Primes to a power.** When n is equal to the product of primes to powers, $b(n)$ satisfies recursion relations relating it to values of $b(n)$ for primes to lower powers. The recursion relations are similar to those for $a(n)$, but more complex.

Theorem 6. *Let p, q and r be prime. Then*

$$\begin{aligned} b(p^c) &= 2b(p^{c-1}) + (p-1)p^{c-1}, \\ b(p^c q^d) &= 2\Big(b(p^{c-1}q^d) + b(p^c q^{d-1}) - b(p^{c-1}q^{d-1})\Big) + (p-1)(q-1)p^{c-1}q^{d-1}, \\ b(p^c q^d r^e) &= 2\Big(b(p^{c-1}q^d r^e) + a(p^c q^{d-1} r^e) + a(p^c q^d r^{e-1}) \\ &- b(p^c q^{d-1} r^{e-1}) - a(p^{c-1} q^d r^{e-1}) - a(p^{c-1} q^{d-1} r^e) \\ &+ b(p^{c-1} q^{d-1} r^{e-1})\Big) + (p-1)(q-1)(r-1)p^{c-1}q^{d-1}r^{e-1}. \end{aligned}$$

Proof. I first prove the case of $n = p^c$. From Definition 2,

$$b(p^c) = p^c + \sum_{i=0}^{c-1} b(p^i). \tag{5}$$

Adding $b(p^c)$ to both sides and with $c \to c-1$,

$$\sum_{i=0}^{c-1} b(p^i) = 2b(p^{c-1}) - p^{c-1},$$

which, when inserted into (5), gives the desired recurrence relation.

I now prove the case of $n = p^c q^d$. From Definition 2,

$$b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1}\sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d-1} b(p^c q^j). \tag{6}$$

Adding $b(p^c q^d)$ to both sides,

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1}\sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d} b(p^c q^j), \tag{7}$$

which we can equally write

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c}\sum_{j=0}^{d} b(p^i q^j). \tag{8}$$

With $d \to d-1$ in (7), we find

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - p^c q^{d-1} - \sum_{j=0}^{c-1}\sum_{i=0}^{d-1} b(p^i q^j). \tag{9}$$

With $c \to c-1$ and $d \to d-1$ in (8), and inserting the result into (9), yields

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - 2b(p^{c-1}q^{d-1}) + (1-p)p^{c-1}q^{d-1}. \tag{10}$$

With $c \to c-1$ in (8), we find

$$\sum_{i=0}^{c-1}\sum_{j=0}^{d} b(p^i q^j) = 2b(p^{c-1}q^d) - p^{c-1}q^d. \tag{11}$$

Inserting (10) and (11) into (6) gives the desired recursion relation.

For $n = p^c q^d r^e$, the proof is similar to that for $n = p^c q^d$ and is omitted here. □

4.2. **Explicit values.** The recursion relations in Theorem 6 can be solved. I only give the results for $n = p^c$ and $n = p^c q^d$. For $n = p^c q^d r^e$, the solution is more intricate but can be solved in a similar way to that for $n = p^c q^d$.

Theorem 7. *Let p and q be prime, and $B(n) = b(n)/n$. Then*

$$\begin{aligned} B(p^c) &= \frac{p-1-(2/p)^c}{p-2} \quad \text{for } p \text{ odd}, \\ B(2^c) &= (c+2)/2, \\ B(p^c q^d) &= \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c}\frac{2^i}{p^i}\sum_{j=0}^{d}\frac{1}{q^j}\sum_{k=0}^{j}\binom{i+k}{k}\binom{j}{k}. \end{aligned}$$

Proof. I first prove the case of $n = p^c$. From Theorem 2,

$$B(p^c) = \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c} A(p^i). \tag{12}$$

From Theorem 4, $a(p^c) = 2^c$ and $A(p^i) = a(p^i)/p^i = (2/p)^i$. Inserting this into (12), we find the desired result. For $p = 2$, by L'Hôpital's rule, $B(2^c) = (c+2)/2$.

I now prove the case of $n = p^c q^d$. From Theorem 2,

$$B(p^c q^d) = \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c}\sum_{j=0}^{d} A(p^i q^j). \tag{13}$$

Theorem 5 gives $a(p^c q^d)$ explicitly. Inserting $A(p^i q^j) = a(p^i q^j)/(p^i q^j)$ into (13) yields the desired result. For $p = 2$, the result simplifies to contain just two sums. □

5. Recursively highly composite and recursively super-abundant numbers

5.1. **Highly composite and super-abundant numbers.** I briefly review highly composite and super-abundant numbers before considering their recursive analogues. A number n is highly composite if it has more divisors than any of its predecessors, that is, $d(n) > d(m)$ for all $m < n$. These are shown in the right side of Table 3. A number n is super-abundant if the sum of its divisors, normalized by n, is greater than that of any of its predecessors, that is, $\sigma(n)/n > \sigma(m)/m$ for all $m < n$. These are the starred numbers in the right side of Table 3. Both types of numbers have been well-studied by Ramanujan and others [4, 5]. For small n, super-abundant numbers are also highly composite, but later this ceases to be the case. The first super-abundant number that is not highly composite is 1,163,962,800 (A166735 [11]), and in fact only 449 numbers have both properties (A166981 [11]).

n			$a(n)$	n			$d(n)$
$*1$	=	1	1	$*1$	=	1	1
$*2$	=	2	$1 \cdot 2$	$*2$	=	2	2
$*4$	=	2^2	$1 \cdot 2^2$	$*4$	=	2^2	3
$*6$	=	$2 \cdot 3$	$3 \cdot 2$	$*6$	=	$2 \cdot 3$	4
8	=	2^3	$1 \cdot 2^3$				
$*12$	=	$2^2 \cdot 3$	$4 \cdot 2^2$	$*12$	=	$2^2 \cdot 3$	6
$*24$	=	$2^3 \cdot 3$	$5 \cdot 2^3$	$*24$	=	$2^3 \cdot 3$	8
$*36$	=	$2^2 \cdot 3^2$	$13 \cdot 2^2$	$*36$	=	$2^2 \cdot 3^2$	9
$*48$	=	$2^4 \cdot 3$	$6 \cdot 2^4$	$*48$	=	$2^4 \cdot 3$	10
				$*60$	=	$2^2 \cdot 3 \cdot 5$	12
72	=	$2^3 \cdot 3^2$	$19 \cdot 2^3$				
96	=	$2^5 \cdot 3$	$7 \cdot 2^5$				
$*120$	=	$2^3 \cdot 3 \cdot 5$	$33 \cdot 2^3$	$*120$	=	$2^3 \cdot 3 \cdot 5$	16
144	=	$2^4 \cdot 3^2$	$26 \cdot 2^4$				
				$*180$	=	$2^2 \cdot 3^2 \cdot 5$	18
192	=	$2^6 \cdot 3$	$8 \cdot 2^6$				
$*240$	=	$2^4 \cdot 3 \cdot 5$	$46 \cdot 2^4$	$*240$	=	$2^4 \cdot 3 \cdot 5$	20
288	=	$2^5 \cdot 3^2$	$34 \cdot 2^5$				
$*360$	=	$2^3 \cdot 3^2 \cdot 5$	$151 \cdot 2^3$	$*360$	=	$2^3 \cdot 3^2 \cdot 5$	24
432	=	$2^4 \cdot 3^3$	$96 \cdot 2^4$				
480	=	$2^5 \cdot 3 \cdot 5$	$61 \cdot 2^5$				
576	=	$2^6 \cdot 3^2$	$43 \cdot 2^6$				
$*720$	=	$2^4 \cdot 3^2 \cdot 5$	$236 \cdot 2^4$	$*720$	=	$2^4 \cdot 3^2 \cdot 5$	30
				$*840$	=	$2^3 \cdot 3 \cdot 5 \cdot 7$	32
864	=	$2^5 \cdot 3^3$	$138 \cdot 2^5$				
960	=	$2^6 \cdot 3 \cdot 5$	$78 \cdot 2^6$				
$*1152$	=	$2^7 \cdot 3^2$	$53 \cdot 2^7$				
				$*1260$	=	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	36
$*1440$	=	$2^5 \cdot 3^2 \cdot 5$	$346 \cdot 2^5$				
				$*1680$	=	$2^4 \cdot 3 \cdot 5 \cdot 7$	40
1728	=	$2^6 \cdot 3^3$	$190 \cdot 2^6$				
1920	=	$2^7 \cdot 3 \cdot 5$	$97 \cdot 2^7$				
$*2160$	=	$2^4 \cdot 3^3 \cdot 5$	$996 \cdot 2^4$				
2304	=	$2^8 \cdot 3^2$	$64 \cdot 2^8$				
				$*2520$	=	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	48
$*2880$	=	$2^6 \cdot 3^2 \cdot 5$	$484 \cdot 2^6$				
3456	=	$2^7 \cdot 3^3$	$253 \cdot 2^7$				
$*4320$	=	$2^5 \cdot 3^3 \cdot 5$	$1590 \cdot 2^5$				
				$*5040$	=	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	60
$*5760$	=	$2^7 \cdot 3^2 \cdot 5$	$653 \cdot 2^7$				
6912	=	$2^8 \cdot 3^3$	$328 \cdot 2^8$				
				7560	=	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	64
$*8640$	=	$2^6 \cdot 3^3 \cdot 5$	$2402 \cdot 2^6$				
				$*10080$	=	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	72
$*11520$	=	$2^8 \cdot 3^2 \cdot 5$	$856 \cdot 2^8$				

TABLE 3. The left side shows the recursively highly composite numbers and the recursively super-abundant numbers (starred) up to one million. All of the recursively super-abundant numbers shown here are also recursively highly composite, apart from one, 181,440. The right column shows the highly composite numbers and the super-abundant numbers (starred) up to one million. All of the super-abundant numbers shown here are also highly composite.

n			$a(n)$
*17280	=	$2^7 \cdot 3^3 \cdot 5$	$3477 \cdot 2^7$
23040	=	$2^9 \cdot 3^2 \cdot 5$	$1096 \cdot 2^9$
*25920	=	$2^6 \cdot 3^4 \cdot 5$	$10368 \cdot 2^6$
*30240	=	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	$20874 \cdot 2^5$
*34560	=	$2^8 \cdot 3^3 \cdot 5$	$4864 \cdot 2^8$
46080	=	$2^{10} \cdot 3^2 \cdot 5$	$1376 \cdot 2^{10}$
*51840	=	$2^7 \cdot 3^4 \cdot 5$	$15979 \cdot 2^7$
*60480	=	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$34266 \cdot 2^6$
*69120	=	$2^9 \cdot 3^3 \cdot 5$	$6616 \cdot 2^9$
86400	=	$2^7 \cdot 3^3 \cdot 5^2$	$28481 \cdot 2^7$
*103680	=	$2^8 \cdot 3^4 \cdot 5$	$23692 \cdot 2^8$
*120960	=	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	$53485 \cdot 2^7$
138240	=	$2^{10} \cdot 3^3 \cdot 5$	$8790 \cdot 2^{10}$
161280	=	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	$17656 \cdot 2^9$
*172800	=	$2^8 \cdot 3^3 \cdot 5^2$	$42520 \cdot 2^8$
*207360	=	$2^9 \cdot 3^4 \cdot 5$	$34026 \cdot 2^9$
*241920	=	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	$80176 \cdot 2^8$
276480	=	$2^{11} \cdot 3^3 \cdot 5$	$11447 \cdot 2^{11}$
311040	=	$2^8 \cdot 3^5 \cdot 5$	$103540 \cdot 2^8$
*345600	=	$2^9 \cdot 3^3 \cdot 5^2$	$61436 \cdot 2^9$
*362880	=	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	$267219 \cdot 2^7$
*414720	=	$2^{10} \cdot 3^4 \cdot 5$	$47576 \cdot 2^{10}$
*483840	=	$2^9 \cdot 3^3 \cdot 5 \cdot 7$	$116256 \cdot 2^9$
552960	=	$2^{12} \cdot 3^3 \cdot 5$	$14652 \cdot 2^{12}$
604800	=	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$480953 \cdot 2^7$
622080	=	$2^9 \cdot 3^5 \cdot 5$	$156278 \cdot 2^9$
691200	=	$2^{10} \cdot 3^3 \cdot 5^2$	$86362 \cdot 2^{10}$
*725760	=	$2^8 \cdot 3^4 \cdot 5 \cdot 7$	$422932 \cdot 2^8$
829440	=	$2^{11} \cdot 3^4 \cdot 5$	$65018 \cdot 2^{11}$
*967680	=	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	$163934 \cdot 2^{10}$

Recursively super-abundant but not recursively highly composite

*181440 = $2^6 \cdot 3^4 \cdot 5 \cdot 7$

n			$d(n)$
*15120	=	$2^4 \cdot 3^3 \cdot 5 \cdot 7$	80
20160	=	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	84
*25200	=	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7$	90
*27720	=	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	96
45360	=	$2^4 \cdot 3^4 \cdot 5 \cdot 7$	100
50400	=	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$	108
*55440	=	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	120
83160	=	$2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	128
*110880	=	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	144
*166320	=	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	160
221760	=	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	168
*277200	=	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	180
*332640	=	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	192
498960	=	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	200
*554400	=	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	216
*665280	=	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	224
*720720	=	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	240

5.2. **Recursively highly composite numbers.** By analogy with highly composite numbers, a number n is recursively highly composite if it has more recursive divisors than any of its predecessors.

Definition 3. *A number is recursively highly composite if* $a(n) > a(m)$ *for all* $m < n$.

These numbers are shown in the left side of Table 3 up to one million. From the third term, they correspond to the sequence records of indices of $g(n)$, the K-champion numbers [10]. In terms of divisor trees, a number is recursively highly composite if its divisor tree has more squares than any of its predecessors' divisor trees. Because $a(n)$ depends only on the exponents in the prime factorization of $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \dots p_j^{\alpha_j}$, the exponents in recursively highly composite numbers must be non-increasing.

5.3. **Recursively super-abundant numbers.** By analogy with super-abundant numbers, a number n is recursively super-abundant if the sum of its recursive divisors, normalized by n, is greater than that of any of its predecessors.

Definition 4. *A number is recursively super-abundant if* $b(n)/n > b(m)/m$ *for all* $m < n$.

These numbers are shown as starred in the left side of Table 3 up to one million. Geometrically, a number is recursively super-abundant if the perimeter of its divisor tree, normalized such that the largest square has length one, is bigger than that of any of its predecessors' divisor trees. For small values of n, recursively super-abundant numbers are recursively highly composite. The first exception is 181,440.

5.4. **Applications.** Recursively divisible numbers are especially well suited to design across multiple length scales, in which the whole must be divided into parts, the parts into subparts, and so on. Recursively highly composite and recursively super-abundant numbers are frequently used in design, technology and displays, examples of which are shown in Table 4.

In design and technology, these numbers are used for the screen resolutions of watches, phones, cameras and computers. They appear in typesetting and in experimental equipment, such as test tube microplates. Using these numbers provides maximal optionality for dividing space into parts in a hierarchical way when building websites, laying out text or planning experiments.

In displays, many standard resolutions use these numbers in the height or width, measured in pixels. Because standard displays tend to preserve certain aspect ratios, such as 16:9, usually just one of the two dimensions is highly recursively divisible.

5.5. **Ten open questions.** There are many open questions about the recursive divisor function and recursively divisible numbers, and I list eight here.

1. For what values of n does a divisor tree overlap itself?
2. For what values of n do divisor trees have an (approximate) fractal dimension?
3. Is the normalized sum of the squares of the recursive divisors, $\kappa_2(n)/n^2$, bounded?
4. Theorem 2 relates $a(n) = \kappa_0(n)$ and $b(n) = \kappa_1(n)$. What about $\kappa_1(n)$ and $\kappa_2(n)$?
5. What is the recursion relation for $b(n)$ when n is the product of k distinct primes?
6. How many numbers are recursively highly composite and recursively super-abundant?
7. Recursively perfect numbers satisfy $a(n) = n$. How dense are they?
8. Recursively abundant numbers satisfy $a(n) > n$. What is the smallest odd one?

n	*Design and technology*		*Standard displays*	
*24	24×16	Biotech 384-well assay		
*48	128×48	TRS 80		
72	72 points/in	Adobe typography point		
96	96×65	Nokia 1100 phone		
*120	120×160	Nokia 100 phone	160×120	QQVGA
144	144×168	Pebble Time watch		
*240	240×64	Atari Portfolio	320×240	Quarter VGA
288	352×288	Video CD	352×288	CIF
*360	360×360	LG Watch Style	640×360	nHD
480	320×480	iPhone 1–3	640×480	VGA
576	576 lines	PAL analog television	1024×576	WSVGA
*720	720×364	Macintosh XL, Hercules	1280×720	HD
864			1152×864	XGA+
960		Facebook website to 2019		
*1152			1152×2048	QWXGA
*1440		3.5” disk block size	2560×1440	Quad HD
1920			1920×1080	Full HD
*2160	2160×1440	Microsoft Surface Pro 3	4096×2160	4K Ultra HD
2304	2304×1440	MacBook Retina		
*2880	2880×1800	15” MacBook Pro Retina	5120×2880	5K
3456		Canon EOS 1100D		
*4320			7680×4320	8K Ultra HD
*8640			15360×8640	16K Ultra HD

Table 4. Some applications of recursively highly composite numbers and recursively super-abundant numbers (starred) in design, technology and standard displays.

I acknowledge Rayan Melwani for helpful discussions and Andriy Fedosyeyev for creating the divisor tree generator, lims.ac.uk/recursively-divisible-numbers.

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