# Hyperbolicity measures democracy in real-world networks

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In this work, we analyze the hyperbolicity of real-world networks, a geometric quantity that measures if a space is negatively curved. We provide two improvements in our understanding of this quantity: first of all, in our interpretation, a hyperbolic network is "aristocratic", since few elements "connect" the system, while a non-hyperbolic network has a more "democratic" structure with a larger number of crucial elements. The second contribution is the introduction of the average hyperbolicity of the neighbors of a given node. Through this definition, we outline an "influence area" for the vertices in the graph. We show that in real networks the influence area of the highest degree vertex is small in what we define "local" networks (i.e., social or peer-to-peer networks), and large in "global" networks (i.e., power grid, metabolic networks, or autonomous system networks).

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## I. INTRODUCTION

The most basic way to describe a graph is to consider its metric quantities as for instance the diameter [1], the degrees, the distances [2]. In case no other information is available, a good choice is to consider randomly drawn edges [3,4] and determine the expected values and distribution of those properties in such random graphs. More recently, the computer revolution and the pervasive presence of the internet and worldwide web, created a whole series of complex networks in technological systems whose properties can be directly measured from data [5,6]. All the real networks show particular structures of edges, making them definitely different from random graphs. Driven by such evidence, researchers recognized analogous structures in a variety of other cases, ranging from biology to economics and finance [7–10]. All these structures show a lack of characteristic scale in the statistical distribution of the degree and small world effect, making it therefore important to understand the basic principles at the basis of their formation [11–14].

To bring order in this huge set of systems, it would be extremely useful if we could classify the various networks by means of some specific quantity differing from case to case. In this quest to distinguish universal from particular behavior we decide to consider the connection with the "curvature" of the graph. Embedding spaces can have negative curvature (hyperbolic spaces), zero value of curvature (Euclidean spaces), or positive curvature (spherical spaces). On the basis of the hyperbolicity measure [15], it is possible to extend such a measure of curvature for manifolds to discrete networks. Hyperbolicity measure [15] defines the curvature for an infinite metric space with bounded local geometry, using a four-points condition. In detail, the hyperbolicity $\delta(x,y,v,w)$ of a quadruple of vertices $\{x,y,v,w\}$ is defined as half the difference between the biggest two of the following sums:

$$d(x,y)+d(v,w),\quad d(x,v)+d(y,w),\quad d(x,w)+d(y,v), \tag{1}$$

where $d$ denotes the distance between two vertices. The hyperbolicity $\delta(G)$ of a graph $G$ is commonly defined as the maximum of the hyperbolicity of a quadruple of vertices [16–20]. However, for the purposes of this work, the average hyperbolicity of a quadruple will also have a significant role: to distinguish the two, we will use $\delta_{\text{worst}}$ and $\delta_{\text{avg}}$, respectively [21]. Informally, a network is hyperbolic (respectively, hyperbolic on average) if $\delta_{\text{worst}}$ ( $\delta_{\text{avg}}$) is "small" (hence, $\delta$ is a measure of "non-hyperbolicity"). See Fig. 1 for an intuition of the meaning of $\delta$.

This approach attracted the interest of the community, both in modeling this phenomenon [22], and in classifying networks from the real world [23]. For example, it has been argued that several properties of complex networks arise naturally, once a negative curvature of the space has been assumed [14,24]. Similarly, others investigated the role of hyperbolicity in a series of different networks [21] ranging from social networks in dolphins, to characters in books, with the aim of discovering essential edges in the path of communication. In addition, by studying structural holes, it has been shown that most of these networks are essentially tree-like [21].

Other works have proved several mathematical results linking hyperbolicity with congestion: if a graph is hyperbolic (that is, $\delta_{\text{worst}}$ is small), then some nodes will have a larger traffic load than others [25,26]. Furthermore, in [27], the authors prove that the maximum vertex congestion scales as $\Theta(n^2)$ on any hyperbolic family of graphs (that is, any family of graphs where $\delta_{\text{worst}}$ is upper bounded by a constant). A similar result also hold if we relax the condition on $\delta_{\text{worst}}$ [28].

 

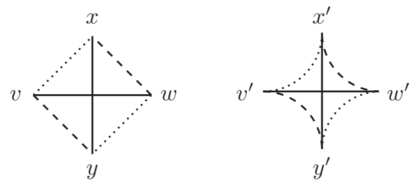


FIG. 1. An intuition of the definition of $\delta$. In both quadruples, assuming the lines are shortest paths, the biggest sum is $d(xy)+d(vw)$ (straight lines), and the other two sums are equal (dashed and dotted lines). However, $\delta(x,y,v,w) > \delta(x',y',v',w')$, because in the second case the two small sums are closer to the two big sums (since the underlying space is hyperbolic).

However, despite the big amount of mathematical analysis of this quantity, researchers have rarely applied it to the analysis of real-world networks. In this paper, based on the aforementioned mathematical results, we will use hyperbolicity to measure how much a network is "democratic": if the network is hyperbolic ($\delta$ is small), it means that there are few vertices with high load, and the network is aristocratic; conversely, if $\delta$ is large, the network is democratic. Our experiments will suggest that the best definition of democracy is $\frac{2\delta_{avg}}{d_{avg}}$, where $d_{avg}$ is the average distance between two nodes: thanks to this rescaling, this parameter is always between 0 and 1.

As far as we know, this is the first measure of democracy in a complex network, apart from assortativity [29,30]. In any case, our measure is quite different from the latter one, because it is based on shortest paths and not on neighbors: consequently, the new measure is global. Moreover, it is more robust: for instance, if we "break" all edges by "adding a vertex in the middle", the hyperbolicity of the graph does not change much, but the assortativity decreases drastically.

After providing this definition of democracy, which is the first contribution of this paper to our understanding of hyperbolicity, we analyze several biological, social, and technological networks, by computing $\delta_{worst}$ and $\delta_{avg}$, and by showing the benefits of our definition. This analysis confirms previous results showing that $\delta_{worst}$ is highly influenced by "random events", and it does not capture a specific characteristic of the network [21]. Differently from previous works, we will also be able to quantify this phenomenon by analyzing more and bigger networks than previous works, showing that the distribution of $\frac{2\delta_{worst}}{D}$ (where $D$ is the diameter of the graph) is approximately normal. Instead, the value of $\delta_{avg}$ is much more robust with respect to random events, and it allows us to effectively distinguish networks of different kind. Our classification will be different from the classification provided by assortativity [29,30]: for instance, a network with few influential hubs not connected to each other is democratic if we consider assortativity, while it is aristocratic in our framework. Finally, we introduce the average hyperbolicity of neighborhoods of a given node, which measures the "importance" of a node (the $k$-neighborhood of a node $v$ is the subgraph induced by the $k$ vertices closest to $v$). Applications include the classification of complex networks (hyperbolic networks have interesting properties, as outlined in the literature), the analysis of nodes in a given network, and possibly the detection of communities using hyperbolicity.

## II. METHODS

There are several formal results that link the hyperbolicity constant with democracy in complex networks. Here, we will just refer to two of them, to provide an intuition. The first one shows that, if for some vertices $v,w$, $\max_{x,y}\delta(x,y,v,w)$ is not high, then there is a set of small diameters that "control" all approximately shortest paths from $x$ to $y$. Consequently, a hyperbolic network is not democratic, because $\delta$ is small, and shortest paths are controlled by small sets.

*Proposition 1 ([31], Lemma 2).* Let $v,w$ be two vertices in a network $G=(V,E)$, let $B_r(v)$ be the $r$-neighborhood of $v$ (that is, the set $\{u \in V : d(u,v) \leqslant r\}$), and let $B_s(w)$ be the $s$-neighborhood of $w$. Then, the diameter of the set $X = B_r(v) \cap B_s(w)$ is at most $2\max_{x,y}\delta(x,y,v,w) + r + s - d(v,w)$.

The second proposition is a sort of converse: if there is a set of vertices controlling the shortest paths of a given quadruple $(x,y,v,w)$, $\delta(x,y,v,w)$ is low. Consequently, if $\delta$ is high, then there is not a small set of vertices controlling many shortest paths, and the network is democratic.

*Proposition 2 ([31], Lemma 2).* Let $x,y,v,w$ be a quadruple of vertices, and let us assume that there exists a set $C \subseteq V$ of diameter $D$ such that all shortest paths between $x,y,v,w$ pass through $C$. Then, $\delta(x,y,v,w) \leqslant D$.

However, analyzing hyperbolicity *per se* makes little sense, because most of the networks we are analyzing are *small world*, that is, distances are usually very small, and consequently $\delta$ is usually small, too. Hence, we rescale our values, and we consider the ratios $\frac{2\delta_{worst}}{D}$ and $\frac{2\delta_{avg}}{d_{avg}}$, by rescaling with respect to the diameter $D$ and the average distance $d_{avg}$, respectively. This way, we obtain a value which is independent from distances in the graph, and measures only how "democratic" a network is. Since we will show that $\frac{2\delta_{worst}}{D}$ is not robust, we will choose $\frac{2\delta_{avg}}{d_{avg}}$ as our definition of democracy in a complex network.

This work will confirm this interpretation by analyzing a dataset of 93 graphs, made by 19 biological networks, 32 social networks, and 42 technological networks.

The first experiment extends some of the activity already done [14,21]: for each network in our dataset, we computed the distribution of $\frac{2\delta_{worst}}{D}$, where $D$ is the diameter of the graph (this value is always between 0 and 1 [32]). With respect to the previous papers, we got more data referring to larger networks, and we therefore deal with lower statistical errors (we have been able to analyze bigger networks thanks to the algorithm in [33], which is able to terminate in reasonable time on networks with up to 100 000 vertices). The second experiment focuses on $\delta_{avg}$, already considered in the literature [21], but not deeply analyzed. In particular, for each graph in the dataset, we have considered the ratio $\frac{2\delta_{avg}}{d_{avg}}$, where $d_{avg}$ is the average distance between two randomly chosen vertices [32] (this parameter takes values in the interval $[0,1]$).

Although the exact computation of $\delta_{avg}$ is also hard, the value $\frac{2\delta_{avg}}{d_{avg}}$ can be easily approximated through sampling, as shown by the following lemma.

*Lemma 1.* Let $G$ be a graph with $\delta_{\text{worst}} = 2.5$, and let us sample the hyperbolicity of $N = 10\,000\,000$ quadruples of vertices, obtaining $\delta_1, \ldots, \delta_N$. Let $h_i := \frac{2\delta_i}{d_{\text{avg}}}$, and let $t = 0.01$ be the tolerance. Then, $\mathbb{P}(|\frac{\sum_{i=1}^N h_i}{N} - \frac{2\delta_{\text{avg}}}{d_{\text{avg}}}| \geqslant t) \leqslant 2e^{-\frac{Nt^2}{2\delta_{\text{worst}}^2}} = 0.07\%$.

*Proof.* By the Azuma-Hoeffding inequality [34] applied with $a_i = 0 \leqslant h_i \leqslant \frac{2\delta_{\text{worst}}}{d_{\text{avg}}} = b_i$, we obtain

$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^N h_i}{N} - \frac{2\delta_{\text{avg}}}{d_{\text{avg}}}\right| \geqslant t\right) \leqslant 2e^{\frac{-2N^2t^2}{\sum_{i=1}^N (b_i - a_i)^2}}$$

$$\leqslant 2e^{-\frac{Nt^2 d_{\text{avg}}^2}{2\delta_{\text{worst}}^2}}$$

$$\leqslant 2e^{-\frac{Nt^2}{2\delta_{\text{worst}}^2}}$$

because $d_{\text{avg}} \geqslant 1$. The previous inequality applied with $N = 10\,000\,000$, $t = 0.01$, and $\delta_{\text{worst}} = 2.5$ yields

$$\mathbb{P}\left(\left|\frac{\sum_{i=1}^{10\,000\,000} \delta_i}{10\,000\,000} - \delta_{\text{avg}}\right| \geqslant 0.01\right) \leqslant 0.07\%.$$

■

Finally, let us define the $k$-neighborhood of a vertex $v$ as the subgraph induced by the $k$ vertices closest to $v$ (in case of a tie, we use a random tie-break). We have analyzed the hyperbolicity of $k$-neighborhoods of $v$, where $k$ ranges between the degree of $v$ and the number $n$ of nodes in the graph, with steps of ten vertices. We have chosen $v$ as the maximum degree vertex, which intuitively should have a large influence area, and we have used as a benchmark of comparison the same results from a random vertex.

We have defined the "influence area" of $v$ as the biggest neighborhood where $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ is at most half of the same value in the whole graph. However, in order to avoid "random deviations" (especially, when the neighborhood is small), in our experiments we have considered the fourth neighborhood where this event has occurred. The purpose of this analysis is twofold: not only do we define and compute the influence area of a vertex, but we also classify networks according to the size of this area.

## III. RESULTS

### A. Worst-case hyperbolicity

Our first experiment computes the ratio $\frac{2\delta_{\text{worst}}}{D}$ in all the networks in our dataset (Fig. 2).

These results show that the distribution of the ratio $\frac{2\delta_{\text{worst}}}{D}$ is approximately Gaussian, both in the whole dataset and in each single kind of network. The average ratio is 0.521, and the standard deviation is 0.085. Moreover, a $\chi$-square goodness of fit test applied to the previous data does not reject the hypothesis that the distribution is Gaussian with mean 0.5 and variance 0.085, with a very high confidence level [35]. This result confirms that the value of $\delta_{\text{worst}}$ in real-world networks is not much "smaller than expected", the result already obtained in the past [21]. This experiment confirms that real-world networks are not hyperbolic, at least in the sense of $\delta_{\text{worst}}$: this is the first main result of this paper. However, we are able to perform a further step: the Gaussian probability distribution makes us think that $\delta_{\text{worst}}$ is influenced by random events. Indeed it does not reflect particular characteristics of the network, since the same distribution arises from networks of different kinds.

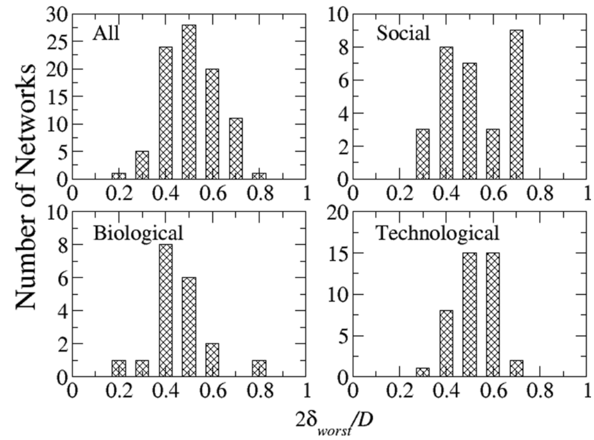


FIG. 2. The distribution of $\frac{2\delta_{\text{worst}}}{D}$ in the graphs in our dataset. The bar corresponding to the value $p$ contains all networks where $p - 0.5 < \frac{2\delta_{\text{worst}}}{D} \leqslant p + 0.5$.

Social networks show a slightly different behavior, since many of them have a larger value of $\frac{2\delta_{\text{worst}}}{D}$, between 0.65 and 0.75. However, this is due to the presence of several financial (e-MID, a platform for interbank lending) networks, where the ratio is often $\frac{2}{3}$ or $\frac{3}{4}$ since the diameter is 3 or 4.

Despite this particular case, we may conclude that the ratio $\frac{2\delta_{\text{worst}}}{D}$ is not a characteristic of the network, but it mainly depends on "random events" that have a deep impact on this value. This conclusion is further confirmed by the particular case of the e-MID networks: this parameter changed from 0.750 in 2011 to 0.286 in 2012, only because a simple path of length 3 increased the diameter from 4 to 7.

### B. Average hyperbolicity

In the past, the average hyperbolicity $\delta_{\text{avg}}$ of a quadruple of vertices was rarely analyzed: the only known result is that it is usually significantly smaller than $\delta_{\text{worst}}$ [21]. In order to fill this gap, we have computed the ratio $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$, where $d_{\text{avg}}$ denotes the average distance in the network (also this parameter lies in the interval $[0,1]$ [32]). Detailed results for each single network are plotted in Fig. 1 of the Supplemental Material [36].

The average hyperbolicity is usually an order of magnitude smaller than the average distance: in this sense, real-world networks are indeed hyperbolic. Moreover, it is possible to make a distinction between networks that are "more democratic", like the e-MID networks or the peer-to-peer networks (where $\delta_{\text{avg}}$ is large), and networks that are "more centralized", like some social networks and most autonomous systems networks.

### C. Hyperbolicity of neighborhoods

Since the hyperbolicity of a graph is closely related to the existence of a small part of the graph controlling most shortest paths, we have analyzed which subgraphs of a given graph have small values of $\delta$. Intuitively, these subgraphs

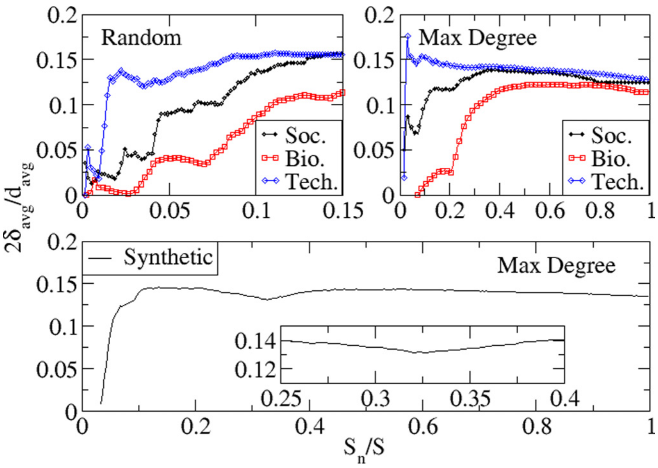


FIG. 3. (Color online) The value of $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ for neighbors of a randomly chosen vertex (up left), or the maximum degree vertex (up right); $S_n$ is the number of vertices in the neighbors, while $S$ is the total number of vertices. Results are shown for a social network, a biological network, a technological network, and (below) a synthetic network.

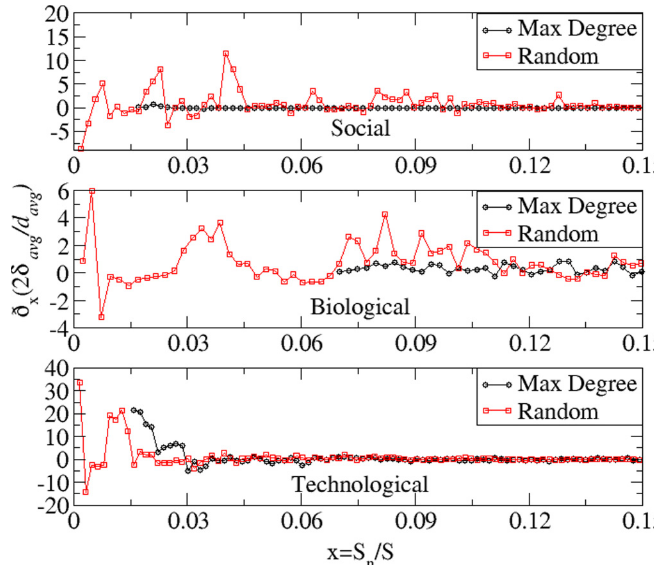


FIG. 4. (Color online) The derivative with respect to the neighborhoods proportion $S_n/S$ of the value of $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$, in neighborhoods of the maximum degree vertex and of random vertices.

should be "less democratic" than the whole graph, in the sense that they are contained in the "influence area" of a small group of vertices. In this analysis, we have tried to spot the influence area of a single vertex, by measuring $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ on neighborhoods of $v$ in increasing order of size. In order to prove the effectiveness of this approach, we have first tested a synthetic power-law graph [37] made by three communities of 1 000 vertices each (see the lowest plot in Fig. 3). We have computed the hyperbolicity of neighborhoods of the vertex $v$ with highest degree: we can see a local minimum close to the size of a community. In our opinion, this minimum appears because the neighborhood is "dominated by the community", and consequently by the center $v$ of the community. This result confirms the link between the value of $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ and the influence area of a vertex.

Finally, we passed to the analysis of neighborhoods in real-world networks. The upper plots in Fig. 3 show the same results for one network of each kind:

(i) a social network, the General Relativity and Quantum Cosmology collaboration network;

(ii) a biological network, the yeast metabolic network;

(iii) a technological network, the peer-to-peer Gnutella network in 2004.

As a benchmark of comparison, we have also considered the hyperbolicity of neighborhoods of a random vertex.

The plots show that the value of $\delta_{\text{avg}}$ in increasing-size neighborhoods of the maximum degree vertex grows almost linearly with the neighborhood size, until it converges to the value of $\delta_{\text{avg}}$ in the whole graph. Convergence time differs from graph to graph. In biological networks, convergence was reached at a size close to $\frac{n}{2}$, while in the social and in the technological networks convergence is reached before. For neighborhoods of a random vertex, we outline a different behavior: at the beginning, the growth is not monotone, like in the previous case, and it is much more irregular. In our opinion, this is due to the fact that, when the neighborhood grows, it reaches more and more "influential" vertices, and the first neighborhood that touches such a vertex corresponds to a local maximum in the plot. This issue is further confirmed by Fig. 4, where the derivative of $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ is shown.

For this reason, we have focused on the maximum degree vertex, and, in order to have more general results, we have analyzed all graphs in the dataset. Figure 2 of the Supplemental Material [36] shows the size of the maximum neighborhood having ratio $\frac{2\delta_{\text{avg}}}{d_{\text{avg}}}$ at least half than the same ratio in the whole graph. Actually, we have plotted the fourth neighborhood where this condition is satisfied, in order to avoid random deviations.

We outline that the influence area of an individual is small in social and peer-to-peer networks, compared to a biological or autonomous system network. This standard behavior has a few exceptions: first of all, protein-protein interaction networks (`string`, `ecoli.interaction`) are different from other biological networks, and the influence area is smaller. Furthermore, the social network `GoogleNW` contains a vertex with an enormous influence area: this network is the set of Google pages, and the central vertex $v$ considered is the page `www.google.com`, which clearly dominates all others. Another particular case is the social network `facebook_combined`: this network is a collection of ego-networks from Facebook, and links are made if common interests are retrieved. We think that this network is different from the others because it is a small subgraph of a bigger graph (where all Facebook users are considered), and the choice of the subgraph has a strong impact on the topology of the network, which does not reflect the standard behavior.

## IV. DISCUSSION

In the literature, several works have analyzed the hyperbolicity of a complex network. They used this quantity in order to classify real-world networks and in order to draw conclusions about the impact of hyperbolicity on the network topology. However, these works are mainly based on the analysis of $\delta_{\text{worst}}$, which has two drawbacks: it is not *robust*, that is, small modifications on the network can have deep impacts on its value (especially if the attach is targeted), and it is not *scalable*, that is, it can be exactly computed only on small networks. In this work, we confirmed and quantified these conclusions, and we proposed a different approach: using $\delta_{\text{avg}}$ instead of $\delta_{\text{worst}}$, a parameter already considered in the literature. We interpreted this parameter as a measure of "democracy" in a network, and we classified different networks according to how democratic they are.

We have shown that technological autonomous system networks are less "democratic" than social or biological networks, in agreement with our intuition (since AS graphs have a "built-in" hierarchy, while in social networks everyone has the same role). Moreover, we have applied this concept to neighborhoods of influential nodes. We have clearly outlined the influence area of a node, whose size strongly depends on the graph considered. Indeed, nodes have a rather small influence area in social and peer-to-peer networks, while in autonomous systems and biological networks the influence area can be close to half the graph. A possible explanation of this behavior is that the former networks are "distributed", in the sense that each node has a goal (downloading in peer-to-peer networks, and creating relationships in social networks), and edges are created locally by nodes that try to reach the goal. On the other hand, the latter networks have global goals (connecting everyone in the network, or making a cell live), and the creation of edges is "centralized".

This work provides an interpretation of the average hyperbolicity. Possible applications include not only the classification of networks according to this parameter, but also the classifications of nodes in a network, or the classification of different communities. These communities might be democratic, if everyone has "the same role" and $\delta_{\text{avg}}$ is high, or not democratic, if there is a group of few nodes that keeps the community together, making $\delta_{\text{avg}}$ small.

## ACKNOWLEDGMENTS

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[36] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.92.032812 for detailed results on the average hyperbolicity and the influence area of the most central vertex.

[37] A. Lancichinetti and S. Fortunato, Benchmarks for testing community detection algorithms on directed and weighted graphs with overlapping communities, Phys. Rev. E **80**, 016118 (2009).