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The height of an infinite parallelotope is infinite ☆

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We show that if no non-trivial linear combinations of independent vectors $f_0, f_1, \ldots, f_m \in \mathbb{R}^\infty$ belongs to ℓ_2, then all the heights of an infinite parallelotope constructed on vectors $f_0, f_1, \ldots, f_m$ are infinite. This result is essential in the proof of the irreducibility of unitary representations of some infinite-dimensional groups.

Contents

☆ This work is dedicated to Ukraine and to all fearless Ukrainians defending their country.

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1. Introduction

This article lies on the border between linear algebra and functional analysis, since we study the behavior of a finite dimensional object namely, parallelotope generated by $m+1$ independent vectors, belonging to an n-dimensional space when dimension $n \to \infty$. Parallelotop is the set which is made up of vectors $\sum C_i f_i$ where $C_i \in [0, 1]$. The main result is proved in Section 4. Some application are given in Section 5. They are the law of large numbers, some generalizations and the proof of the irreducibility. To prove the irreducibility for some infinite-dimensional groups [6], we need to approximate a rich space of functions by combinations of generators of one-parameter subgroups. The desired approximation on the n-th step is of the order of the inverse height h_n of the projections of vectors f_k and the parallelotope to some finite-dimensional space $\mathbb{R}^n$. Namely, we show that if no non-trivial linear combinations of $m+1$ vectors belongs to ℓ_2 then all the heights h of an infinite parallelotope generated by vectors $f_0, f_1, \ldots, f_m$ are infinite.[1] This allows us to prove the approximation and the irreducibility. In the proof we used also the explicit formulas obtained in [6,9] for $\det B(\lambda)$, $B^{-1}(\lambda)$ and $(B^{-1}(\lambda)a, a)$ where $B(\lambda) = B + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and B is an $n \times n$ matrix, for details, see Section 3.

We use notations C and C_k for an absolute constants. Also we denote by ℓ_2 the real Hilbert space $\ell_2 = \{x = (x_k)_{k=1}^{\infty} : \|x\|_{\ell_2}^2 = \sum_{k=1}^{\infty} |x_k|^2 < \infty\}$.

[1] To define correctly the height, see details in Lemma 4.1.

Definition 1.1. Let us recall the definition of the Gram determinant and the Gram matrix (see [3], Chap IX, §5). Given the vectors $x_1, x_2, ..., x_m$ in some Hilbert space H the *Gram matrix* $\gamma(x_1, x_2, ..., x_m)$ is defined by the formula

$$\gamma(x_1, x_2, ..., x_m) = \big((x_k, x_n)\big)_{k,n=1}^{m},$$

where (x_k, x_n) is a scalar product in H. The determinant of this matrix is called the *Gram determinant* for the vectors $x_1, x_2, ..., x_m$ and is denoted by $\Gamma(x_1, x_2, ..., x_m)$. Thus,

$$\Gamma(x_1, x_2, \dots, x_m) := \det \gamma(x_1, x_2, \dots, x_m). \tag{1.1}$$

We start with a general problem, see Sections 5.1 and 5.2 below.

Problem 1.1. For a fixed vector f_0 in a Hilbert space H and an infinite sequence of vectors $(f_n)_{n\in\mathbb{N}}$ in H, when

$$f_0 \in V = \langle f_n, n \in \mathbb{N}\rangle? \tag{1.2}$$

Here $\langle f_n, n \in \mathbb{N}\rangle$ is the completion of all finite linear combinations of vectors from a family $(f_n)_{n\in\mathbb{N}}$. Let us denote by V_n the subspace generated by the first n vectors, then the square of the distance $d^2(f_0, V_n)$ of the vector f_0 from the hyperplane V_n is given by the ratio of two Gram determinants, see (2.1) below. Finally, $f_0 \in V$ if and only if $\lim_{n\to\infty} d^2(f_0, V_n) = 0$.

2. How far is a vector from a hyperplane?

2.1. The distance of a vector from a hyperplane

In this section we follow [9]. We start with a classical result, see, e.g. [3]. Consider the hyperplane V_n generated by n arbitrary independent vectors $f_1, \dots, f_n$ in some Hilbert space H.

Lemma 2.1. *The square of the distance $d(f_0, V_n)$ of a vector f_0 from the hyperplane V_n is given by the ratio of two Gram determinants, see Definition 1.1*

$$d^2(f_0, V_n) = \frac{\Gamma(f_0, f_1, f_2, \dots, f_n,)}{\Gamma(f_1, f_2, \dots, f_n)}. \tag{2.1}$$

Proof. We follow closely the book by Axiezer and Glazman [1]. Set $f = \sum_{k=1}^{n} t_k f_k \in V_n$ and $h = f - f_0$. Since h should be orthogonal to V_n we conclude that $f_r \perp h$, i.e., $(f_r, h) = 0$ for all r, or

$$\sum_{k=1}^{n} t_k (f_r, f_k) = (f_r, f_0), \quad 1 \le r \le n. \tag{2.2}$$

Set $A = \gamma(f_1, f_2, \ldots, f_n)$ and $b = (f_k, f_0)_{k=1}^n \in \mathbb{R}^n$. By definition we have

$$d^2 = \min_{f \in V_n} \|f - f_0\|^2 = (At, t) - 2(t, b) + (f_0, f_0). \tag{2.3}$$

Since $d^2 = (h, h) = (f_0, h)$ we conclude that $d^2 = \sum_{k=1}^n t_k(f_0, f_k) - (f_0, f_0)$ or

$$\sum_{k=1}^n t_k(f_0, f_k) = (f_0, f_0) - d^2. \tag{2.4}$$

So we have the system of equations:

$$\begin{cases} t_1(f_1, f_1) + t_2(f_1, f_2) + \cdots + t_n(f_1, f_n) &= \quad (f_1, f_0) \\ t_1(f_2, f_1) + t_2(f_2, f_2) + \cdots + t_n(f_2, f_n) &= \quad (f_2, f_0) \\ \qquad \qquad \ldots & \\ t_1(f_n, f_1) + t_2(f_n, f_2) + \cdots + t_n(f_n, f_n) &= \quad (f_n, f_0) \\ t_1(f_0, f_1) + t_2(f_0, f_2) + \cdots + t_n(f_0, f_n) &= (f_0, f_0) - d^2 \end{cases}. \tag{2.5}$$

Excluding t_k from the system we get $d^2 = \frac{\Gamma(f_0, f_1, f_2, \ldots, f_n,)}{\Gamma(f_1, f_2, \ldots, f_n)}$. □

Remark 2.1. From the system (2.5) we conclude that $At = b$, where $b = (f_k, f_0)_{k=1}^n \in \mathbb{R}^n$, hence $t = A^{-1}b$. By (2.3) we get

$$d^2 = (f_0, f_0) - (A^{-1}b, b) = \frac{\Gamma(f_0, f_1, f_2, \ldots, f_n)}{\Gamma(f_1, f_2, \ldots, f_n)}. \tag{2.6}$$

See also [6, Chap. 4.3, Lemma 4.3.2].

2.2. Cramer's rule reformulated

Consider two *Gram matrices*, see Definition 1.1:

$$A_m = \gamma(f_1, \ldots, f_m) = \big((f_k, f_r)\big)_{k,r=1}^m, \quad B_m = \gamma(f_0, f_1, \ldots, f_m), \tag{2.7}$$

and a vector $b = (f_k, f_0)_{k=1}^m \in \mathbb{R}^m$. The solution of the equation $A_m t = b$ is as follows.

Recall that $A_s^r(B)$ denote *cofactors* of the matrix B, see Definition 3.2.

Lemma 2.2. *We have*

$$t = A_m^{-1}b = \frac{1}{A_0^0(B_m)} \begin{pmatrix} -A_1^0(B_m) \\ -A_2^0(B_m) \\ \ldots \\ -A_m^0(B_m) \end{pmatrix}. \tag{2.8}$$

Proof. By *Cramer's rule* the solutions of a system of linear equations

$$At = b \tag{2.9}$$

where $A \in \mathrm{Mat}(m, \mathbb{C})$ with $\det A \neq 0$ and $t, b \in \mathbb{C}^m$, are given by the following formulas:

$$t_k = \frac{\det(A_k)}{\det(A)}, \quad 1 \le k \le m, \tag{2.10}$$

where A_k is the matrix formed by replacing the k-th column of $A := A_m$ by the column vector b. Consider the matrix B_m defined by (2.7). We have

$$\det(A) = A_0^0(B_m), \quad \det(A_k) = -A_k^0(B_m), \quad 1 \le k \le m, \tag{2.11}$$

thus implying (2.8). □

2.3. Some estimates

We use some facts from [6], Section 1.4.1, pp. 24-25.

Lemma 2.3. *For a strictly positive operator A (i.e., $(Af, f) > 0, \ f \neq 0$) acting in $\mathbb{R}^n$ and a vector $b \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\min_{x \in \mathbb{R}^n} \Big((Ax, x) \mid (x, b) = 1 \Big) = \frac{1}{(A^{-1}b, b)}. \tag{2.12}$$

The minimum is assumed for $x = \frac{A^{-1}b}{(A^{-1}b, b)}$.

Lemma 2.3 is a direct generalization of the well known result (see, for example, [2], Chap. I, §52): for $a_k > 0, \ 1 \le k \le n$ we have

$$\min_{x \in \mathbb{R}^n} \Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k = 1 \Big) = \Big(\sum_{k=1}^n \frac{1}{a_k} \Big)^{-1}. \tag{2.13}$$

We will also use the same result in a slightly different form:

$$\min_{x \in \mathbb{R}^n} \Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k b_k = 1 \Big) = \Big(\sum_{k=1}^n \frac{b_k^2}{a_k} \Big)^{-1}. \tag{2.14}$$

The minimum is assumed for $x_k = \frac{b_k}{a_k} \Big(\sum_{k=1}^n \frac{b_k^2}{a_k} \Big)^{-1}$. We can prove more general statement. Denote by $D(B)$ the domain of the definition of an operator B acting on some Hilbert space H.

Lemma 2.4. *For a strictly positive operator A in an infinite-dimensional Hilbert space H and a vector $b \in H\backslash\{0\}$ such that $b \in D(A^{-1})$, we have*

$$\min_{x\in H}\Big((Ax,x) \mid (x,b)=1\Big) = \frac{1}{(A^{-1}b,b)}. \tag{2.15}$$

The minimum is reached for $x_0 = \frac{1}{(A^{-1}b,b)}A^{-1}b$.

Proof. Consider a new scalar product in H defined as follows

$$(f,g)_A := (Af,g)_H, \quad f,g \in H. \tag{2.16}$$

Since

$$(Ax,x)_H = (x,x)_A = \|x\|_A^2 \quad \text{and} \quad 1 = (b,x)_H = (A^{-1}b,x)_A,$$

the minimum $\|x\|_A^2$ will be achieved on the vector $x_0 = sA^{-1}b$ generating hyperplane $1 = (A^{-1}b,x)_A$ and belongs to this hyperplane. We get

$$1 = (b, sA^{-1}b), \quad \text{therefore} \quad s = \frac{1}{(A^{-1}b,b)}, \quad x_0 = \frac{1}{(A^{-1}b,b)}A^{-1}b.$$

Finally, we get $(Ax_0,x_0) = \frac{1}{(A^{-1}b,b)}$. □

Below we will provide a counterexample showing that condition $b \in D(A^{-1})$ is essential in the previous lemma.

Counterexample 2.5. Consider a positive definite operator $A = \operatorname{diag}(\lambda_k)_{k=1}^{\infty}$ defined in a Hilbert space ℓ_2, where $\lambda_k = \frac{1}{k}$. Take $b = (b_k)_{k\in\mathbb{N}} \in \ell_2$ with $b_k = \frac{1}{k}$. Then we have $b \notin D(A^{-1})$, since $(A^{-1}b)_k \equiv 1$ for all $k \in \mathbb{N}$ hence, $A^{-1}b \notin \ell_2$. In this case $\frac{1}{(A^{-1}b,b)} = 0$. Indeed, for the corresponding projections $A_n,\ b_n$ on $\mathbb{R}^n$ we have

$$(A_n^{-1}b_n, b_n) = \sum_{k=1}^{n}\frac{1}{k} \to \infty.$$

3. The generalized characteristic polynomial and its properties

Let $\operatorname{Mat}(m,\mathbb{C})$ be the set of all $m \times m$ complex matrices.

Definition 3.1 *([6], Ch.1.4.3).* For a matrix $B \in \operatorname{Mat}(m,\mathbb{C})$ and $\lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{C}^m$ define the *generalization of the characteristic polynomial*, $p_B(t) = \det(tI - B)$, $t \in \mathbb{C}$ as follows:

$$P_B(\lambda) = \det B(\lambda), \quad \text{where} \quad B(\lambda) = \operatorname{diag}(\lambda_1,\ldots,\lambda_m) + B. \tag{3.1}$$

Lemma 3.1 *([6], Ch.1.4.3). For the generalized characteristic polynomial $P_B(\lambda)$ of $B \in \mathrm{Mat}(m,\mathbb{C})$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{C}^m$ we have*

$$P_B(\lambda) = \det B + \sum_{r=1}^{m} \sum_{1\le i_1<i_2<...<i_r\le m} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r} A^{i_1 i_2 ... i_r}_{i_1 i_2 ... i_r}(B). \tag{3.2}$$

Definition 3.2. For a matrix $B \in \mathrm{Mat}(n,\mathbb{R})$, $a \in \mathbb{R}^n$ and fixed $1 \le i_1 < i_2 < \cdots < i_r \le n$ rows and $1 \le j_1 < j_2 < \cdots < j_r \le n$ columns $1 \le r \le n$ denote by

$$M^{i_1 i_2 ... i_r}_{j_1 j_2 ... j_r}(B) \quad \text{and} \quad A^{i_1 i_2 ... i_r}_{j_1 j_2 ... j_r}(B)$$

the corresponding *minors* and *cofactors* of the matrix B. Set $M(i_1 i_2 \dots i_r)(B) = M^{i_1 i_2 ... i_r}_{i_1 i_2 ... i_r}(B)$ and $a_{i_1 i_2 ... i_r} = (a_{i_1}, a_{i_2}, \dots, a_{i_r})$. Let also $B_{i_1 i_2 ... i_r}$ be the corresponding *submatrix* of the matrix B. The elements of this matrix are on the intersection of $i_1, i_2, \dots, i_r$ rows and column of the matrix B. Denote by $A(B_{i_1 i_2 ... i_r})$ the matrix of the cofactors of the first order of the matrix $B_{i_1 i_2 ... i_r}$, another name is *adjugate matrix*, occasionally known as *adjunct matrix*:

$$A(B_{i_1 i_2 ... i_r}) = (A^i_j(B_{i_1 i_2 ... i_r}))_{1\le i,j\le r}. \tag{3.3}$$

Minor of order zero is often defined to be 1, therefore, set $A(B_k) = 1$ for $1 \le k \le n$. As usual, denote by B^T the *matrix transposed* to B.

Remark 3.1. If we set $\lambda_\alpha = \lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}$, where $\alpha = (i_1, i_2, ..., i_r)$ and $A^\alpha_\alpha(B) = A^{i_1 i_2 ... i_r}_{i_1 i_2 ... i_r}(B)$, $M^\alpha_\alpha(B) = M^{i_1 i_2 ... i_r}_{i_1 i_2 ... i_r}(B)$, $\lambda_\emptyset = 1$, $A^\emptyset_\emptyset(B) = \det B$ (see Definition 3.2) we may write (3.2) as follows:

$$P_B(\lambda) = \det B(\lambda) = \sum_{\emptyset\subseteq\alpha\subseteq\{1,2,...,m\}} \lambda_\alpha A^\alpha_\alpha(B), \tag{3.4}$$

$$P_B(\lambda) = \det B(\lambda) = \left(\prod_{k=1}^{n} \lambda_k\right) \sum_{\emptyset\subseteq\alpha\subseteq\{1,2,...,m\}} \frac{M^\alpha_\alpha(B)}{\lambda_\alpha}, \tag{3.5}$$

Let

$$X = X_{mn} = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ ... & ... & ... & ... \\ x_{m1} & x_{m2} & ... & x_{mn} \end{pmatrix}. \tag{3.6}$$

Setting

$$x_k = (x_{1k}, x_{2k}, ..., x_{mk}) \in \mathbb{R}^m, \quad y_r = (x_{r1}, x_{r2}, ..., x_{rn}) \in \mathbb{R}^n, \tag{3.7}$$

we get

$$X^*X=\begin{pmatrix}(x_1,x_1) & (x_1,x_2) & ... & (x_1,x_n)\\ (x_2,x_1) & (x_2,x_2) & ... & (x_2,x_n)\\ ... & ... & ... & ...\\ (x_n,x_1) & (x_n,x_2) & ... & (x_n,x_n)\end{pmatrix}=\gamma(x_1,x_2,...,x_n), \tag{3.8}$$

$$XX^*=\begin{pmatrix}(y_1,y_1) & (y_1,y_2) & ... & (y_1,y_m)\\ (y_2,y_1) & (y_2,y_2) & ... & (y_2,y_m)\\ ... & ... & ... & ...\\ (y_m,y_1) & (y_m,y_2) & ... & (y_m,y_m)\end{pmatrix}=\gamma(y_1,y_2,...,y_m), \tag{3.9}$$

therefore, we obtain

$$\Gamma(x_1,x_2,...,x_n)=\det(X^*X)=\det(XX^*)=\Gamma(y_1,y_2,...,y_m). \tag{3.10}$$

3.1. The explicit expression for $B^{-1}(\lambda)$ and $(B^{-1}(\lambda)a,a)$

In this section we follow [9]. Fix $B\in\mathrm{Mat}(n,\mathbb{R})$, $a\in\mathbb{R}^n$ and $\lambda\in\mathbb{C}^n$. Our aim is to find the explicit formulas for $B^{-1}(\lambda)$ and $(B^{-1}(\lambda)a,a)$, where $B(\lambda)$ is defined by (3.1).

Let $n=3$, then $A(B_{123})=A(B)$ is the following matrix:

$$A(B)=A(B_{123})=\begin{pmatrix}A^1_1 & A^1_2 & A^1_3\\ A^2_1 & A^2_2 & A^2_3\\ A^3_1 & A^3_2 & A^3_3\end{pmatrix}=\begin{pmatrix}M^{23}_{23} & -M^{23}_{13} & M^{23}_{12}\\ -M^{13}_{23} & M^{13}_{13} & -M^{13}_{12}\\ M^{12}_{23} & -M^{12}_{13} & M^{12}_{12}\end{pmatrix}, \tag{3.11}$$

where we write M^{ij}_{rs} instead of $M^{ij}_{rs}(B)$ and A^i_j instead of $A^i_j(B)$.

Remark 3.2. Let A^T be the transposed matrix of A. Then

$$A^T(B_{i_1i_2\dots i_r})=\det B_{i_1i_2\dots i_r}\Big(B_{i_1i_2\dots i_r}\Big)^{-1}. \tag{3.12}$$

In what follows we will consider the submatrix $B_{i_1i_2\dots i_r}$ of the matrix $B\in\mathrm{Mat}(n,\mathbb{R})$ as an appropriate element of $\mathrm{Mat}(n,\mathbb{R})$.

Theorem 3.2. *For the matrix $B(\lambda)$ defined by (3.1) $a\in\mathbb{R}^n$ and $\lambda\in\mathbb{C}^n$ we have*

$$P_B(\lambda)=\Big(\prod_{k=1}^n\lambda_k\Big)\sum_{r=1}^n\sum_{1\le i_1<i_2<\dots<i_r\le n}\frac{M(i_1i_2\dots i_r)}{\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_r}}, \tag{3.13}$$

$$B^{-1}(\lambda)=\frac{1}{P_B(\lambda)}\Big(\prod_{k=1}^n\lambda_k\Big)\sum_{r=1}^n\sum_{1\le i_1<i_2<\dots i_r\le n}\frac{A(B_{i_1i_2\dots i_r})}{\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_r}}, \tag{3.14}$$

$$(B^{-1}(\lambda)a,a)=\frac{1}{P_B(\lambda)}\Big(\prod_{k=1}^{n}\lambda_k\Big)\sum_{r=1}^{n}\sum_{1\le i_1<i_2<\dots i_r\le n}\frac{(A(B_{i_1i_2\dots i_r})a_{i_1i_2\dots i_r},a_{i_1i_2\dots i_r})}{\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_r}}. \tag{3.15}$$

Below we will provide an example of calculation of $\det B(\lambda)$ and $B(\lambda)^{-1}$.

Example 3.1. For the matrix $B(\lambda)$ defined below we have by (3.5)

$$B(\lambda)=\begin{pmatrix}1+\lambda_1 & 1 & \dots & 1\\ 1 & 1+\lambda_2 & \dots & 1\\ & & \dots & \\ 1 & 1 & \dots & 1+\lambda_n\end{pmatrix}, \tag{3.16}$$

$$\det B(\lambda)=\Big(\prod_{k=1}^{n}\lambda_k\Big)\Big(1+\sum_{k=1}^{n}\frac{1}{\lambda_k}\Big), \tag{3.17}$$

$$B(\lambda)^{-1}=\left(1+\sum_{k=1}^{n}\frac{1}{\lambda_k}\right)^{-1}\Big[\sum_{k=1}^{n}\frac{A^T(B_k)}{\lambda_k}+\sum_{1\le k<r\le n}\frac{A^T(B_{kr})}{\lambda_k\lambda_r}\Big], \tag{3.18}$$

where $A^T(B_{kr})=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ and $A^T(B_{krs})=0$ for $1\le k<r<s\le n$.

3.2. The case where B is the Gram matrix

Fix the matrix X_{mn} defined by (3.6). Denote by B the Gram matrix $\gamma(x_1,x_2,...,x_n)$, i.e.,

$$B=\gamma(x_1,x_2,...,x_n), \tag{3.19}$$

where $(x_1,x_2,...,x_n)$ are defined by (3.7) and $\gamma(x_1,x_2,...,x_n)$ by (3.8). In what follows we consider the operator $B(\lambda)$ defined by (3.1).

Remark 3.3. In this case we have

$$\begin{aligned}P_B(\lambda)&=\det\Big(\sum_{k=1}^{n}\lambda_kE_{kk}+\gamma(x_1,x_2,...,x_n)\Big) \qquad (3.20)\\ &=\prod_{k=1}^{n}\lambda_k\Big(1+\sum_{r=1}^{n}\sum_{1\le i_1<i_2<\dots<i_r\le m}\Big(\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}\Big)^{-1}\Gamma(x_{i_1},x_{i_2},...,x_{i_r})\Big)\\ &=\prod_{k=1}^{n}\lambda_k\Big(1+\sum_{r=1}^{n}\sum_{\substack{1\le i_1<i_2<\dots<i_r\le n;\\ 1\le j_1<j_2<\dots<j_r\le n}}\Big(\lambda_{i_1}\lambda_{i_2}...\lambda_{i_r}\Big)^{-1}\Big(M^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}(X)\Big)^2\Big),\end{aligned}$$

where we have used the following formula (see [3], Chap IX, §5 formula (25)):

$$\Gamma(x_{i_1}, x_{i_2}, ..., x_{i_r}) = \sum_{1\le j_1<j_2<...<j_r\le m} \left(M^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_r}(X)\right)^2. \tag{3.21}$$

Fix two natural numbers $n, m \in \mathbb{N}$ with $m \le n$, two matrices A_{mn} and X_{mn}, vectors $g_k \in \mathbb{R}^{m-1}$, $1 \le k \le n$ and $a \in \mathbb{R}^n$ as follows

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \ g_k = \begin{pmatrix} a_{2k} \\ a_{3k} \\ \dots \\ a_{mk} \end{pmatrix} \in \mathbb{R}^{m-1}, \ \ a = (a_{1k})_{k=1}^n \in \mathbb{R}^n. \tag{3.22}$$

Set $B = \gamma(g_1, g_2, \dots, g_n)$. We calculate $P_B(\lambda)$ and $(B^{-1}(\lambda)a, a)$ for an arbitrary n. Consider the matrix (3.6)

$$X_{mn} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & & \dots & \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}, \ \text{where} \ x_k = \left(\frac{a_{rk}}{\sqrt{\lambda_k}}\right)_{r=1}^m, \quad y_r = (x_{rk})_{k=1}^n \in \mathbb{R}^n. \tag{3.23}$$

3.3. Case $m = 2$

Lemma 3.3 *([7]). For $m = 2$ we have*

$$(B^{-1}(\lambda)a, a) = \frac{\det\left(I_2 + \gamma(y_1, y_2)\right)}{\det\left(I_1 + \gamma(y_2)\right)} - 1 = \frac{\Gamma(y_1) + \Gamma(y_1, y_2)}{1 + \Gamma(y_2)}, \tag{3.24}$$

where y_1 and y_2 are defined as follows

$$y_1 = \left(\frac{a_{1k}}{\sqrt{\lambda_k}}\right)_{k=1}^n \quad y_2 = \left(\frac{a_{2k}}{\sqrt{\lambda_k}}\right)_{k=1}^n. \tag{3.25}$$

3.4. Case $m = 3$

Lemma 3.4 *([9]). For $m = 3$ we have*

$$(B^{-1}(\lambda)a, a) = \frac{\det\left(I_3 + \gamma(y_1, y_2, y_3)\right)}{\det\left(I_2 + \gamma(y_2, y_3)\right)} - 1, \tag{3.26}$$

where the y_r are defined as follows:

$$y_r = \left(\frac{a_{rk}}{\sqrt{\lambda_k}}\right)_{k=1}^n \in \mathbb{R}^n, \quad 1 \le r \le 3. \tag{3.27}$$

3.5. General case

Lemma 3.5 *([9]). For the general m we have*

$$(B^{-1}(\lambda)a,a)=\frac{\det\bigl(I_m+\gamma(y_1,\dots,y_m)\bigr)}{\det\bigl(I_{m-1}+\gamma(y_2,\dots,y_m)\bigr)}-1 \tag{3.28}$$

where the y_r are defined as follows:

$$y_r=\left(\frac{a_{rk}}{\sqrt{\lambda_k}}\right)_{k=1}^{n}\in\mathbb{R}^n,\quad 1\le r\le m. \tag{3.29}$$

4. The height of an infinite parallelotope

Lemma 4.1. *Consider vectors $f_0,f_1\dots,f_m\in\mathbb{R}^\infty$ such that $f_0,f_1\dots,f_m\notin\ell_2$. Denote by $f_r^{(n)}\in\mathbb{R}^n$ the* projections *of the vectors f_r on the subspace $\mathbb{R}^n$. Then for all s with $0\le s\le m$*

$$\frac{\Gamma(f_0,f_1,\dots,f_m)}{\Gamma(f_0,\dots,\hat{f}_s,\dots,f_m)}:=\lim_{n\to\infty}\frac{\Gamma(f_0^{(n)},f_1^{(n)}\dots,f_m^{(n)})}{\Gamma(f_0^{(n)},\dots,\widehat{f_s^{(n)}},\dots,f_m^{(n)})}=\infty, \tag{4.1}$$

if and only if for all $\left(C_k\right)_{k=0}^{m+1}\in\mathbb{R}^{m+1}\setminus\{0\}$ holds

$$\sum_{r=0}^{m}C_rf_r\notin\ell_2,\qquad\sum_{r=0,r\ne s}^{m}C_rf_r\notin\ell_2. \tag{4.2}$$

Here $\hat{f}_s$ means that the vector f_s is absent and $\Gamma(f_0,f_1,\dots,f_m)$ is the Gram determinant*.*

Before proving Lemma 4.1 let us formulate one more statement.

Lemma 4.2. *Consider vectors $f_0,f_1\dots,f_m\in\mathbb{R}^\infty$ such that $\sum_{k=0}^m C_kf_k\notin\ell_2$ for any non-trivial combination $(C_k)_{k=0}^m$. Then for any s with $0\le s\le m$ holds*

$$\frac{\det\bigl(I_{m+1}+\gamma(f_0,\dots,f_m)\bigr)}{\det\bigl(I_m+\gamma(f_0,\dots,\hat{f}_s,\dots,f_m)\bigr)}=\lim_{n\to\infty}\frac{\det\bigl(I_{m+1}+\gamma(f_0^{(n)},\dots,f_m^{(n)})\bigr)}{\det\bigl(I_m+\gamma(f_0^{(n)},\dots,\widehat{f_s^{(n)}},\dots,f_m^{(n)})\bigr)}=\infty. \tag{4.3}$$

Here I_m is identity matrix and $\gamma(f_0,\dots,f_m)$ is the Gram matrix*.*

Proof. The proof follows from Lemma 4.1 and (3.20). □

4.1. Particular cases of Lemma 4.1

4.1.1. Case $m = 1$

Lemma 4.3 *([6,7]). Consider vectors $f_0, f_1 \in \mathbb{R}^\infty$ such that $f_0, f_1 \notin \ell_2$, then*

$$\frac{\Gamma(f_0, f_1)}{\Gamma(f_1)} = \lim_{n\to\infty} \frac{\Gamma(f_0^{(n)}, f_1^{(n)})}{\Gamma(f_1^{(n)})} = \infty, \quad \frac{\Gamma(f_0, f_1)}{\Gamma(f_0)} = \infty, \tag{4.4}$$

$$\text{if and only if} \quad C_0 f_0 + C_1 f_1 \notin \ell_2 \quad \text{for all} \quad (C_0, C_1) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.5}$$

Proof. The initial proof can be found in [6], Lemma 10.4.20 or [7]. Here we give a different proof that *can be generalized for an arbitrary $m \in \mathbb{N}$*. For $t \in \mathbb{R}$ and $f_0, f_1 \in H$, where H is some Hilbert space, define the quadratic form[2]

$$F_1(t) = \|tf_1 - f_0\|_H^2 = t^2(f_1, f_1)_H - 2t(f_1, f_0)_H + (f_0, f_0)_H =$$
$$(A_1 t, t)_\mathbb{R} - 2(t, b)_\mathbb{R} + (f_0, f_0)_H, \text{ where } b = (f_1, f_0)_H \in \mathbb{R}. \quad \text{We have}$$
$$\min_{t\in\mathbb{R}} F_1(t) = F_1(t_0) = \frac{\Gamma(f_0, f_1)}{\Gamma(f_1)}, \tag{4.6}$$

where $t_0 = \frac{(f_1,f_0)}{(f_1,f_1)}$, [3] and A_1 is a *Gram matrix* (see Definition 1.1)

$$A_1 = \gamma(f_1) = (f_1, f_1). \tag{4.7}$$

Consider another Gram matrix

$$B_1 := \gamma(f_0, f_1) = \begin{pmatrix} (f_0, f_0) & (f_0, f_1) \\ (f_1, f_0) & (f_1, f_1) \end{pmatrix}, \quad B_1^{(n)} := \gamma(f_0^{(n)}, f_1^{(n)}), \tag{4.8}$$

then

$$t_0 = \frac{(f_1, f_0)}{(f_1, f_1)} = \frac{-A_1^0(B_1)}{A_0^0(B_1)}, \tag{4.9}$$

where $A_0^0(B_1)$ and $A_1^0(B_1)$ are the corresponding minors of the matrix B_1, see Definition 3.2. If we replace the vectors f_0, f_1 with $f_0^{(n)}, f_1^{(n)}$, the formulas (4.6) and (4.9) become

$$\min_{t\in\mathbb{R}} F_1^{(n)}(t) = F_1^{(n)}(t_0^{(n)}) = \frac{\Gamma(f_0^{(n)}, f_1^{(n)})}{\Gamma(f_1^{(n)})}, \quad t_0^{(n)} = \frac{(f_1^{(n)}, f_0^{(n)})}{(f_1^{(n)}, f_1^{(n)})} = \frac{-A_1^0(B_1^{(n)})}{A_0^0(B_1^{(n)})}. \tag{4.10}$$

[2] we prefer to write $(A_1 t, t)_\mathbb{R}$ to find a general pattern in the case $\mathbb{R}^m$, see below (4.9), (4.19) and (4.36).

[3] **important notation:** in general, for $t, b \in \mathbb{R}^m$ and $A_m \in \text{Mat}(m, \mathbb{R})$ we denote by t_0 the solution of the equation $A_m t = b$, see below (4.34), see also Remark 2.1.

Suppose that there exists an absolute constant C such that for all $n \in \mathbb{N}$

$$\Gamma(f_0^{(n)}, f_1^{(n)})/\Gamma(f_1^{(n)}) \le C. \tag{4.11}$$

Without loss of generality, we can assume that for all $n \in \mathbb{N}$

$$\Gamma(f_0^{(n)}) \le C_1 \Gamma(f_1^{(n)}), \tag{4.12}$$

we choose some subsequence n_k, if necessary. Then it is sufficient to verify the first part of (4.4). Indeed, denote by $a_n = \Gamma(f_0^{(n)})$ and $b_n = \Gamma(f_1^{(n)})$. We have two sequences of positive numbers (a_n) and (b_n) with property $\lim_n a_n = \lim_n b_n = \infty$. Let for some absolute constant C_1 holds $a_n \le C_1 b_n$ for all $n \in \mathbb{N}$, then (4.12) holds. Suppose the opposite. Denote for all $N \in \mathbb{N}$ the set $E_N = \{n \in \mathbb{N} : a_n > N b_n\}$. Then this set E_N is infinite for all N. Now fix N and denote all the elements of E_N by $(n_k)_{k=1}^\infty$, then $b_{n_k} < N^{-1} a_{n_k}$ for all $k \in \mathbb{N}$ or $\Gamma(f_1^{(n_k)}) \le C_1 \Gamma(f_0^{(n_k)})$.

We prove that the sequence $t_0^{(n)}$ defined by (4.10) is bounded. Since all the matrices $\gamma(f_0^{(n)}, f_1^{(n)})$ are positively defined, we have

$$1 \ge \frac{(f_0^{(n)}, f_1^{(n)})^2}{(f_0^{(n)}, f_0^{(n)})(f_1^{(n)}, f_1^{(n)})} \overset{(4.12)}{\ge} \frac{(f_0^{(n)}, f_1^{(n)})^2}{C_1 (f_1^{(n)}, f_1^{(n)})^2} = \frac{1}{C_1}\left(t_0^{(n)}\right)^2.$$

Hence, the sequence $t_0^{(n)}$ is bounded. Therefore, there exists a subsequence $(t_0^{(n_k)})_{k\in\mathbb{N}}$ that converges to some $t \in \mathbb{R}$. This contradicts (4.11). Indeed

$$\lim_{n\to\infty} F_1^{(n)}(t) = \infty, \quad F_1^{(n)}(t_0^{(n)}) \le C, \quad \lim_{k\to\infty} t_0^{(n_k)} = t.$$

To prove the *necessity* condition suppose that $C_0 f_0 + C_1 f_1 \in \ell_2$ for some C_0, C_1. Let $f_1 = c_0 f_0 + h$, where $h \in \ell_2$. We have

$$\Gamma(f_0, f_1) = \Gamma(f_0, h) \le \Gamma(f_0)\Gamma(h).$$

Since $h \in \ell_2$ and $f \notin \ell_2$ we conclude that $\frac{\Gamma(f_0, f_1)}{\Gamma(f_0)}$ is bounded. □

4.1.2. Case $m = 2$

Lemma 4.4. *Consider vectors $f_0, f_1, f_2 \in \mathbb{R}^\infty$ such that $f_0, f_1, f_2 \notin \ell_2$, then for all r, s with $0 \le r < s \le 2$ holds*

$$\frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_r, f_s)} := \lim_{n\to\infty} \frac{\Gamma(f_0^{(n)}, f_1^{(n)}, f_2^{(n)})}{\Gamma(f_r^{(n)}, f_s^{(n)})} = \infty, \tag{4.13}$$

if and only if $\sum_{r=0}^2 C_r f_r \notin \ell_2$ for all $(C_0, C_1, C_2) \in \mathbb{R}^3 \setminus \{0\}$ and $C_r f_r + C_s f_s \notin \ell_2$ for all $(C_r, C_s) \in \mathbb{R}^2 \setminus \{0\}$.

Proof. Suppose that for all $n \in \mathbb{N}$

$$\frac{\Gamma(f_0^{(n)}, f_1^{(n)}, f_2^{(n)})}{\Gamma(f_1^{(n)}, f_2^{(n)})} \leq C. \tag{4.14}$$

Without loss of generality, we can assume that for all $n \in \mathbb{N}$

$$\Gamma(f_0^{(n)}, f_1^{(n)}) \leq C_2\Gamma(f_1^{(n)}, f_2^{(n)}), \quad \Gamma(f_0^{(n)}, f_2^{(n)}) \leq C_1\Gamma(f_1^{(n)}, f_2^{(n)}), \tag{4.15}$$

we choose some subsequence n_k, if necessary. See explanations of (4.12) before. Then it is sufficient to verify (4.13) for $(r, s) = (1, 2)$.

For $t \in \mathbb{R}^2$ and $f_0, f_1, f_2 \in H$, where H is some Hilbert space, define the quadratic form $F_2(t)$ as follows:

$$F_2(t) = \| \sum_{r=1}^{2} t_r f_r - f_0\|_H^2 = \sum_{k,r=1}^{2} t_k t_r (f_k, f_r)_H - 2\sum_{k=1}^{2} t_k (f_k, f_0)_H + (f_0, f_0)_H$$
$$= (A_2 t, t)_{\mathbb{R}^2} - 2(t, b)_{\mathbb{R}^2} + (f_0, f_0)_H,$$

where $b = (f_k, f_0)_{k=1}^2 \in \mathbb{R}^2$ and A_2 is the *Gram matrix*

$$A_2 = \gamma(f_1, f_2) = \big((f_k, f_r)\big)_{k,r=1}^2. \tag{4.16}$$

For t_0 defined by $A_2 t_0 = b$ we have by Lemma 2.1

$$F_2(t) = (A_2 t, t) - 2(t, b) + (f_0, f_0) = (A_2(t - t_0), (t - t_0)) + \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)},$$

and therefore,

$$F_2(t_0) = \frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_1, f_2)}. \tag{4.17}$$

Consider the following matrix

$$B_2 := \gamma(f_0, f_1, f_2) = \begin{pmatrix} (f_0, f_0) & (f_0, f_1) & (f_0, f_2) \\ (f_1, f_0) & (f_1, f_1) & (f_1, f_2) \\ (f_2, f_0) & (f_2, f_1) & (f_2, f_2) \end{pmatrix}. \tag{4.18}$$

By Cramer's rule (see Lemma 2.2) we have

$$t_0 = \frac{1}{A_0^0(B_2)} \begin{pmatrix} -A_1^0(B_2) \\ -A_2^0(B_2) \end{pmatrix}. \tag{4.19}$$

Let we have $f_0, f_1, f_2 \in \mathbb{R}^\infty$ and let $f_0^{(n)}, f_1^{(n)}, f_2^{(n)}$ be their projections on the subspace $\mathbb{R}^n$. Define for $t \in \mathbb{R}^2$ the quadratic form $F_2^{(n)}(t)$ as follows:

$$F_2^{(n)}(t) = \| \sum_{r=1}^{2} t_r f_r^{(n)} - f_0^{(n)} \|^2_{\mathbb{R}^2} = (A_2^{(n)} t, t)_{\mathbb{R}^2} - 2(t, b)_{\mathbb{R}^2} + (f_0^{(n)}, f_0^{(n)})_{\mathbb{R}^n},$$

where $b = (f_k^{(n)}, f_0^{(n)})_{k=1}^2 \in \mathbb{R}^2$ and $A_2^{(n)}$ is the *Gram matrix*

$$A_2^{(n)} = \gamma(f_1^{(n)}, f_2^{(n)}) = \big((f_k^{(n)}, f_r^{(n)})\big)_{k,r=1}^2. \tag{4.20}$$

In what follows we will use a scalar product $(f, g)_H$ without referring to a specific space H. By (4.35) we have

$$F_2^{(n)}(t_0^{(n)}) = \min_{t \in \mathbb{R}^2} F_2^{(n)}(t) = \frac{\Gamma(f_0^{(n)}, f_1^{(n)}, f_2^{(n)})}{\Gamma(f_1^{(n)}, f_2^{(n)})}.$$

We prove that the sequence $t_0^{(n)}$ is bounded. If we replace the vectors f_0, f_1, f_2 with $f_0^{(n)}, f_1^{(n)}, f_2^{(n)}$, we will get the following expressions:

$$t_0^{(n)} = \frac{1}{A_0^0(B_2^{(n)})} \begin{pmatrix} -A_1^0(B_2^{(n)}) \\ -A_2^0(B_2^{(n)}) \end{pmatrix}, \tag{4.21}$$

where $B^{(n)}(2)$ is defined by

$$B_2^{(n)} := \gamma(f_0^{(n)}, f_1^{(n)}, f_2^{(n)}) = \begin{pmatrix} (f_0^{(n)}, f_0^{(n)}) & (f_0^{(n)}, f_1^{(n)}) & (f_0^{(n)}, f_2^{(n)}) \\ (f_1^{(n)}, f_0^{(n)}) & (f_1^{(n)}, f_1^{(n)}) & (f_1^{(n)}, f_2^{(n)}) \\ (f_2^{(n)}, f_0^{(n)}) & (f_2^{(n)}, f_1^{(n)}) & (f_2^{(n)}, f_2^{(n)}) \end{pmatrix}. \tag{4.22}$$

Since all the matrices $B_2^{(n)}$ defined by (4.22) are positively defined, the inverse matrices $\left(B_2^{(n)}\right)^{-1}$ are also positively defined. We have the following expression for them, here we denote by B^T the *matrix transposed* to B,

$$\left(B_2^{(n)}\right)^{-1} = \frac{1}{\det B_2^{(n)}} \begin{pmatrix} A_0^0(B_2^{(n)}) & A_1^0(B_2^{(n)}) & A_2^0(B_2^{(n)}) \\ A_0^1(B_2^{(n)}) & A_1^1(B_2^{(n)}) & A_2^1(B_2^{(n)}) \\ A_0^2(B_2^{(n)}) & A_1^2(B_2^{(n)}) & A_2^2(B_2^{(n)}) \end{pmatrix}^T. \tag{4.23}$$

We prove that the sequence $t_0^{(n)}$ defined by (4.21)

$$t_0^{(n)} = \frac{1}{A_0^0(B_2^{(n)})} \begin{pmatrix} -A_1^0(B_2^{(n)}) \\ -A_2^0(B_2^{(n)}) \end{pmatrix} \tag{4.24}$$

is bounded when (4.15) holds. Set

$$t_{rr}^{(n)} = A_r^r(B_2^{(n)}), \quad 0 \le r \le 2, \quad t_{rs}^{(n)} = -A_s^r(B_2^{(n)}), \quad 0 \le r \ne s \le 2. \tag{4.25}$$

Then $t_0^{(n)} = \left(\frac{t_{01}^{(n)}}{t_{00}^{(n)}}, \frac{t_{02}^{(n)}}{t_{00}^{(n)}}\right)$. Since the matrix $\left(B_2^{(n)}\right)^{-1}$ is positively defined and (4.15) holds, we have

$$\left(t_{01}^{(n)}\right)^2 \le t_{00}^{(n)} t_{11}^{(n)}, \quad \left(t_{02}^{(n)}\right)^2 \le t_{00}^{(n)} t_{22}^{(n)}, \tag{4.26}$$

$$t_{22}^{(n)} \le C_2 t_{00}^{(n)}, \quad t_{11}^{(n)} \le C_1 t_{00}^{(n)}, \tag{4.27}$$

$$\|t_0^{(n)}\|^2 := \frac{|t_{01}^{(n)}|^2 + |t_{02}^{(n)}|^2}{|t_{00}^{(n)}|^2} \overset{(4.26)}{\le} \frac{t_{11}^{(n)} + t_{22}^{(n)}}{t_{00}^{(n)}} \tag{4.28}$$

$$\overset{(4.27)}{\le} \frac{C_1 t_{00}^{(n)} + C_2 t_{00}^{(n)}}{t_{00}^{(n)}} = C_1 + C_2. \tag{4.29}$$

Hence, the sequence $t_0^{(n)} \in \mathbb{R}^2$ is bounded. Therefore, there exists a subsequence $(t_0^{(n_k)})_{k\in\mathbb{N}}$ that converges to some $t \in \mathbb{R}^2$. This contradicts (4.14). Indeed

$$\lim_{n\to\infty} F_2^{(n)}(t) = \infty, \quad F_2^{(n)}(t_0^{(n)}) \le C, \quad \lim_{k\to\infty} t_0^{(n_k)} = t.$$

We prove the *necessity* condition for $(r, s) = (0, 1)$, the general case is similar. Suppose that $\sum_{k=0}^{2} C_k f_k \in \ell_2$ for some $(C_k)_k$ but $C_0 f_0 + C_1 f_1 \notin \ell_2$ for all $(C_0, C_1) \in \mathbb{R}^2 \setminus \{0\}$. Let $f_2 = c_0 f_0 + c_1 f_1 + h$, where $h \in \ell_2$. We have

$$\Gamma(f_0, f_1, f_2) = \Gamma(f_0, f_1, h) \le \Gamma(f_0, f_1)\Gamma(h).$$

Since $h \in \ell_2$ but $C_0 f_0 + C_1 f_1 \notin \ell_2$ for all $(C_0, C_1) \in \mathbb{R}^2 \setminus \{0\}$ we conclude that $\frac{\Gamma(f_0,f_1,f_2)}{\Gamma(f_0,f_1)}$ is bounded. □

4.2. The proof of Lemma 4.1

Proof. Suppose that for all $n \in \mathbb{N}$ we have

$$\frac{\Gamma(f_0^{(n)}, f_1^{(n)} \dots, f_m^{(n)})}{\Gamma(f_1^{(n)}, f_2^{(n)} \dots, f_m^{(n)})} \le C. \tag{4.30}$$

Without loss of generality, we can assume that for all $n \in \mathbb{N}$ and $1 \le s \le m$ hold

$$\Gamma(f_0^{(n)}, \dots, \widehat{f_s^{(n)}}, \dots, f_m^{(n)}) \le C_s \Gamma(f_1^{(n)}, f_2^{(n)}, \dots, f_m^{(n)}), \tag{4.31}$$

we choose some subsequence n_k, if necessary. Consider the following quadratic forms for $t = (t_r)_{r=1}^m \in \mathbb{R}^m$

$$F_m^{(n)}(t) = \|f_0^{(n)} - \sum_{r=1}^{m} t_r f_r^{(n)}\|^2. \tag{4.32}$$

The forms $F_m^{(n)}(t)$ defined by (4.32) have the following properties, for any fixed $t \in \mathbb{R}^m$ we have $\lim_{n\to\infty} F_m^{(n)}(t) = \infty$. By Lemma 2.1 there exists some $t_0^{(n)} \in \mathbb{R}^m$ such that

$$F_m^{(n)}(t_0^{(n)}) = \min_{t\in\mathbb{R}^m} F_m^{(n)}(t) = \frac{\Gamma(f_0^{(n)}, f_1^{(n)}, \dots, f_m^{(n)})}{\Gamma(f_1^{(n)}, f_2^{(n)} \dots, f_m^{(n)})}. \tag{4.33}$$

We prove that the sequence $t_0^{(n)}$ is bounded. To find $t_0^{(n)}$ explicitly in (4.33) we introduce some notations. For $t \in \mathbb{R}^m$ and $f_0, f_1, \dots, f_m \in H$ define the function

$$\begin{aligned} F_m(t) &= \|\sum_{k=1}^{m} t_k f_k - f_0\|^2 = \sum_{k,r=1}^{m} t_k t_r (f_k, f_r)_H - 2\sum_{k=1}^{m} t_k (f_k, f_0)_H + (f_0, f_0)_H \\ &= (A_m t, t)_{\mathbb{R}^m} - 2(t, b)_{\mathbb{R}^m} + (f_0, f_0)_H, \end{aligned}$$

where $b = \big((f_k, f_0)_H\big)_{k=1}^m \in \mathbb{R}^m$ and A_m is the *Gram matrix*, see Definition 1.1:

$$A_m = \gamma(f_1, \dots, f_m) = \big((f_k, f_r)_H\big)_{k,r=1}^m. \tag{4.34}$$

The minimum of $F_m(t)$ is attained at t_0 defined by $A_m t_0 = b$. By Remark 2.1, (2.6) and (2.3) we get (for details see Section 2.1)

$$F_m(t) = (A_m t, t) - 2(t, b) + (f_0, f_0) = (A_m(t - t_0), (t - t_0)) + \frac{\Gamma(f_0, f_1, \dots, f_m)}{\Gamma(f_1, \dots, f_m)},$$

$$\text{and therefore,} \quad F_m(t_0) = \frac{\Gamma(f_0, f_1, \dots, f_m)}{\Gamma(f_1, \dots, f_m)}. \tag{4.35}$$

Consider the following matrix

$$B_m = \gamma(f_0, f_1, \dots, f_m) = \begin{pmatrix} (f_0, f_0) & (f_0, f_1) & (f_0, f_2) & \dots & (f_0, f_m) \\ (f_1, f_0) & (f_1, f_1) & (f_1, f_2) & \dots & (f_1, f_m) \\ (f_2, f_0) & (f_2, f_1) & (f_2, f_2) & \dots & (f_2, f_m) \\ & & & \dots & \\ (f_m, f_0) & (f_m, f_1) & (f_m, f_2) & \dots & (f_m, f_m) \end{pmatrix}.$$

By Cramer's rule (see Lemma 2.2) the solution of $A_m t_0 = b$ is as follows:

$$t_0 = \frac{1}{A_0^0(B_m)} \begin{pmatrix} -A_1^0(B_m) \\ -A_2^0(B_m) \\ \dots \\ -A_m^0(B_m) \end{pmatrix}. \tag{4.36}$$

If we replace the vectors $(f_k)_{k=0}^m$ with $\big(f_k^{(n)}\big)_{k=0}^m$ we will get the following expression

$$t_0^{(n)} = \frac{1}{A_0^0(B_m^{(n)})} \begin{pmatrix} -A_1^0(B_m^{(n)}) \\ \dots \\ -A_m^0(B_m^{(n)}) \end{pmatrix}, \tag{4.37}$$

where $B^{(n)}(m)$ is defined by

$$B_m^{(n)} = \gamma(f_0^{(n)}, \dots, f_m^{(n)}) = \begin{pmatrix} (f_0^{(n)}, f_0^{(n)}) & (f_0^{(n)}, f_1^{(n)}) & \dots & (f_0^{(n)}, f_m^{(n)}) \\ (f_1^{(n)}, f_0^{(n)}) & (f_1^{(n)}, f_1^{(n)}) & \dots & (f_1^{(n)}, f_m^{(n)}) \\ & & \dots & \\ (f_m^{(n)}, f_0^{(n)}) & (f_m^{(n)}, f_1^{(n)}) & \dots & (f_m^{(n)}, f_m^{(n)}) \end{pmatrix}.$$

Since all the matrices $B_m^{(n)}$ are positively defined, the inverse matrices $\Big(B_m^{(n)}\Big)^{-1}$ are also positively defined. We have the following expression for them

$$\Big(B_m^{(n)}\Big)^{-1} = \frac{1}{\det B_m^{(n)}} \begin{pmatrix} A_0^0(B_m^{(n)}) & A_1^0(B_m^{(n)}) & \dots & A_m^0(B_m^{(n)}) \\ A_0^1(B_m^{(n)}) & A_1^1(B_m^{(n)}) & \dots & A_m^1(B_m^{(n)}) \\ & & \dots & \\ A_0^m(B_m^{(n)}) & A_1^m(B_m^{(n)}) & \dots & A_m^m(B_m^{(n)}) \end{pmatrix}^T. \tag{4.38}$$

We prove that the sequence $\big(t_0^{(n)}\big)_n$ is bounded when (4.31) holds.

$$t_0^{(n)} = \frac{1}{A_0^0(B_m^{(n)})} \begin{pmatrix} -A_1^0(B_m^{(n)}) \\ \dots \\ -A_m^0(B_m^{(n)}) \end{pmatrix}. \tag{4.39}$$

$$\text{Set } \; t_{rr}^{(n)} = A_r^r(B_m^{(n)}), \; 0 \le r \le m, \; t_{rs}^{(n)} = -A_s^r(B_m^{(n)}), \; 0 \le r \ne s \le m. \tag{4.40}$$

Then $t_0^{(n)} = \Big(\frac{t_{01}^{(n)}}{t_{00}^{(n)}}, \frac{t_{02}^{(n)}}{t_{00}^{(n)}}, \dots \frac{t_{0m}^{(n)}}{t_{00}^{(n)}}\Big)$. Since the matrix $\Big(B_m^{(n)}\Big)^{-1}$ is positively defined and (4.15) holds, we have

$$\big(t_{rs}^{(n)}\big)^2 \le t_{rr}^{(n)} t_{ss}^{(n)}, \quad \text{for all} \quad 0 \le r < s \le m, \tag{4.41}$$

$$t_{ss}^{(n)} \le C_s t_{00}^{(n)}, \quad \text{for all} \quad 0 \le s \le m-1, \tag{4.42}$$

$$\|t_0^{(n)}\|^2 := \frac{\sum_{s=1}^m |t_{0s}^{(n)}|^2}{|t_{00}^{(n)}|^2} \overset{(4.41)}{\le} \frac{\sum_{s=1}^m t_{ss}^{(n)}}{t_{00}^{(n)}} \tag{4.43}$$

$$\overset{(4.42)}{\le} \frac{\sum_{s=1}^m C_s t_{00}^{(n)}}{t_{00}^{(n)}} = \sum_{s=1}^m C_s. \tag{4.44}$$

Hence, the sequence $t_0^{(n)} \in \mathbb{R}^m$ is bounded. Therefore, there exists a subsequence $(t_0^{(n_k)})_{k\in\mathbb{N}}$ that converges to some $t \in \mathbb{R}^m$. This contradicts (4.30). Indeed

$$\lim_{n\to\infty} F_m^{(n)}(t) = \infty, \quad F_m^{(n)}(t_0^{(n)}) \le C, \quad \lim_{k\to\infty} t_0^{(n_k)} = t.$$

The necessity is proved as in the case $m = 1$ and $m = 2$. □

5. Some applications

5.1. The law of large numbers

In this and the following subsection we explain how Lemma 2.3 and the study of the generalized characteristic polynomials $\det B(\lambda)$ (especially the explicit expression for $B^{-1}(\lambda)$) where $B(\lambda)$ is defined by (3.1)

$$P_B(\lambda) = \det C(\lambda), \quad \text{where} \quad B(\lambda) = \mathrm{diag}(\lambda_1, \dots, \lambda_m) + B,$$

could be used in other fields. Consider $\mathbb{R}^\infty$ with infinite product of a *standard Guassian measures*

$$\mu_1(x) = \otimes_{n=1}^\infty \mu(x_n), \quad \text{where} \quad d\mu(x_n) = \sqrt{\frac{1}{2\pi}} \exp\Big(-\frac{x_n^2}{2}\Big) dx_n. \tag{5.1}$$

Set $f_0(x) \equiv 1$ and $f_n = x_n^2$. The law of large numbers can be reformulated in this particular case as follows:

Lemma 5.1. *We have $f_0 \in \langle f_n, n \in \mathbb{N}\rangle$ moreover, in $H = L_2(\mathbb{R}^\infty, \mu_1)$ holds*

$$s.\lim_n \frac{1}{n} \sum_{k=1}^n x_k^2 = f_0, \tag{5.2}$$

where $s.\lim$ means a strong limit in a Hilbert space H.

Proof. To prove this lemma we should show that $\lim_{n\to\infty} d^2(f_0, V_n) = 0$ where $d^2(f_0, V_n)$ is defined by (2.1). We should calculate $\Gamma(f_0, f_1, f_2, \dots, f_n)$ and $\Gamma(f_1, f_2, \dots, f_n)$. For the corresponding *Gram matrices* $\gamma(f_1, f_2, \dots, f_n)$ we get (we denote $\lambda_k = 2$ for all $1 \le k \le n$)

$$\gamma(f_0, f_1, f_2, \dots, f_n) = \Big((f_i, f_j)\Big)_{i,j=0}^n = \begin{pmatrix} 1 & 1 & 1 \dots & 1 \\ 1 & 3 & 1 \dots & 1 \\ 1 & 1 & 3 \dots & 1 \\ & & \dots & \\ 1 & 1 & 1 \dots & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \dots & 1 \\ 1 & 1+\lambda_1 & 1 \dots & 1 \\ 1 & 1 & 1+\lambda_2 \dots & 1 \\ & & \dots & \\ 1 & 1 & 1 \dots & 1+\lambda_n \end{pmatrix},$$

$$\gamma(f_1, f_2, \dots, f_n) = \Big((f_i, f_j)\Big)_{i,j=1}^n = \begin{pmatrix} 3 & 1 \dots & 1 \\ 1 & 3 \dots & 1 \\ & & \dots \\ 1 & 1 \dots & 3 \end{pmatrix} = \begin{pmatrix} 1+\lambda_1 & 1 \dots & 1 \\ 1 & 1+\lambda_2 \dots & 1 \\ & & \dots \\ 1 & 1 \dots & 1+\lambda_n \end{pmatrix},$$

$$\Gamma(f_0, f_1, f_2, \dots, f_n) = \det \gamma(f_0, f_1, f_2, \dots, f_n) \overset{(3.17)}{=} \prod_{k=1}^{n} \lambda_k,$$

$$\Gamma(f_0, f_1, f_2, \dots, f_n) = \det \gamma(f_0, f_1, f_2, \dots, f_n) \overset{(3.17)}{=} \prod_{k=1}^{n} \lambda_k \Big(1 + \sum_{k=1}^{n} \frac{1}{\lambda_k}\Big),$$

$$d^2(f_0, V_n) = \frac{\det\big(\gamma(f_0, f_1, \dots, f_n)\big)}{\det\big(\gamma(f_1, f_2, \dots, f_n)\big)} = \Big(1 + \sum_{k=1}^{n} \frac{1}{2}\Big)^{-1} \to 0.$$

By Lemma 2.3 and (3.18) we conclude that on the step n the coefficient should be $t_k \equiv \frac{1}{n}$. □

5.2. Some generalization

Consider $\mathbb{R}^\infty \times \mathbb{R}^\infty$ with infinite product of a standard Guassian measures

$$\mu_2(x) = \otimes_{k=1}^{2} \otimes_{n=1}^{\infty} \mu(x_{kn}). \tag{5.3}$$

Set $f_0(x) \equiv 1,\ \ f_n = x_{1n}^2 + a_n x_{1n} x_{2n}$.

Lemma 5.2 *([9]). We have $f_0 \in \langle f_n, n \in \mathbb{N}\rangle$ if and only if $\sum_{k=3}^{\infty} \frac{1}{a_k^2} = \infty$.*

Proof. To prove this lemma we should show that $\lim_{n\to\infty} d^2(f_0, V_n) = 0$ where $d^2(f_0, V_n)$ is defined by (2.1), if and only if $\sum_{k=3}^{\infty} \frac{1}{a_k^2} = \infty$. Further,

$$\gamma(f_0, f_3, f_4, \dots, f_{n+2}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1+a_3^2 & \dots & 1 \\ & & \dots & \\ 1 & 1 & \dots & 1+a_{n+2}^2 \end{pmatrix}, \tag{5.4}$$

$$\gamma(f_3, f_4, \dots, f_{n+2}) = \begin{pmatrix} 1+a_3^2 & 1 & \dots & 1 \\ 1 & 1+a_4^2 & \dots & 1 \\ & & \dots & \\ 1 & 1 & \dots & 1+a_{n+2}^2 \end{pmatrix}, \tag{5.5}$$

$$\det\big(\gamma(f_0, f_3, f_4, \dots, f_{n+2})\big) \overset{(3.17)}{=} \Big(\prod_{k=3}^{n+2} a_k^2\Big), \tag{5.6}$$

$$\det\big(\gamma(f_3, f_4, \dots, f_{n+2})\big) \overset{(3.17)}{=} \Big(\prod_{k=3}^{n+2} a_k^2\Big)\Big(1 + \sum_{k=3}^{n+2} \frac{1}{a_k^2}\Big). \tag{5.7}$$

Finally, by (2.1) we get

$$d^2(f_0, V_n) = \frac{\det\big(\gamma(f_0, f_3, f_4, \dots, f_{n+2})\big)}{\det\big(\gamma(f_3, f_4, \dots, f_{n+2})\big)} = \Big(1 + \sum_{k=3}^{n+2} \frac{1}{a_k^2}\Big)^{-1} \to 0.$$

5.3. *Idea of the proof of irreducibility*

By Lemma 2.3, for a strictly positive operator A acting in $\mathbb{R}^n$ and a vector $b \in \mathbb{R}^n \setminus \{0\}$ we have

$$\min_{x\in\mathbb{R}^n}\Big((Ax,x)\mid (x,b)=1\Big)=\frac{1}{(A^{-1}b,b)}.$$

In the concrete examples considered in [4–9] we prove the irreducibility as follows. First, we can approximate a lot of functions in $L^\infty(X,\mu)$ using Lemma 2.3. Second, the measure μ is ergodic with respect to the action of the group. The approximations follows from the fact

$$\lim_{n\to\infty}(B_n(\lambda)^{-1}a_n,a_n)=\infty. \tag{5.8}$$

Here $B_n = \gamma(g_1, g_2, \ldots, g_n)$, where $\gamma(g_1, g_2, \ldots, g_n)$ is Gram matrix of vectors $g_1, g_2, \ldots, g_n \in H$, (see Definition 1.1) and $B_n(\lambda) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) + B_n$, $\lambda \in \mathbb{C}^n$. By Lemma 3.5 proved in [9] we have (for notations see Section 3.2)

$$\big(B_n(\lambda)^{-1}a_n,a_n\big)=\frac{\det\big(I_m+\gamma(y_1^{(n)},y_2^{(n)},\ldots,y_m^{(n)})\big)}{\det\big(I_{m-1}+\gamma(y_2^{(n)},\ldots,y_m^{(n)})\big)}-1. \tag{5.9}$$

Finally, by Lemma 4.2 we have

$$\lim_{n\to\infty}\frac{\det\big(I_m+\gamma(y_1^{(n)},y_2^{(n)},\ldots,y_m^{(n)})\big)}{\det\big(I_{m-1}+\gamma(y_2^{(n)},\ldots,y_m^{(n)})\big)}=\infty.$$

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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