Three Families and Pontryagin Class

Zheyan Wan*[1], Juven Wang†[2], and Shing-Tung Yau‡[3,1]

[1]Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, China
[2]London Institute for Mathematical Sciences, Royal Institution, W1S 4BS, UK
[3]Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

January 29, 2026

Abstract

The family puzzle asks why the Standard Model (SM) features exactly 3 families of quarks and leptons. Motivated by topological constraints suggested by recent works (including modular invariance, framing anomaly cancellation, and cobordism constraints), we study 4-dimensional fermionic anomalies with discrete $\mathbb{Z}_n$ symmetry, classified by the 5d spin bordism group. We show that only the group-cohomology subclass $\mathrm{H}^5(\mathbb{Z}_n, \mathrm{U}(1)) \cong \mathbb{Z}_n$ can be canceled by an anomalous $\mathbb{Z}_n$-symmetric 4d $\mathbb{Z}_n$-gauge topological quantum field theory (TQFT), while beyond-group-cohomology contributions Ap_1 involving the Pontryagin class p_1 cannot. More generally, we prove that any cocycle $\alpha_d \in \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1))$ in odd spacetime dimension $d \geqslant 3$ is trivialized by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$, and we construct explicitly the corresponding symmetric anomalous boundary TQFT. For $d = 5$ and $n = 3$, this yields a $\mathrm{Spin} \times \mathbb{Z}_3$-symmetric 4d $\mathbb{Z}_3$-gauge TQFT that cancels the mixed discrete $(\mathbf{B}+\mathbf{L})$-gauge-gravitational anomaly of the SM in the absence of 3 "sterile" right-handed neutrinos ν_R. We further analyze a generalized SM with N_c colors and N_f families and argue that missing N_f copies of the ν_R can be naturally replaced by that 4d anomalous $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}$ symmetric $\mathbb{Z}_{N_c}$-gauge TQFT under the anomaly cancellation, via an appropriate $\mathbb{Z}_{N_c}$-color symmetry extension construction $1 \to \mathbb{Z}_{N_c} \to \mathrm{Spin} \times \mathbb{Z}_{N_c N_f} \to \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2N_f}$ of anomalous topological order. If N_c and N_f are minimal nonzero positive integers, then we find minimal color extensions:

$$\begin{cases} N_c = 3, & N_f \geqslant 3, \\ N_c = 4, & N_f \geqslant 2, \\ N_c = 12, & N_f \geqslant 6. \end{cases}$$

If we further require that an SM baryon is a fermion so N_c is odd, then we prove that 3 families and 3 colors, $N_c = N_f = 3$, is the unique case that stands out. We also prove that $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ for the mod 3 cohomology class in an appropriate context.

Contents

*wanzheyan@bimsa.cn
†jw@lims.ac.uk
‡styau@tsinghua.edu.cn

1 Introduction

One of the long-standing mysteries of theoretical physics is the origin of the three-family structure of fermion replication in the Standard Model (SM): quarks and leptons appear in exactly three families with identical gauge quantum numbers but differing masses and mixings, observed in particle physics since the 1970s [1]. While the SM itself places no restriction on the number of fermion families or generations, experimental observations indicate the family number $N_f = 3$, a fact often referred to as the Family Puzzle or Generation Problem. Understanding whether this number $N_f = 3$ is accidental or enforced by deeper consistency conditions remains an open theoretical puzzle in high-energy particle physics.

Experimentally, several independent lines of evidence establish that the number of fermion families in the SM is $N_f = 3$. The most direct constraint comes from precision measurements of the invisible decay width of the Z boson at LEP, which determine the number of light neutrino species to be 3. Since each SM family contains one left-handed neutrino, this implies three fermion families. Additional families are further strongly constrained by electroweak precision data, including the absence of deviations in Z-boson decays that would signal additional light quarks, as well as by flavor physics and the observed unitarity of the CKM matrix. Cosmological observations, such as Big Bang Nucleosynthesis and Cosmic Microwave Background measurements, independently support this conclusion by constraining the effective number of relativistic fermion species (primarily the three active light left-handed neutrinos) to be consistent with three.

Theoretically, renewed attention to this problem has been prompted by proposals based on topological and nonperturbative global anomaly constraints, suggesting that the $N_f = 3$ family structure may follow from fundamental mathematical consistency requirements rather than from model-dependent dynamics. In particular, there are two recent proposals based on topological constraints that have drawn our attention [2, 3]. Ref. [2, 3] approach this open problem in particle physics using tools from topology and topological quantum field theory, which have been rapidly developed in recent years to describe topological quantum matter [4], thereby going beyond conventional model-building frameworks in particle physics — in other words, thinking outside the box of conventional particle physics approaches.

1. Ref. [2] proposes that when the family number is a multiple of 3,

$$N_f = 0 \mod 3, \text{ namely, } \quad N_f \in 3\mathbb{Z}, \tag{1}$$

   the multiple of 3 families of 16 Weyl fermions per family/generation in the SM, with total $(N_f = 3) \times 16 = 48$ Weyl fermions in the 3+1d spacetime dimensions, are topologically constrained. This is due to

   (a) Modular Invariance: The dimensional-reduced 1+1d theory has a chiral central charge

$$c_- = c_L - c_R = \frac{N_f \times 16}{2} = \frac{N_f}{3} \times 48 = 0 \mod 24. \tag{2}$$

   (b) Hirzebruch signature: for a spacetime 4-manifold with a special orthogonal (SO) structure and purely bosonic gauge-invariant matter content, the signature $\sigma(M)$ of the manifold M and its first Pontryagin class $p_1(M)$ follow an integer-quantized relation $\sigma(M) = \frac{p_1(M)}{3} \in \mathbb{Z}$.

   (c) Rokhlin's theorem: for a spacetime 4-manifold with a Spin structure (the fermion parity $\mathbb{Z}_2^{\mathrm{F}}$ graded lift of the special orthogonal group SO, so $\mathrm{Spin}/\mathbb{Z}_2^{\mathrm{F}} = \mathrm{SO}$), and fermionic gauge-invariant matter content, the signature $\sigma(M)$ of the manifold M becomes $\sigma(M) = \frac{p_1(M)}{3} \in 16\mathbb{Z}$.

   (d) Cobordism mapping: The 48 Weyl fermions of the Standard Model, organized according to the bosonic SO and fermionic Spin structures, can be mapped to a trivial class in String cobordism (related to the framing anomaly-free [5]), or more generally to a trivial class in w_1-p_1 cobordism (related to the 2-framing anomaly-free [6]). This mapping argument holds independently of any internal global symmetry or gauge structure of the Standard Model.

Namely, the observation in Ref. [2] is mainly a **purely gravitational anomaly argument** (in the dimensionally reduced 1+1d theory), or a trivial cobordism class for a consistent quantum gravity theory [7].

Namely, the observation in Ref. [2] is primarily a gravitational anomaly argument (in the dimensionally reduced 1+1d theory) or corresponds to a trivial cobordism class in a consistent quantum gravity theory [7].

2. Ref. [3] proposes a unique interplay between the family and color numbers, with $N_f = N_c = 3$. This approach introduces an additional internal $\mathbb{Z}_3$ symmetry, naturally arising from the discrete Baryon plus Lepton $\mathbf{B}+\mathbf{L}$ symmetry in the SM [8, 9],

$$\mathbb{Z}^{\mathrm{F}}_{6,\mathbf{B}+\mathbf{L}} = \mathbb{Z}^{\mathrm{F}}_2 \times \mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}. \tag{3}$$

In the absence of the three right-handed sterile neutrinos ν_R, the Standard Model exhibits a mixed $(\mathbf{B}+\mathbf{L})$-gauge–gravitational nonperturbative global anomaly. The corresponding anomaly index for this SM (up to a $\pm$ sign),

$$N_f = 3 \in \mathbb{Z}_9 = \Omega_5^{\mathrm{Spin}\times\mathbb{Z}_3},$$

can be canceled via an appropriate color symmetry extension via

$$1 \to \mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_c N_f=9} \to \mathbb{Z}_{N_f=3} \to 1. \tag{4}$$

Here the symmetry extension refers to a particular Ref. [10]'s symmetry extension (e.g., group extension) construction, by trivializing the nontrivial anomaly index in $G = \mathrm{Spin} \times \mathbb{Z}_{N_f=3}$ by pulling it back to $G_{\mathrm{Tot}} = \mathrm{Spin} \times \mathbb{Z}_{N_c N_f=9}$ as a trivial anomaly class in G_{Tot}.

This means that the Standard Model without the $3\nu_R$ can still preserve the full $\mathbb{Z}^{\mathrm{F}}_{6,\mathbf{B}+\mathbf{L}}$ symmetry by replacing $3\nu_R$ by a finite gauge $\mathbb{Z}_{N_c=3}$ TQFT at low-energy. This demonstrates the uniqueness of

$$N_f = N_c = 3, \tag{5}$$

corresponding to $N_f = 3$ families with $N_c = 3$ colors, which represent the number of quarks in a baryon ($\mathbf{B}$). Note that this $\mathbb{Z}^{\mathrm{F}}_{6,\mathbf{B}+\mathbf{L}} = \mathbb{Z}^{\mathrm{F}}_2 \times \mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$ symmetry is independent of the choice of the SM gauge group, explained in [8, 9],

$$G_{\mathrm{SM}_{\mathrm{q}}} \equiv \frac{\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_{\mathrm{q}}}, \qquad \mathrm{q} = 1, 2, 3, 6$$

(see [11] for an explanation of $G_{\mathrm{SM}_{\mathrm{q}}}$). Consequently, the argument in [3] that $N_f = N_c = 3$ is also independent of the choice of the SM gauge group $G_{\mathrm{SM}_{\mathrm{q}}}$, which holds for any $\mathrm{q} = 1, 2, 3, 6$.

In this work, on one hand, we follow the setup in Ref. [3], to prove some of its observations more mathematically; on the other hand, we obtain some generalized theorems, and we derive some general topological constraints for the hidden topologically ordered dark matter sector of the N_f-family N_c-color generalized SM.

We bring the reader's attention to the fact that some other previous works also use potential nonperturbative global anomalies to constrain the $N_f = 3$ families of the SM, but these works are fundamentally different than our approach:

- Ref. [12] introduces an additional $\mathbb{Z}_3$ symmetry with a $\mathbb{Z}_9$ nonperturbative global anomaly in 4d, motivated by baryon triality or proton hexality. However, this extra $\mathbb{Z}_3$ symmetry relies on the structure of the more sophisticated supersymmetric Standard Model. Instead, Ref. [3] and our present work consider the simpler discrete $\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$ arising from the universal discrete $\mathbf{B}+\mathbf{L}$ symmetry of the non-supersymmetric SM [8, 9].

- Ref. [13] uses the 6d homotopy group analysis, $\pi_6(\mathrm{SU}(2)) = \mathbb{Z}_{12}$, $\pi_6(\mathrm{SU}(3)) = \mathbb{Z}_6$ and $\pi_6(G_2) = \mathbb{Z}_3$, to argue the nonperturbative global anomaly constraints from 6d to the 4d SM. However, we find that the cobordism classification of nonperturbative global anomaly constraints shows $\Omega_7^{\mathrm{Spin}\times G_{\mathrm{SM_q}}} = 0$ vanishes [14, 15], thus this means no these 6d nonperturbative global anomalies to constrain the 4d SM.

In summary, we believe that the $\mathbb{Z}_{3,\mathbf{B+L}}$ symmetry with a $\mathbb{Z}_9$ nonperturbative global anomaly offers a more robust trustworthy argument than the 3-family arguments in the older literature Ref. [12, 13].

1.1 Symmetry Extension

Let us explain the symmetry extension method in [10]. For 't Hooft anomalies of some global symmetry G to be nonperturbative global anomalies, we can potentially apply the appropriate symmetry-extension trivialization method [10], making a nonperturbative global anomaly in G becomes anomaly-free in an appropriate G_{Tot} via an appropriate group extension

$$1 \to K \to G_{\mathrm{Tot}} \xrightarrow{r} G \to 1. \tag{6}$$

Namely, the precise mathematical check is, given a G, we search for what an appropriate finite K and an appropriate extended G_{Tot} are, such that a nonperturbative global anomaly index

$$\nu_G \in \mathrm{TP}_d^G \tag{7}$$

in the Freed-Hopkins version [16] of cobordism group TP becomes the trivial anomaly class

$$(r^*\nu)_{G_{\mathrm{Tot}}} = 0 \in \mathrm{TP}_d^{G_{\mathrm{Tot}}} \tag{8}$$

for the cobordism group TP of the pulled back G_{Tot}. Here r is the reduction map from $G_{\mathrm{Tot}} \xrightarrow{r} G$, then the r^* with a $*$ denote the pullback. According to [10], this provides a (3+1)d anomalous G-symmetric K-gauge topological order construction whose low-energy theory is a (3+1)d finite K-gauge TQFT, which is designed to carry the original nontrivial 't Hooft anomaly index in G, namely $\nu_G \in \mathrm{TP}_d(G)$.

Note that

$$\mathrm{TP}_d(G) = (\Omega_d^G)_{\mathrm{torsion}} \oplus (\Omega_{d+1}^G)_{\mathrm{free}}. \tag{9}$$

When we focus on the nonperturbative global anomaly of 3+1d quantum field theory (QFT) with G symmetry, we could use the classification of anomalies at the $4 + 1 = 5$d cobordism group as

$$\mathrm{TP}_5(G) = (\Omega_5^G)_{\mathrm{torsion}}$$

in the case that $(\Omega_6^G)_{\mathrm{free}} = 0$. Then we need to check the anomaly index a nonperturbative global anomaly index

$$\nu_G \in \mathrm{TP}_5(G) = (\Omega_5^G)_{\mathrm{torsion}}, \tag{10}$$

which becomes the trivial anomaly class

$$(r^*\nu)_{G_{\mathrm{Tot}}} = 0 \in (\Omega_5^{G_{\mathrm{Tot}}})_{\mathrm{torsion}} \tag{11}$$

for the bordism group of the pulled back G_{Tot}. In summary, when we refer to the symmetry extension, or the symmetry extension trivialization of the ('t Hooft) anomaly, what we really mean is exactly the check done in this subsection, Sec. 1.1.

1.2 The Plan

The plan of this article goes as follows:

In Sec. 2.1, to warm up, we derive the 3+1d nonperturbative global anomaly of a Weyl fermion in $\text{Spin} \times \mathbb{Z}_3$ symmetry (with $\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$ in mind) from the reduction of the perturbative local anomaly in $\text{Spin} \times \text{U}(1)$ symmetry.

In Sec. 2.2, we derive the 3+1d nonperturbative global anomaly of a Weyl fermion in $\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_6^{\text{F}}$ symmetry (with $\mathbb{Z}_{6,\mathbf{B}+\mathbf{L}}^{\text{F}}$ in mind) from the reduction of the perturbative local anomaly in $\text{Spin}^c \equiv \text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \text{U}(1)$.

In Sec. 3, we show the global anomaly trivialization via the symmetry extension $1 \to \mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_c N_f=9} \to \mathbb{Z}_{N_f=3} \to 1$, and we explicitly construct 3+1d anomalous $\mathbb{Z}_3$-gauge TQFT as the low-energy theory of topologically ordered dark matter.

In Sec. 4, we generalize to a generic N_c-color and N_f-family SM such that we aim to trivialize the nonperturbative global anomaly associated with discrete $\mathbf{B}+\mathbf{L}$ symmetry $\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}^{\text{F}}$ that involves the family number N_f via the appropriate $\mathbb{Z}_{N_c}$-color symmetry extension

$$1 \to \mathbb{Z}_{N_c} \to \mathbb{Z}_{N_c N_f} \to \mathbb{Z}_{N_f} \to 1, \tag{12}$$

or more precisely involving the spacetime-internal symmetry together:

$$1 \to \mathbb{Z}_{N_c} \to \text{Spin} \times \mathbb{Z}_{N_c N_f,\mathbf{B}+\mathbf{L}} \to \text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}^{\text{F}} \to 1. \tag{13}$$

The symmetry-extension construction of anomalous $\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2N_c N_f,\mathbf{B}+\mathbf{L}}$-symmetry $\mathbb{Z}_{N_c}$-gauge TQFT in 3+1d follows the general approach outlined in [10].

In addition, we will prove various mathematical theorems in the Appendices that will be implemented in the main text.

In Appendix A, we prove that any group cocycle $\alpha_d \in \text{H}^d(\mathbb{Z}_n, \text{U}(1))$ is trivialized by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for odd $d \geqslant 3$ and any $n \geqslant 2$.

In Appendix B, following Ref. [10]'s symmetry-extension approach, we derive the explicit $(d-1)$-cochain $\tilde{\beta}_{d-1}$ that splits the d-cocycle $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$ as a coboundary in $\text{H}^d(\mathbb{Z}_{n^2}, \text{U}(1))$ by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for any odd $d \geqslant 3$ and any $n \geqslant 2$.

In Appendix C, we construct the path integral of dd-bulk/$(d-1)$d-boundary coupled invertible topological field theory/symmetric anomalous gapped TQFT by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for any odd $d \geqslant 3$ and any $n \geqslant 2$.

In Appendix D, we derive a 3+1d nonperturbative global anomaly formula of Weyl fermion in $\text{Spin} \times \mathbb{Z}_n$ symmetry for integer n with $2 \nmid n$ and $3 \nmid n$.

In Appendix E, we derive a 3+1d nonperturbative global anomaly formula of Weyl fermion $\text{Spin} \times \mathbb{Z}_{3^r} = \text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2\cdot 3^r}^{\text{F}}$ symmetry.

In Appendix F, we provide a proof of $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ for the mod 3 cohomology class in an appropriate context.

Below we outline a physical main theorem that we will prove in Sec. 4,

Theorem 1.1. *If the following conditions hold,*

1. *The color number N_c and the family number N_f are minimal nonzero positive integers,*

2. *The anomaly of N_f copies of a 4d charge-1 Weyl fermion (namely the unit charge "sterile" right-handed neutrinos ν_R) with symmetry $\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2N_f}$ is trivialized by a minimal $\mathbb{Z}_{N_c}$-extension,*[1]

[1] We can read this statement also as:

The anomaly of the N_f-family Standard Model missing all sterile right-handed neutrinos with symmetry $\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2N_f}$ is trivialized by a minimal $\mathbb{Z}_{N_c}$-extension.

3. N_c is odd, so that an SM baryon is a fermion (as it is in our SM),

then we prove that 3 families and 3 colors, $N_c = N_f = 3$, is the unique case that stands out.

Below we sketch the proof of Theorem 1.1, the detailed proof can be found in the proof of Theorem 4.1. Assume that the anomaly of N_f copies of a 4d charge-1 Weyl fermion (namely unit charge ν_R) with symmetry Spin $\times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2N_f}$ is trivialized by a minimal $\mathbb{Z}_{N_c}$-extension. Writing a generic family number

$$N_f = 2^p 3^r s, \qquad p, r \geqslant 0, \quad 2 \nmid s, \quad 3 \nmid s,$$

the anomaly index decomposes as the subgroup of the cobordism group as

$$2^p \cdot \mathbb{Z}_{2^{p+3}} \ \oplus \ 3^r \cdot \mathbb{Z}_{3^{r+1}}.$$

By [17], the 2-power factor is trivialized by a $\mathbb{Z}_4$-extension, and (see Appendix E), the 3-power factor is trivialized by a $\mathbb{Z}_3$-extension, where the fact that $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ is used. Hence, the minimal extension order is

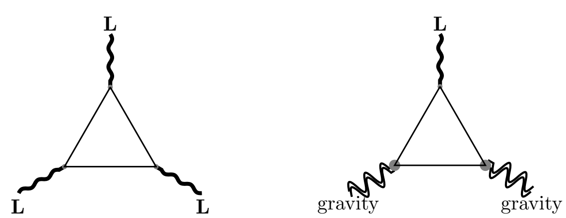
$$\begin{cases} N_c = 3, & \text{if } p = 0, r \geqslant 1, \quad \text{so } N_f \geqslant 3, \\ N_c = 4, & \text{if } p \geqslant 1, r = 0, \quad \text{so } N_f \geqslant 2, \\ N_c = 12, & \text{if } p \geqslant 1, r \geqslant 1, \quad \text{so } N_f \geqslant 6. \end{cases}$$

If the baryon is a fermion, then the color number N_c must be odd. So out of the above three choices, only $N_c = 3$ holds, whence the minimal family number is $N_f = 3$. So under the stated conditions, we prove that

$$N_c = N_f = 3.$$

2 Perturbative Local Anomaly to Nonperturbative Global Anomaly

In the 3+1d Standard Model (SM), each family contains 15 Weyl fermions in the absence of the 16th Weyl fermion sterile right-handed neutrino ν_R. This SM suffers from the perturbative local mixed-gauge-gravitational anomalies [18–20] between the lepton number $\mathbf{L}$ symmetry and gravitational background fields, in 3+1d (or simply 4d) spacetime. Namely these anomalies are computable via perturbative triangle Feynman diagrams $\mathrm{U}(1)^3_{\mathbf{L}}$ and $\mathrm{U}(1)_{\mathbf{L}}$-gravity^2,



with the anomaly index coefficient

$$-N_f + n_{\nu_R},$$

counting the difference between the family or generation number N_f (typically $N_f = 3$) and the total right-hand neutrino number n_{ν_R}. See recent related expositions about this anomaly index $-N_f + n_{\nu_R}$ for examples in [9, 21–26]. However, because of the analogous Adler-Bell-Jackiw anomalies [27, 28] via the SM electroweak gauge instanton [29–32], instead of thinking of the classical lepton number $\mathbf{L}$ symmetry, only the baryon number plus or minus lepton number $\mathbf{B} \pm \mathbf{L}$ symmetries are physically meaningful quantum mechanical symmetries of the SM [8, 9]:

For n_{ν_R} the number of types of right-handed neutrinos, so we have $n_{\nu_R} = 0$ to make the anomaly index of the SM (missing all sterile right-handed neutrinos) as $-N_f + n_{\nu_R} = -N_f$.

- For the gauge-invariant baryons, a full faithful combined symmetry of $\mathbf{B}-\mathbf{L}$ and $\mathbf{B}+\mathbf{L}$ with the Lorentz spacetime Spin group symmetry is

$$\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{2N_f,\mathbf{B}+\mathbf{L}}. \tag{14}$$

  Here $\times_N$ means the direct product mod out the common normal subgroup N here $N=\mathbb{Z}_2^{\mathrm{F}}$ fermion parity.

- For the free quarks, a full faithful combined symmetry of $\mathbf{Q}-N_c\mathbf{L}$ and $\mathbf{Q}+N_c\mathbf{L}$ with the Lorentz spacetime Spin group symmetry is

$$\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathrm{U}(1)_{\mathbf{Q}-N_c\mathbf{L}}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{2N_cN_f,\mathbf{Q}+N_c\mathbf{L}}, \tag{15}$$

  but it is unfaithful for the gauge-invariant baryons. Here in the conventional SM, the family number is $N_f=3$, and the color number is $N_c=3$.

For the conventional SM with, below we determine the $\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}=\mathrm{Spin}\times\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$ mixed gauge-gravitational anomaly for the right-handed "sterile" neutrino ν_R (sterile to the SM gauge force but not sterile to $\mathbf{B}\pm\mathbf{L}$ gauge field). In fact, we shall treat the anti-particle of right-handed neutrino $\bar{\nu}_R$ as the left-handed particle,

	$\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$	$\mathbb{Z}_{6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}$	$\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$	$\mathbb{Z}_2^{\mathrm{F}}$	$\mathbb{Z}_{18,\mathbf{Q}+3\mathbf{L}}^{\mathrm{F}}$	$\mathbb{Z}_{9,\mathbf{Q}+3\mathbf{L}}$
$\bar{\nu}_R$	1	-1	-1	1	-3	-9

(16)

So the charge $Q_{\mathbb{Z}_{6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}}=Q_{\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}}\mod 3$, and $Q_{\mathbb{Z}_{18,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}}=Q_{\mathbb{Z}_{9,\mathbf{B}+\mathbf{L}}}\mod 9$.[2]

In Subsection 2.1, we start with a perturbative local anomaly of U(1) charge $q=1$ left-handed Weyl fermion in $\mathrm{Spin}\times\mathrm{U}(1)$ to derive the the nonperturbative global anomaly of $\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$ charge $q=1$ Weyl fermion in $\mathrm{Spin}\times\mathbb{Z}_{3,\mathbf{B}+\mathbf{L}}$.

In Subsection 2.2, we will make a comparison to a perturbative local anomaly in $\mathrm{Spin}^c\equiv\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathrm{U}(1)$ and a nonperturbative global anomaly in $\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{6,\mathbf{B}+\mathbf{L}}^{\mathrm{F}}$.

2.1 $\mathrm{Spin}\times\mathrm{U}(1)$ to $\mathrm{Spin}\times\mathbb{Z}_3$

The perturbative local anomaly of U(1) charge $q=1$ left-handed Weyl fermion of $\mathrm{Spin}\times\mathrm{U}(1)$ symmetry in 3+1d or 4d is captured by a 5d invertible field theory (iTFT) with the anomaly index $k=1$ [19, 25]:

$$\exp(\mathrm{i}k\int_{M^5}A\frac{c_1^2}{6}-A\frac{p_1}{24}), \tag{17}$$

with the first Chern class c_1 and the first Pontryagin class p_1. Now we redefine the U(1) gauge field A as a $\mathbb{Z}_3$ gauge field $A_{\mathbb{Z}_3}\in\mathrm{H}^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)=\mathbb{Z}_3$ with the following replacement:

$$\begin{aligned} A &\mapsto \frac{2\pi}{3}A_{\mathbb{Z}_3}.\\ c_1=\frac{\mathrm{d}A}{2\pi} &\mapsto \frac{\mathrm{d}A_{\mathbb{Z}_3}}{3}\equiv\beta_{(3,3)}A_{\mathbb{Z}_3}. \end{aligned} \tag{18}$$

[2]We label $n_6\in\mathbb{Z}_6=\mathbb{Z}_6^{\mathrm{F}}\supset\mathbb{Z}_2^{\mathrm{F}}$ in terms of a doublet $(n_2^{\mathrm{F}},n_3)\in\mathbb{Z}_2^{\mathrm{F}}\times\mathbb{Z}_3$, such that the bosons have $n_2^{\mathrm{F}}=0$ and the fermions have $n_2^{\mathrm{F}}=1$. In addition, without loss of generality, we assign the charge $q=1\in\mathbb{Z}_6^{\mathrm{F}}$ fermion to the $(n_2^{\mathrm{F}},n_3)=(1,1)\in\mathbb{Z}_2^{\mathrm{F}}\times\mathbb{Z}_3$. This constrains the map as $n_6=3n_2^{\mathrm{F}}-2n_3$, so $n_6=n_3\mod 3$.
We label $n_{18}\in\mathbb{Z}_{18}=\mathbb{Z}_{18}^{\mathrm{F}}\supset\mathbb{Z}_2^{\mathrm{F}}$ in terms of a doublet $(n_2^{\mathrm{F}},n_9)\in\mathbb{Z}_2^{\mathrm{F}}\times\mathbb{Z}_9$, such that the bosons have $n_2^{\mathrm{F}}=0$ and the fermions have $n_2^{\mathrm{F}}=1$. In addition, without loss of generality, we assign the charge $q=1\in\mathbb{Z}_{18}^{\mathrm{F}}$ fermion to the $(n_2^{\mathrm{F}},n_9)=(1,1)\in\mathbb{Z}_2^{\mathrm{F}}\times\mathbb{Z}_9$. This constrains the map as $n_{18}=9n_2^{\mathrm{F}}-8n_9$, so $n_{18}=n_9\mod 9$.

The $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \mapsto \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$. Thus we get the 5d topological invariant of the Spin $\times \mathbb{Z}_3$ that captures the 4d anomaly as:

$$
\begin{aligned}
& \exp\big(\mathrm{i} 2\pi k \int_{M^5} (\frac{1}{18} A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}) - \frac{1}{3\cdot 24} A_{\mathbb{Z}_3} p_1)\big) \\
= & \exp\big(\mathrm{i}\frac{2\pi}{9} k \int_{M^5} (\frac{1}{2} A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}) - \frac{1}{8} A_{\mathbb{Z}_3} p_1)\big) \\
= & \exp\big(\mathrm{i}\frac{2\pi}{9} k \int_{M^5} (-4 A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}) + 3\cdot \frac{A_{\mathbb{Z}_3} p_1}{3})\big) \\
= & \exp\big(\mathrm{i}\frac{2\pi}{9} (-4k) \int_{M^5} (A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}) - 3\cdot \frac{A_{\mathbb{Z}_3} p_1}{3})\big).
\end{aligned} \tag{19}
$$

Here $\beta_{(3,3)} : \mathrm{H}^1(-, \mathbb{Z}_3) \to \mathrm{H}^2(-, \mathbb{Z}_3)$ is the Bockstein homomorphism. Here $\frac{A_{\mathbb{Z}_3} p_1}{3}$ is a mod 3 class that involves Pontryagin class because $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ [33, 34] (see Appendix F for the proof), while $A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3})$ is a mod 3 class.

In (19), the first quality rewrites the coefficients $\frac{1}{18} = \frac{1}{9} \cdot \frac{1}{2}$ and $-\frac{1}{3\cdot 24} = \frac{1}{9} \cdot (-\frac{1}{8})$, since the anomaly of 4d Weyl fermion with symmetry Spin $\times \mathbb{Z}_3$ contains only 3-torsion [12, 35–39], we can regard 2 and 8 as invertible in $\mathbb{Z}_9$.
The second equality uses the fact that $1 = -8 \mod 9$ to obtain $\frac{1}{2} = -4 \mod 9$ and $-\frac{1}{8} = 1 \mod 9$ and uses the fact that $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ [33, 34] to rewrite $A_{\mathbb{Z}_3} p_1 = 3 \cdot \frac{A_{\mathbb{Z}_3} p_1}{3}$.
The third equality uses the fact that $3 = 4 \cdot 3 \mod 9$ to rewrite $3 \cdot \frac{A_{\mathbb{Z}_3} p_1}{3} = 4 \cdot 3 \cdot \frac{A_{\mathbb{Z}_3} p_1}{3} \mod 9$ and factors out the common factor $-4k$ of the two terms.

Thus the 4d fermionic anomaly has the anomaly index $k \in \mathbb{Z}_9$, agreeing with the bordism group classification by $\Omega_5^{\mathrm{Spin}\times\mathbb{Z}_3} = \mathbb{Z}_9$ [12, 35–39].

When we have three right-handed neutrinos ($3\nu_R$), we need to consider $k = 3$ instead of $k = 1$, so eq. (19), with $4k = -12 = -3 \mod 9$, becomes

$$
\exp\big(\mathrm{i}\frac{2\pi}{3}(-1) \int_{M^5} (A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big), \tag{20}
$$

which is the generator of the bosonic group cohomology 5d iTFT from $\mathrm{H}^5(\mathbb{Z}_n, \mathrm{U}(1)) \cong \mathbb{Z}_n$ [40] with $n = 3$. Later Sec. 3 shows that for this specific case with $N_f = 3$, the symmetry-extension eq. (4) can be used to construct the 3+1d $\mathbb{Z}_{N_c=3}$-gauge topological order with a low-energy 3+1d fermionic $\mathbb{Z}_{N_c=3}$-gauge TQFT (as a hypothetical sector of 3+1d dark matter).

2.2 $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)$ to $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_6^{\mathrm{F}}$

In this section, we start with a perturbative local anomaly of U(1) charge $q = 1$ left-handed Weyl fermion in $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)$ to derive the the nonperturbative global anomaly of $\mathbb{Z}_{6,\mathbf{B+L}}^{\mathrm{F}}$ charge $q = 1$ Weyl fermion in $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{6,\mathbf{B+L}}^{\mathrm{F}}$.

First, we compare the Spin^c gauge field and the U(1) gauge field. For Spin^c, the $\mathrm{U}(1) \supset \mathbb{Z}_2^{\mathrm{F}}$ contains the fermion parity as a normal subgroup.

For the original U(1) with $c_1(\mathrm{U}(1))$, the gauge bundle constraint is $w_2(TM) = 2c_1 \mod 2$. In the original U(1), fermions have odd charges under U(1), while bosons have even charges under U(1). Call the original U(1) gauge field A, then $c_1 = \frac{\mathrm{d}A}{2\pi} \in \frac{1}{2}\mathbb{Z}$.

For the new $\mathrm{U}(1)' = \frac{\mathrm{U}(1)}{\mathbb{Z}_2^{\mathrm{F}}}$ with $c_1(\mathrm{U}(1)')$, the gauge bundle constraint is $w_2(TM) = c_1' = 2c_1 \mod 2$. Call the new $\mathrm{U}(1)'$ gauge field A', then $c_1' = \frac{\mathrm{d}A'}{2\pi} = \frac{\mathrm{d}(2A)}{2\pi} = 2c_1 \in 2\frac{1}{2}\mathbb{Z} = \mathbb{Z}$.

To explain why $A' = 2A$ or $c_1' = 2c_1$, we look at the Wilson line operator $\exp(\mathrm{i}q'\oint A')$ and $\exp(\mathrm{i}q\oint A)$. The original U(1) has charge transformation $\exp(\mathrm{i}q\theta)$ with $\theta \in [0, 2\pi)$, while the new $\mathrm{U}(1)'$ has charge transformation $\exp(\mathrm{i}q'\theta')$ with $\theta' \in [0, 2\pi)$. But the $\mathrm{U}(1)' = \frac{\mathrm{U}(1)}{\mathbb{Z}_2^{\mathrm{F}}}$, so the $\theta = \pi$ in the old U(1) is identified as $\theta' = 2\pi$ as a trivial zero in the new $\mathrm{U}(1)'$. In the original U(1), the $q \in \mathbb{Z}$ to be compatible with $\theta \in [0, 2\pi)$. In the new $\mathrm{U}(1)'$, the original q is still allowed to have $2\mathbb{Z}$ to be compatible with $\theta \in [0, \pi)$; but the new $q' = \frac{1}{2}q \in \mathbb{Z}$ and the new $\theta' = 2\theta \in [0, 2\pi)$ are scaled accordingly. Since the new $q' = \frac{1}{2}q \in \mathbb{Z}$, we show the new $A' = 2A$.

The perturbative local anomaly of charge $q = 1$ left-handed Weyl fermion of Spin^c symmetry in 3+1d or 4d is captured by a 5d invertible field theory (iTFT) with the anomaly index $k' = 1$:

$$\exp(\mathrm{i}k'\int_{M^5} A'\frac{(2c_1)^2}{48} - A'\frac{p_1}{48}). \tag{21}$$

Now we redefine the U(1) gauge field A' as a $\mathbb{Z}_3$ gauge field $A'_{\mathbb{Z}_3} \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3) = \mathbb{Z}_3$ with the following replacement:

$$\begin{aligned} A' &\mapsto \frac{2\pi}{3}A'_{\mathbb{Z}_3}. \\ 2c_1 = c_1' = \frac{\mathrm{d}A'}{2\pi} &\mapsto \frac{\mathrm{d}A'_{\mathbb{Z}_3}}{3} \equiv \beta_{(3,3)}A'_{\mathbb{Z}_3}. \end{aligned} \tag{22}$$

The $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \mapsto \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n \overset{\cdot m}{\to} \mathbb{Z}_{nm} \to \mathbb{Z}_m$. Thus we get the 5d topological invariant of the $\mathrm{Spin} \times \mathbb{Z}_3$ as:

$$\begin{aligned} &\exp\big(\mathrm{i}2\pi k'\int_{M^5}(\frac{1}{144}A'_{\mathbb{Z}_3}(\beta_{(3,3)}A'_{\mathbb{Z}_3})(\beta_{(3,3)}A'_{\mathbb{Z}_3}) - \frac{1}{3\cdot 48}A'_{\mathbb{Z}_3}p_1)\big) \\ =\ &\exp\big(\mathrm{i}\frac{2\pi}{9}k'\int_{M^5}(\frac{1}{16}A'_{\mathbb{Z}_3}(\beta_{(3,3)}A'_{\mathbb{Z}_3})(\beta_{(3,3)}A'_{\mathbb{Z}_3}) - \frac{1}{16}A'_{\mathbb{Z}_3}p_1)\big) \\ =\ &\exp\big(\mathrm{i}\frac{2\pi}{9}k'\int_{M^5}(4A'_{\mathbb{Z}_3}(\beta_{(3,3)}A'_{\mathbb{Z}_3})(\beta_{(3,3)}A'_{\mathbb{Z}_3}) - 4\cdot 3\cdot\frac{A'_{\mathbb{Z}_3}p_1}{3})\big) \\ =\ &\exp\big(\mathrm{i}\frac{2\pi}{9}(4k')\int_{M^5}(A'_{\mathbb{Z}_3}(\beta_{(3,3)}A'_{\mathbb{Z}_3})(\beta_{(3,3)}A'_{\mathbb{Z}_3}) - 3\cdot\frac{A'_{\mathbb{Z}_3}p_1}{3})\big). \end{aligned} \tag{23}$$

Here $\beta_{(3,3)} : \mathrm{H}^1(-, \mathbb{Z}_3) \to \mathrm{H}^2(-, \mathbb{Z}_3)$ is the Bockstein homomorphism. Here $\frac{A'_{\mathbb{Z}_3}p_1}{3}$ is a mod 3 class that involves Pontryagin class because $A'_{\mathbb{Z}_3}p_1 = 0 \mod 3$ [33, 34] (see Appendix F for the proof), while $A'_{\mathbb{Z}_3}(\beta_{(3,3)}A'_{\mathbb{Z}_3})(\beta_{(3,3)}A'_{\mathbb{Z}_3})$ is a mod 3 class.

In (23), the first equality rewrites the coefficients $\frac{1}{144} = \frac{1}{9}\cdot\frac{1}{16}$ and $-\frac{1}{3\cdot 48} = \frac{1}{9}\cdot(-\frac{1}{16})$ since the anomaly of 4d Weyl fermion with symmetry $\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_6$ contains only 3-torsion [12, 35–39], we can regard 16 as invertible in $\mathbb{Z}_9$.
The second equality uses the fact that $1 = 64 \mod 9$ to obtain $\frac{1}{16} = 4 \mod 9$ and uses the fact that $A'_{\mathbb{Z}_3}p_1 = 0 \mod 3$ [33, 34] to rewrite $A'_{\mathbb{Z}_3}p_1 = 3\cdot\frac{A'_{\mathbb{Z}_3}p_1}{3}$.
The third equality factors out the common factor $4k'$ of the two terms.

Eq.(19) and (23) give the same 5d topological invariant since

$$A'_{\mathbb{Z}_3} = -A_{\mathbb{Z}_3} \mod 3, \text{ and } k' = k. \tag{24}$$

Here, $A'_{\mathbb{Z}_3} = 2A_{\mathbb{Z}_3} = -A_{\mathbb{Z}_3} \mod 3$, because $c_1' = 2c_1$ and $A' = 2A$.
Here, $k' = k$, because we derive from the same perturbative anomaly from the same charge $q = 1$ Weyl fermion for both $\mathrm{Spin} \times \mathrm{U}(1)$ and $\mathrm{Spin}^c \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathrm{U}(1)$ symmetries.

Thus, the conclusion here in Sec. 2.2 follows the same as Sec. 2.1.

3 Symmetry Extension $1 \to \mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_c N_f=9} \to \mathbb{Z}_{N_f=3} \to 1$, Anomaly Trivialization, and 3+1d Anomalous $\mathbb{Z}_3$-Gauge Topologically Ordered Dark Matter

The fermionic anomaly in 3+1d or 4d spacetime with $\mathbb{Z}_n$ symmetry is classified by fermionic Spin bordism group $\Omega_5^{\text{Spin}\times\mathbb{Z}_n}$ [12, 35–39] which is isomorphic to bosonic SO bordism $\Omega_5^{\text{SO}}(\text{B}\mathbb{Z}_n)$ up to 2-torsion term when $2 \nmid n$, namely

$$\Omega_5^{\text{Spin}}(\text{B}\mathbb{Z}_n) \cong \tilde{\Omega}_5^{\text{SO}}(\text{B}\mathbb{Z}_n), \quad 2 \nmid n \tag{25}$$

$$\Omega_5^{\text{Spin}}(\text{B}\mathbb{Z}_{3^r \cdot s}) \cong \tilde{\Omega}_5^{\text{SO}}(\text{B}\mathbb{Z}_{3^r \cdot s}) \cong \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}} \oplus \mathbb{Z}_s \oplus \mathbb{Z}_s, \quad 2 \nmid s, \quad 3 \nmid s. \tag{26}$$

Here $\tilde{\Omega}_5^{\text{SO}}(\text{B}G) := \Omega_5^{\text{SO}}(\text{B}G)/\Omega_5^{\text{SO}}$ is the reduced bordism group, modding out the $\Omega_5^{\text{SO}} = \Omega_5^{\text{SO}}(pt)$.

In Appendix A, we prove that only the group cohomology subclass ($\text{H}^5(\mathbb{Z}_n, \text{U}(1)) \cong \mathbb{Z}_n$) anomaly[3] can be canceled by anomalous G-symmetric $K = \mathbb{Z}_n$-gauge 4d TQFTs, via the appropriate symmetry-extension construction [10] of

$$1 \to K \to G_{\text{Tot}} \to G \to 1. \tag{27}$$

as

$$1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1. \tag{28}$$

On the other hand, the beyond-group-cohomology subclass anomaly that involves Ap_1 (the first cohomology class A and the first Pontryagin class p_1) allows no such symmetric TQFTs.

More generally and mathematically, in this work, for odd $d \geqslant 3$ and any $n \geqslant 2$, we prove that any group cohomology cocycle

$$\alpha_d \in \text{H}^d(\mathbb{Z}_n, \text{U}(1)) \cong \mathbb{Z}_n \tag{29}$$

is trivialized by the group extension as eq. (28)'s $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ [10].

In Appendix B, we find an explicit $(d-1)$-cochain β_{d-1} that splits the d-cocycle α_d by that extension for odd $d \geqslant 3$ and any $n \geqslant 2$. Namely, $\alpha_d = \delta\beta_{d-1}$ holds when pulling back the quotient $\mathbb{Z}_n$ to the total $\mathbb{Z}_{n^2}$ group, from the cocycle α_d in $\text{H}^d(\mathbb{Z}_n, \text{U}(1))$ to the coboundary

$$\alpha_d = \delta\beta_{d-1} \in \text{H}^d(\mathbb{Z}_{n^2}, \text{U}(1)). \tag{30}$$

3.1 Explicit construction of 3+1d anomalous TQFT by the symmetry extension $1 \to \mathbb{Z}_3 \to \mathbb{Z}_9 \to \mathbb{Z}_3 \to 1$

As an application, for $d = 5$ and $n = 3$, we prove Ref. [3]'s statement that the symmetry-extension via eq. (4)'s

$$1 \to \mathbb{Z}_{N_c=3} \to \mathbb{Z}_{N_c N_f=9} \to \mathbb{Z}_{N_f=3} \to 1$$

can construct a $G = \text{Spin} \times \mathbb{Z}_{N_f=3}$-symmetric $K = \mathbb{Z}_{N_c=3}$-gauge 4d low-energy TQFT of gapped anomalous topologically ordered dark matter via canceling the missing $N_f = 3$ right-handed neutrinos ν_R's $\mathbb{Z}^{\text{F}}_{6,\mathbf{B+L}}$- or $\mathbb{Z}_{3,\mathbf{B+L}}$-gauge-gravitational anomaly in the 4d SM. This proves a claim in Ref. [3].

[3]For $n = 3$ as $\text{H}^5(\mathbb{Z}_3, \text{U}(1)) \cong \mathbb{Z}_3$, that generator is the anomaly of three right-handed neutrinos $3\nu_R$, which has the anomaly index $3 \in \Omega_5^{\text{Spin}\times\mathbb{Z}_3} \cong \Omega_5^{\text{Spin}}(B\mathbb{Z}_3) \cong \mathbb{Z}_9$. That is also the anomaly of the SM missing the $3\nu_R$, up to a negative sign -1 for the anomaly index.

More explicitly, for the α_5 given in eq. (20),

$$\alpha_5 = \exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^5}(A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big), \tag{31}$$

we have β_4 with $\alpha_5 = \delta\beta_4$, obtained in Appendix B, which suggests a construction of the 5d iTFT on the bulk 5-manifold M^5 and the 4d noninvertible TQFT on the 4d boundary $M^4 = \partial M^5$ with dynamical 1-cochain gauge field $a_{\mathbb{Z}_3} \in C^1(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$ and 2-cochain (dual) gauge field $b_{\mathbb{Z}_3} \in C^2(\mathrm{B}\mathbb{Z}_3, \mathbb{Z}_3)$, such that the full 5d/4d coupled path integral is given by

$$\begin{aligned}
&\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^5}(A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_3}\in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)\\ b_{\mathbb{Z}_3}\in C^2(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)}}\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^4=\partial M^5}(b_{\mathbb{Z}_3}\mathrm{d}a_{\mathbb{Z}_3} - b_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3} - a_{\mathbb{Z}_3}A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3})\big)\\
=\;&\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^5}(A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_3}\in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)\\ b_{\mathbb{Z}_3}\in C^2(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)}}\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^4=\partial M^5}(a_{\mathbb{Z}_3}(\mathrm{d}b_{\mathbb{Z}_3} - A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}) - b_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3})\big).
\end{aligned} \tag{32}$$

This 5d/4d coupled path integral analogously matches the discrete cocycle forms or cochain forms (e.g., [40]) derived in Appendix B, as the 5-cocycle

$$\alpha_5(g_1,g_2,g_3,g_4,g_5) = \zeta_n^{g_1[\frac{g_2+g_3}{n}][\frac{g_4+g_5}{n}]}$$

and the 4-cochain

$$\beta_4(h_1,h_2,h_3,h_4) = \zeta_n^{g_1k_2[\frac{g_3+g_4}{n}]}$$

at $n=3$, where ζ_n is an n-th root of unity such as $\zeta_n = \exp(\frac{2\pi\mathrm{i}}{n})$, with variables $g \in \mathbb{Z}_n$ and $k \in \mathbb{Z}_n$.

The 5d bulk partition function on a 5d manifold with a 4d boundary is not gauge-invariant, but the full 5d/4d coupled path integral eq. (32) is gauge-invariant under the following gauge transformation:

$$\begin{aligned}
&A_{\mathbb{Z}_3} \mapsto A_{\mathbb{Z}_3} + \mathrm{d}\lambda_{0,\mathbb{Z}_3},\\
&a_{\mathbb{Z}_3} \mapsto a_{\mathbb{Z}_3} + \mathrm{d}\mu_{0,\mathbb{Z}_3},\\
&b_{\mathbb{Z}_3} \mapsto b_{\mathbb{Z}_3} + \lambda_{0,\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3} + \mathrm{d}\mu_{1,\mathbb{Z}_3},\\
&A_{\mathbb{Z}_3} \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3) = \mathbb{Z}_3,\\
&a_{\mathbb{Z}_3} \in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3),\\
&b_{\mathbb{Z}_3} \in C^2(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3),\\
&\lambda_{0,\mathbb{Z}_3} \in C^0(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3),\\
&\mu_{0,\mathbb{Z}_3} \in C^0(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3),\\
&\mu_{1,\mathbb{Z}_3} \in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3).
\end{aligned} \tag{33}$$

Below we check that (32) is gauge-invariant under (33). Because $\beta_{(3,3)}\mathrm{d}\lambda_{0,\mathbb{Z}_3} = 0$, $\beta_{(3,3)}A_{\mathbb{Z}_3}$ and $\mathrm{d}b_{\mathbb{Z}_3} - A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}$ are gauge-invariant under the gauge transformation (33), hence (32) transforms under (33) as

$$\begin{aligned}
&\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^5}(A_{\mathbb{Z}_3}(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_3}\in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)\\ b_{\mathbb{Z}_3}\in C^2(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)}}\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^4=\partial M^5}(a_{\mathbb{Z}_3}(\mathrm{d}b_{\mathbb{Z}_3} - A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}) - b_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3})\big)
\end{aligned}$$

$$
\begin{aligned}
\mapsto\ & \exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^5}((A_{\mathbb{Z}_3}+\mathrm{d}\lambda_{0,\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3})(\beta_{(3,3)}A_{\mathbb{Z}_3}))\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_3}\in C^1(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)\\ b_{\mathbb{Z}_3}\in C^2(\mathrm{B}\mathbb{Z}_3,\mathbb{Z}_3)}}\exp\big(\mathrm{i}\,\frac{2\pi}{3}\int_{M^4=\partial M^5}((a_{\mathbb{Z}_3}+\mathrm{d}\mu_{0,\mathbb{Z}_3})(\mathrm{d}b_{\mathbb{Z}_3}-A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3})\\
&-(b_{\mathbb{Z}_3}+\lambda_{0,\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}+\mathrm{d}\mu_{1,\mathbb{Z}_3})\beta_{(3,3)}A_{\mathbb{Z}_3})\big).
\end{aligned}
\tag{34}
$$

Since by the Stokes theorem, we have

$$\int_{M^4=\partial M^5}(\mathrm{d}\mu_{0,\mathbb{Z}_3})(\mathrm{d}b_{\mathbb{Z}_3}-A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3})=0, \tag{35}$$

$$\int_{M^4=\partial M^5}(\mathrm{d}\mu_{1,\mathbb{Z}_3})\beta_{(3,3)}A_{\mathbb{Z}_3}=0, \tag{36}$$

and

$$\int_{M^4=\partial M^5}\lambda_{0,\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}=\int_{M^5}(\mathrm{d}\lambda_{0,\mathbb{Z}_3})\beta_{(3,3)}A_{\mathbb{Z}_3}\beta_{(3,3)}A_{\mathbb{Z}_3}, \tag{37}$$

(32) is gauge-invariant under (33).

In Appendix C, we construct the dd-bulk/$(d-1)$d-boundary coupled invertible topological field theory/symmetric anomalous gapped TQFT by the symmetry extension $1\to\mathbb{Z}_n\to\mathbb{Z}_{n^2}\to\mathbb{Z}_n\to 1$ for any odd $d\geqslant 3$ and any $n\geqslant 2$ explicitly. Ref. [41] constructs low dimensional coupled bulk–boundary TQFTs via symmetry extension; examples include a $(2+1)$d bulk with a $(1+1)$d boundary and a $(3+1)$d bulk with a $(2+1)$d boundary.

Many more 3+1d anomalous fermionic TQFTs, which carry a mixed gauge-gravitational nonperturbative global anomaly of the $G=\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{2m}$ or $G=\mathrm{Spin}\times\mathbb{Z}_n$ symmetry, can be found in: Cheng-Wang-Yang's 3+1d anomalous fermionic $\mathbb{Z}_4$-gauge theory [42] (see also the bosonic analogous discussion in [43]), the recent work of Décoppet-Yu [44] and Debray-Ye-Yu [45], and Wan-Wang [17]. General obstructions and constraints on the existence of these anomalous symmetric (3+1)d TQFTs are discussed in Cordova-Ohmori [46]. General properties of the anomalies of $G=\mathrm{Spin}\times_{\mathbb{Z}_2^{\mathrm{F}}}\mathbb{Z}_{2m}^{\mathrm{F}}$ or $\mathrm{Spin}\times\mathbb{Z}_n$ are discussed in Hsieh [37] and Wan [39], and other related nonperturbative global anomalies are discussed in Brennan-Intriligator [47].

These TQFTs can have BSM applications [3, 21, 23] for canceling SM's nonperturbative global anomalies [12, 37, 48, 49]. For future directions, it will be interesting to explore how other nonperturbative global anomalies can constrain other QFT-coupling-to-TQFT systems, with other potential BSM applications in mind.

4 Conclusion and Discussions: General N_f family and General N_c color Standard Model: Topologically Ordered Dark Matter via symmetry extension $1\to\mathbb{Z}_{N_c}\to\mathbb{Z}_{N_cN_f}\to\mathbb{Z}_{N_f}\to 1$

Consider the following N_f family and N_c color version of the Standard Model (SM), which is a 4d chiral gauge theory with Yang-Mills spin-1 gauge fields of the Lie algebra

$$\mathcal{G}_{\mathrm{SM}}\equiv su(N_c)\times su(2)\times u(1)_Y \tag{38}$$

coupling to N_f families of 15 or 16 Weyl fermions (spin-$\frac{1}{2}$ Weyl spinor is in the $\mathbf{2}_L^{\mathbb{C}}$ representation of the spacetime symmetry Spin(1,3), written as a left-handed 15- or 16-plet ψ_L) in the following $\mathcal{G}_{\mathrm{SM}}$ representation

[50, 51]

$$(\psi_L)_{\rm I} = (\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R)_{\rm I} \oplus n_{\nu_{{\rm I},R}} \bar{\nu}_{{\rm I},R}$$
$$\sim \big((\overline{\mathbf{N}}_c, \mathbf{1})_{-(1-r)h} \oplus (\mathbf{1}, \mathbf{2})_{-N_c h} \oplus (\mathbf{N}_c, \mathbf{2})_h \oplus (\overline{\mathbf{N}}_c, \mathbf{1})_{-(1+r)h} \oplus (\mathbf{1}, \mathbf{1})_{2N_c h}\big)_{\rm I} \oplus n_{\nu_{{\rm I},R}} (\mathbf{1}, \mathbf{1})_0$$
$$\sim \big((\overline{\mathbf{N}}_c, \mathbf{1})_{N_c - 1} \oplus (\mathbf{1}, \mathbf{2})_{-N_c} \oplus (\mathbf{N}_c, \mathbf{2})_1 \oplus (\overline{\mathbf{N}}_c, \mathbf{1})_{-(N_c+1)} \oplus (\mathbf{1}, \mathbf{1})_{2N_c}\big)_{\rm I} \oplus n_{\nu_{{\rm I},R}} (\mathbf{1}, \mathbf{1})_0 \quad (39)$$

for each family, while h is an overall normalization and r is the splitting parameter that can be solved by $u(1)_Y^3$ cubic anomaly cancellation to find $h = \pm N_c$ where $h = N_c$ gives the correct choice of u and d quark charges. Here our generic $u(1)_Y$ hypercharges are solved by the following anomaly-cancellation conditions

$$\begin{array}{rcl}
u(1)_Y\text{-}su(N_c)^2 & : & 2Y_{q_L} + Y_{\bar{u}_R} + Y_{\bar{d}_R} = 0, \\
u(1)_Y\text{-}su(2)^2 & : & N_c Y_{q_L} + Y_{l_L} = 0, \\
u(1)_Y\text{-}(\text{gravity})^2 & : & 2N_c Y_{q_L} + N_c Y_{\bar{u}_R} + N_c Y_{\bar{d}_R} + 2Y_{l_L} + Y_{\bar{e}_R} + Y_{\bar{\nu}_R} = 0, \\
u(1)_Y^3 & : & 2N_c Y_{q_L}^3 + N_c Y_{\bar{u}_R}^3 + N_c Y_{\bar{d}_R}^3 + 2Y_{l_L}^3 + Y_{\bar{e}_R}^3 + Y_{\bar{\nu}_R}^3 = 0, \\
(\mathbf{B}-\mathbf{L})\text{-}u(1)_Y^2 & : & (2Y_{q_L}^2 - Y_{\bar{u}_R}^2 - Y_{\bar{d}_R}^2) - (2Y_{l_L}^2 - Y_{\bar{e}_R}^2) = 0,
\end{array} \quad (40)$$

and the solution when $Y_{\nu_{{\rm I},R}} = 0$ is given by

$$(Y_{\bar{d}_R}, Y_{l_L}, Y_{q_L}, Y_{\bar{u}_R}, Y_{\bar{e}_R}, Y_{\nu_{{\rm I},R}}) = h \times (N_c - 1, -N_c, 1, -(N_c+1), 2N_c, 0) \quad (41)$$

At $h = 1$, $N_c = 3$, we get

$$(Y_{\bar{d}_R}, Y_{l_L}, Y_{q_L}, Y_{\bar{u}_R}, Y_{\bar{e}_R}, Y_{\nu_{{\rm I},R}}) = (2, -3, 1, -4, 6, 0). \quad (42)$$

So $N_c = 3$ typically goes as

$$(\psi_L)_{\rm I} = (\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R)_{\rm I} \oplus n_{\nu_{{\rm I},R}} \bar{\nu}_{{\rm I},R} \sim \big((\overline{\mathbf{N}}_c, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{2})_{-3} \oplus (\mathbf{N}_c, \mathbf{2})_1 \oplus (\overline{\mathbf{N}}_c, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_6\big)_{\rm I} \oplus n_{\nu_{{\rm I},R}} (\mathbf{1}, \mathbf{1})_0 \quad (43)$$

The total number of Weyl fermions in N_f family for the whole multiplet of eq. (39) is

$$N_f(4N_c + 3) + \sum_{\rm I} n_{\nu_{{\rm I},R}}. \quad (44)$$

For $N_f = N_c = 3$, this total number becomes $3 \cdot 15 + \sum_{\rm I} n_{\nu_{{\rm I},R}}$.

Now the Witten SU(2) anomaly free [52] demands that the total number of **2** dimensional representation of SU(2) Weyl fermions need to be an even integer:

$$\begin{array}{l}
N_f(N_c+1) \in 2\mathbb{Z} \text{ for Witten SU(2) anomaly free} \\
\text{so either} \begin{cases} N_f \in \mathbb{Z}_{\rm odd}, N_c \in \mathbb{Z}_{\rm odd}, \text{ thus baryon is a fermion.} \\ N_f \in \mathbb{Z}_{\rm even}, N_c \in \mathbb{Z}, \text{ thus baryon can be a fermion } (N_c \in \mathbb{Z}_{\rm odd}) \text{ or a boson } (N_c \in \mathbb{Z}_{\rm even}). \end{cases}
\end{array} \quad (45)$$

Again, we treat the anti-particle of right-handed neutrino $\bar{\nu}_R$ as the left-handed particle,

	$\mathrm{U}(1)_{\mathbf{L}}$	$\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$	$\mathbb{Z}^{\rm F}_{2N_f,\mathbf{B}+\mathbf{L}}$	$\mathbb{Z}_2^{\rm F}$	$\mathbb{Z}^{\rm F}_{2N_c N_f,\mathbf{Q}+3\mathbf{L}}$
$\bar{\nu}_R$	-1	1	-1	1	$-N_c$

(46)

Moreover, only when N_f and 2 are coprime, namely their greatest common divisor is $\gcd(N_f, 2) = 1$, such as $N_f = 3, 5, 7, \ldots$, then we further have $\mathbb{Z}^{\rm F}_{2N_f} = \mathbb{Z}_2^{\rm F} \times \mathbb{Z}_{N_f}$, such that $\bar{\nu}_R$ has a well-defined $\mathbb{Z}_{N_f,\mathbf{B}+\mathbf{L}}$ charge -1:

	$\mathbb{Z}^{\rm F}_{2N_f,\mathbf{B}+\mathbf{L}} = \mathbb{Z}_2^{\rm F} \times \mathbb{Z}_{N_f,\mathbf{B}+\mathbf{L}}$
$\bar{\nu}_R$	$-1 \sim 1 \cdot -1$

, $\gcd(N_f, 2) = 1.$ (47)

Furthermore, only when $N_c N_f$ and 2 are coprime, namely their greatest common divisor is $\gcd(N_c N_f, 2) = 1$, then we further have $\mathbb{Z}^{\rm F}_{2N_c N_f} = \mathbb{Z}_2^{\rm F} \times \mathbb{Z}_{N_c N_f}$, such that $\bar{\nu}_R$ has a well-defined $\mathbb{Z}_{N_f,\mathbf{B+L}}$ charge $-N_c$:

$$
\begin{array}{|c|c|}
\hline
 & \mathbb{Z}^{\rm F}_{2N_f,\mathbf{B+L}} = \mathbb{Z}_2^{\rm F} \times \mathbb{Z}_{N_f,\mathbf{B+L}} \\
\hline
\bar{\nu}_R & -N_c \sim 1 \cdot -N_c \\
\hline
\end{array}, \quad \gcd(N_c N_f, 2) = 1. \tag{48}
$$

For a generic N_f-family SM missing some $\bar{\nu}_R$, we have to determine its anomaly index in $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}^{\rm F}_{2N_f,\mathbf{B+L}}}$, which we require the following bordism group classification (let $N_f = 2^p \cdot 3^r \cdot s$)

$$
\begin{split}
&\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2^{p+1} \cdot 3^r \cdot s}} \cong \Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2^{p+1}}} \oplus \tilde{\Omega}_5^{\rm SO}(\mathrm{B}\mathbb{Z}_{3^r \cdot s}) \\
&= \mathbb{Z}_{2^{p+3}} \oplus \mathbb{Z}_{2^{p-1}} \oplus \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}} \oplus \mathbb{Z}_s \oplus \mathbb{Z}_s, \quad p \geqslant 1, \quad r \geqslant 1, \quad 2 \nmid s, \quad 3 \nmid s.
\end{split} \tag{49}
$$

Here $\tilde{\Omega}_5^{\rm SO}(\mathrm{B}G) := \Omega_5^{\rm SO}(\mathrm{B}G)/\Omega_5^{\rm SO}$ is the reduced bordism group, modding out the $\Omega_5^{\rm SO} = \Omega_5^{\rm SO}(pt)$.

Now we ask two general questions relevant for high-energy phenomenology for $G = \text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$ symmetry:

1. For a single charge $q = -1 \in \mathbb{Z}_{2N_f,\mathbf{B+L}} \subset \text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$-symmetry $\bar{\nu}_R$, can there exists a symmetric-gapped 4d TQFT matching the $\bar{\nu}_R$'s symmetry and anomaly in the full $\text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$?

   The answer to this question is the same as asking in the case of a single charge $q = 1 \in \mathbb{Z}_{2N_f} \subset \text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f}$-symmetry Weyl fermion, up to a -1 sign of the chosen basis. The answer is no, there exists no such symmetric-gapped 4d TQFT matching $q = 1$ or -1 Weyl fermion's anomaly in general.

2. For N_f copies of charge $q = -1 \in \mathbb{Z}_{2N_f,\mathbf{B+L}} \subset \text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$-symmetry $\bar{\nu}_R$, can there exists a symmetric-gapped 4d TQFT matching the $\bar{\nu}_R$'s symmetry and anomaly in the full $\text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$?

   The answer to this question is the same as asking in the case of N_f copies of charge $q = 1 \in \mathbb{Z}_{2N_f} \subset \text{Spin} \times_{\mathbb{Z}_2^{\rm F}} \mathbb{Z}_{2N_f}$-symmetry Weyl fermions, up to a -1 sign of the chosen basis. The answer is in general yes, there exists such symmetric-gapped 4d TQFT matching N_f copies of $q = 1$ or -1 Weyl fermion's anomaly in general.

   But there is a refined question: Is this N_f copies of Weyl fermion anomaly within a group cohomology (GC) class or beyond a group cohomology (BGC) class?

   (a) For group cohomology (GC) class, Ref. [10] shows that there always exists a symmetric anomalous gapped boundary with a finite abelian gauge group as the low-energy TQFT (here in 4d) to cancel the GC class

   (b) For beyond a group cohomology (BGC) class, Ref. [10] cannot show that a symmetric anomalous gapped boundary with a finite abelian gauge group exists or not. But we are able to determine what are the minimal finite group K symmetry extension that can trivialize the anomaly in G via pulling back through $1 \to K \to G_{\rm Tot} \to G \to 1$ to the anomaly-free in $G_{\rm Tot}$.

Due to Witten's SU(2) anomaly free constraint in eq. (45), we summarize the results in two cases, eq. (50) and eq. (51). Here we determine the **minimal finite abelian K-gauge group extension** for the generalized SM with N_f family number (in the column) and N_c color number (in the row). The symmetry-extension trivialization via K means that we can replace N_f copies of $\bar{\nu}_R$ by K-gauge symmetric-gapped 4d TQFT (namely with a gauge group K). The color index N_c does *not* directly affect the minimal K-gauge TQFT. But for a certain appropriate N_c, when $K = \mathbb{Z}_{N_c}$, there is an interesting interplay between N_c color and N_f family.

1. The $N_f \in \mathbb{Z}_{\text{odd}}$ and $N_c \in \mathbb{Z}_{\text{odd}}$ case gives rise to the following relation in a table:

K-group extension	$N_f = 1$		$N_f = 3$		$N_f = 5$		$N_f = 7$		$N_f = 9$
$N_c = 3$	No		$\mathbb{Z}_{3=N_c}$ GC		Trivial		Trivial		$\mathbb{Z}_3$ GC ($9\ \bar{\nu}_R$)
$N_c = 5$	No		$\mathbb{Z}_3$ GC		Trivial		Trivial		
$N_c = 7$	No		$\mathbb{Z}_3$ GC		Trivial		Trivial		
$N_c = 9$	No		$\mathbb{Z}_3$ GC		Trivial		Trivial		$\mathbb{Z}_9$ GC ($3\ \bar{\nu}_R$)

. (50)

(a) For $N_f = 1$, there is no anomaly. We consider the SM missing $N_f = 1$ $\bar{\nu}_R$, thus "No" means "No anomaly" (even for a single Weyl fermion) and "No TQFT."

(b) For $N_f = 3$, there is a $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_6^{\text{F}}} = \Omega_5^{\text{Spin} \times \mathbb{Z}_3} = \mathbb{Z}_9$ class anomaly. We consider the SM missing $N_f = 3$ $\bar{\nu}_R$, there "$\mathbb{Z}_3$ GC" means $K = \mathbb{Z}_3$-gauge TQFT can match the group cohomology $\mathbb{Z}_3$ subclass anomaly. When $N_c = 3$, we have a $K = \mathbb{Z}_{N_c=3}$-gauge TQFT that the K-extension matches an intriguing $\mathbb{Z}_{N_c=3}$-color extension, because surprisingly $N_c = N_f = 3$ in this case.

(c) For $N_f = 5$, $N_f = 7$, or other N_f such that $2 \nmid N_f$ and $3 \nmid N_f$, there is a $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{2N_f}} = \Omega_5^{\text{Spin} \times \mathbb{Z}_{N_f}} = \mathbb{Z}_{N_f} \oplus \mathbb{Z}_{N_f}$ class anomaly. Although a generic number of $\bar{\nu}_R$ can contribute an anomaly, when we consider the SM missing N_f $\bar{\nu}_R$, the total anomaly class is trivial, thus we write "Trivial" in this case and there is also no need to have any 4d TQFT to cancel the anomaly.

(d) For $N_f = 9$, there is a $\Omega_5^{\text{Spin} \times_{\mathbb{Z}_2^{\text{F}}} \mathbb{Z}_{18}^{\text{F}}} = \Omega_5^{\text{Spin} \times \mathbb{Z}_9} = \mathbb{Z}_{27} \oplus \mathbb{Z}_3$ class anomaly.
We consider the SM missing $N_f = 9$ $\bar{\nu}_R$, there "$\mathbb{Z}_3$ GC" means $K = \mathbb{Z}_3$-gauge TQFT can match the group cohomology $\mathbb{Z}_3$ subclass anomaly. When $N_c = 3$, we have a $K = \mathbb{Z}_{N_c=3}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=3}$-color extension, but $N_c = 3 \neq N_f = 9$ in this case.
Instead if we consider the $(N_f = 9)$-SM missing 3 $\bar{\nu}_R$, there "$\mathbb{Z}_9$ GC" means $K = \mathbb{Z}_9$-gauge TQFT can match the group cohomology $\mathbb{Z}_9$ subclass anomaly. When $N_c = 9$, we have a $K = \mathbb{Z}_{N_c=9}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=9}$-color extension, although $N_c = N_f = 9$ in this case, we need to have 6 extra $\bar{\nu}_R$ added into the SM.
Thus we show that $N_c = N_f = 3$ case is more natural in terms of the $\mathbb{Z}_{N_c}$-color extension.
While $N_f = 9$ or higher 3-torsions

$$N_f = 3^r, r \geq 2,$$

for the $(N_f = 3^r)$-SM missing $N_f = 3^r$ copies of sterile neutrinos $\bar{\nu}_R$, there we also have "$\mathbb{Z}_3$ GC" means $K = \mathbb{Z}_3$-gauge TQFT can match the group cohomology $\mathbb{Z}_3$ subclass anomaly. When $N_c = 3$, we have a $K = \mathbb{Z}_{N_c=3}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=3}$-color extension, but $N_c = 3 \neq N_f = 3^r$, with $r \geq 2$ in this case.
Here we need to quote our results derived in Appendices A and E, the $k = 3^r$ anomaly of the 4d Weyl fermion for $k = 3 \in \mathbb{Z}_{3^{r+1}} \subset \Omega_5^{\text{Spin}}(\mathrm{B}\mathbb{Z}_{3^r}) \cong \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}}$ eq. (108) with $\text{Spin} \times \mathbb{Z}_{3^r}$ symmetry can be trivialized by a $\mathbb{Z}_3$ extension.

2. The $N_f \in \mathbb{Z}_{\text{even}}$ and $N_c \in \mathbb{Z}$ case gives rise to the following K-extension:

K-group extension		$N_f = 2$		$N_f = 4$		$N_f = 6$	
$N_c = 2$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12}$ BGC	
$N_c = 3$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12}$ BGC	
$N_c = 4$		$\mathbb{Z}_{4=N_c}$ BGC		$\mathbb{Z}_{4=N_c}$ BGC		$\mathbb{Z}_{12}$ BGC	
$N_c = 5$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12}$ BGC	
$N_c = 6$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12}$ BGC	
$N_c = 7$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12}$ BGC	
$\dots$		$\dots$		$\dots$		$\dots$	
$N_c = 12$		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_4$ BGC		$\mathbb{Z}_{12=N_c}$ BGC	

. (51)

(a) For $N_f = 2$, there is a $\Omega_5^{\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_4^{\text{F}}} = \mathbb{Z}_{16}$ class anomaly. We consider the SM missing $N_f = 2$ $\bar{\nu}_R$, there "$\mathbb{Z}_4$ BGC" means $K = \mathbb{Z}_4$-gauge TQFT can match the beyond-the-group-cohomology (BGC) $\mathbb{Z}_8$ subclass anomaly. When $N_c = 4$, we have a $K = \mathbb{Z}_{N_c=4}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=4}$-color extension, but $N_c = 4 \neq N_f = 2$ in this case.

(b) For $N_f = 4$, there is a $\Omega_5^{\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_8^{\text{F}}} = \mathbb{Z}_{32}\oplus\mathbb{Z}_2$ class anomaly. We consider the SM missing $N_f = 4$ $\bar{\nu}_R$, there "$\mathbb{Z}_4$ BGC" means $K = \mathbb{Z}_4$-gauge TQFT can match the beyond-the-group-cohomology (BGC) $\mathbb{Z}_8$ subclass anomaly. When $N_c = 4$, we have a $K = \mathbb{Z}_{N_c=4}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=4}$-color extension, so $N_c = N_f = 4$ in this case.

(c) For $N_f = 6$, there is a $\Omega_5^{\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_{12}^{\text{F}}} = \mathbb{Z}_{16}\oplus\mathbb{Z}_9$ class anomaly. We consider the SM missing $N_f = 6$ $\bar{\nu}_R$, there "$\mathbb{Z}_{12}$ BGC" means $\mathbb{Z}_4$-gauge TQFT can match the beyond-the-group-cohomology (BGC) $\mathbb{Z}_8 \in \mathbb{Z}_{16}$ subclass anomaly and an additional $\mathbb{Z}_3$-gauge TQFT can match the group-cohomology (GC) $\mathbb{Z}_3 \in \mathbb{Z}_9$ subclass anomaly. When $N_c = 12$, we have a $K = \mathbb{Z}_{N_c=12}$-gauge TQFT that the K-extension matches a $\mathbb{Z}_{N_c=12}$-color extension, but $N_c = 12 \neq N_f = 6$ in this case.

(d) For $N_f \in \mathbb{Z}_{\text{even}}$, we can go through similar discussions like the above.

To conclude, for the general N_f family and general N_c color Standard Model (SM), we find the following case of $\mathbb{Z}_{N_c}$ color symmetry-extension

$$1 \to \mathbb{Z}_{N_c} \to \text{Spin}\times\mathbb{Z}_{N_cN_f} \to \text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_{2N_f} \to 1 \tag{52}$$

can trivialize the N_f copies of sterile neutrinos $\bar{\nu}_R$ (and also its corresponding SM complement) such that the following three cases are the most intriguing:

$$\begin{cases} N_c = 3,\ N_f = 3^r s,\ r \geqslant 1,\ 2\nmid s,\ 3\nmid s,\ \text{the group-cohomology anomaly. Baryon is a fermion.} \\ N_c = 4,\ N_f = 2^p s,\ p \geqslant 1,\ 2\nmid s,\ 3\nmid s,\ \text{beyond-the-group-cohomology anomaly. Baryon is a boson.} \\ N_c = 12,\ N_f = 2^p 3^r s,\ p \geqslant 1,\ r \geqslant 1,\ 2\nmid s,\ 3\nmid s,\ \text{beyond-the-group-cohomology anomaly. Baryon is a boson.} \end{cases} \tag{53}$$

We can prove the following theorem, which is a restatement of Theorem 1.1.

Theorem 4.1. *If the anomaly of N_f copies of the 4d charge $q = 1$ Weyl fermion with symmetry $\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_{2N_f}$ is trivialized by a $\mathbb{Z}_{N_c}$-extension where N_c is minimal. If N_f is also minimal and N_c is odd (so the baryon is a fermion), then $N_f = N_c = 3$.*

Proof. Under our assumption, the anomaly of N_f copies of the 4d charge $q = 1$ Weyl fermion with symmetry $\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_{2N_f}$ is trivialized by a (minimal) $\mathbb{Z}_{N_c}$-extension. Let $N_f = 2^p \cdot 3^r \cdot s$ where $p \geqslant 0$, $r \geqslant 0$, $2 \nmid s$, and $3 \nmid s$. By (49), the anomaly of N_f copies of the 4d charge $q = 1$ Weyl fermion with symmetry $\text{Spin}\times_{\mathbb{Z}_2^{\text{F}}}\mathbb{Z}_{2N_f}$ is

$$2^p \cdot \mathbb{Z}_{2^{p+3}} \oplus 3^r \cdot \mathbb{Z}_{3^{r+1}}. \tag{54}$$

Here, $\mathbb{Z}_{2^{p+3}}$ is the anomaly of the 4d charge $q=1$ Weyl fermion with symmetry Spin $\times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2^{p+1}}$, and $\mathbb{Z}_{3^{r+1}}$ is the anomaly of the 4d charge $q=1$ Weyl fermion with symmetry Spin $\times \mathbb{Z}_{3^r}$.

By the results in [17], $2^p \cdot \mathbb{Z}_{2^{p+3}}$ is trivialized by a $\mathbb{Z}_4$-extension. In Appendix E, we prove that $3^r \cdot \mathbb{Z}_{3^{r+1}}$ is trivialized by a $\mathbb{Z}_3$-extension. Therefore, $N_c = 4$ if $p \geqslant 1$ and $r = 0$, $N_c = 3$ if $p = 0$ and $r \geqslant 1$, and $N_c = 12$ if $p \geqslant 1$ and $r \geqslant 1$, .

If the baryon is a fermion, then N_c is odd. Therefore, $N_c = 3$ and $N_f = 3^r \cdot s$ where $r \geqslant 1$, $2 \nmid s$, and $3 \nmid s$. So the minimal N_f is $N_f = 3$. □

In summary, if we restrict to the more familiar generalized SM with baryon as a fermion, and if we require that N_c and N_f are minimal, then $N_c = N_f = 3$ emerges as the unique case for constructing a 4d anomalous Spin $\times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2N_f,\mathbf{B+L}}$**-symmetric gapped topological order with low-energy TQFT**, such that the missing N_f copies of the sterile neutrinos $\bar{\nu}_R$ can be naturally replaced by a 4d $\mathbb{Z}_{N_c}$-gauge fermionic TQFT.

Finally, we remark that our scenario shall be *different* from the Dark Dimension [53] scenario involving:

- Higgs mechanism on the $\mathbf{B-L}$ gauge field sector — note that the Higgs mechanism involves the symmetry-breaking mechanism.
- 3 right-handed neutrinos propagate in the 5th Dark Dimension.

In our case and in one of authors parallel work, we emphasize:

- Symmetry-extension construction of anomalous topological order is beyond the Higgs mechanism, different from the symmetry-breaking mechanism.
- Massive 4+1d Dirac fermion with a relative sign ± 1 of mass flips can give rise to 4+1d invertible topological field theory (iTFT)/ Symmetry-Protected Topological states (SPTs) in the 4+1d bulk. However, the anomalous topological order lives on 3+1d, which can be attached to the boundary of 4+1d bulk; or simply attached to the 3+1d SM without the need of the 4+1d bulk at all.

5 Acknowledgment

JW thanks Dan Freed, Constantin Teleman, and Matthew Yu for the discussions on the related topics. ZW is supported by the NSFC Grant No. 12405001. JW is supported by LIMS and Ben Delo Fellowshop. JW would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Diving Deeper into Defects: On the Intersection of Field Theory, Quantum Matter, and Mathematics, where work on this paper was undertaken. This work was supported by EPSRC grant EP/Z000580/1. JW also thanks Simons Foundation Collaboration on Global Categorical Symmetries Annual Meetings in 2024 and 2025, where this work is discussed and performed during the meetings.

A Any cocycle $\alpha_d \in \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1))$ is trivialized by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for odd $d \geqslant 3$ and any $n \geqslant 2$

In this appendix, we show that any cocycle $\alpha \in \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1))$ is trivialized by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for odd $d \geqslant 3$ and any $n \geqslant 2$.

We consider the Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{Z}_n, \mathrm{H}^q(\mathbb{Z}_n, \mathrm{U}(1))) \Rightarrow \mathrm{H}^{p+q}(\mathbb{Z}_{n^2}, \mathrm{U}(1)) \tag{55}$$

associated with the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$.

Since

$$\mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1)) = \begin{cases} \mathbb{Z}_n & d \text{ odd} \\ 0 & d \text{ even} > 0 \\ \mathrm{U}(1) & d = 0 \end{cases} \tag{56}$$

and

$$\mathrm{H}^d(\mathbb{Z}_n, \mathbb{Z}_n) = \mathbb{Z}_n \quad \forall d \geqslant 0, \tag{57}$$

the E_2 page of the Serre spectral sequence (55) is shown in Fig. 1.

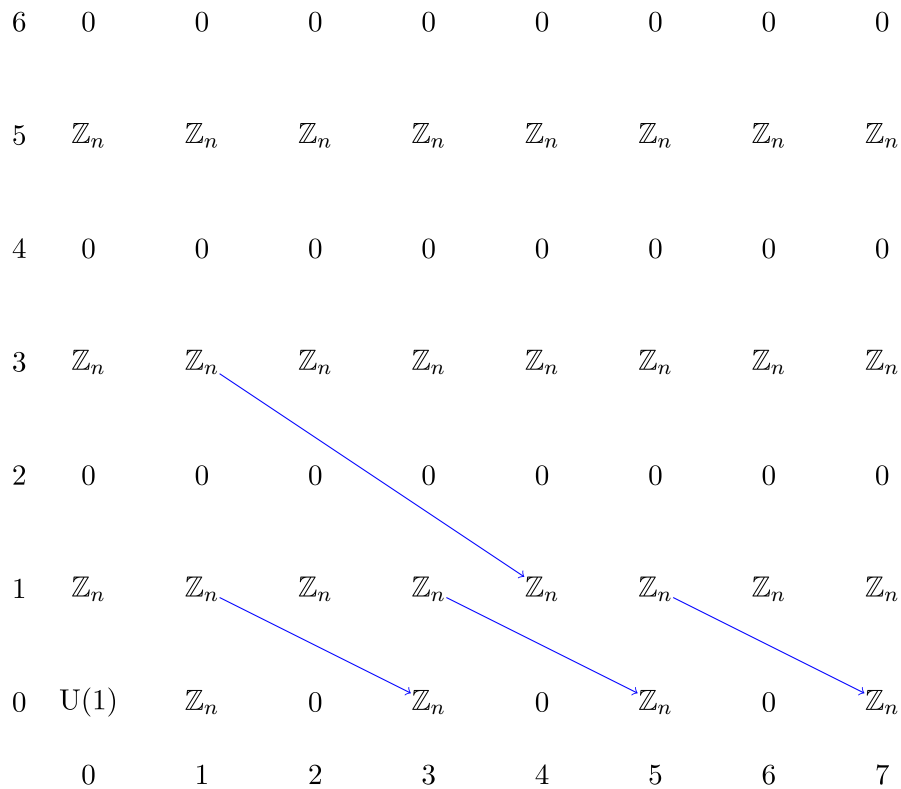


Figure 1: The E_2 page of the Serre spectral sequence (55). The differentials will be explained later.

The differentials in the Serre spectral sequence (55) are

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1} \text{ for } r \geqslant 2, \tag{58}$$

and the pages E_r are defined inductively from E_2 by

$$E_{r+1}^{p,q} = \frac{\text{Ker } d_r^{p,q}}{\text{Im } d_r^{p-r,q+r-1}}. \tag{59}$$

The differentials d_r vanish and the pages E_r stabilize for sufficiently large $r \geqslant N$. The page E_N is denoted E_∞.

The homomorphism $\mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1)) \to \mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ induced from the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ is the composition

$$E_2^{d,0} = \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1)) \twoheadrightarrow E_\infty^{d,0} \hookrightarrow \mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1)). \tag{60}$$

For $p+q=d$, there is a filtration

$$F^{-1}=0\subset F^0\subset F^1\subset\cdots\subset F^d=\mathrm{H}^d(\mathbb{Z}_{n^2},\mathrm{U}(1)) \tag{61}$$

of $\mathrm{H}^d(\mathbb{Z}_{n^2},\mathrm{U}(1))$ with

$$F^q/F^{q-1}=E_\infty^{d-q,q}. \tag{62}$$

We will show that $E_\infty^{d,0}=0$ for odd $d\geqslant 3$ and any $n\geqslant 2$, hence by (60), any cocycle $\alpha\in\mathrm{H}^d(\mathbb{Z}_n,\mathrm{U}(1))$ is trivialized by the extension $1\to\mathbb{Z}_n\to\mathbb{Z}_{n^2}\to\mathbb{Z}_n\to 1$ for odd $d\geqslant 3$ and any $n\geqslant 2$.

Since $\mathrm{H}^2(\mathbb{Z}_{n^2},\mathrm{U}(1))=0$, the group $E_2^{1,1}=\mathbb{Z}_n$ is eliminated by the differential

$$d_2^{1,1}:E_2^{1,1}\longrightarrow E_2^{3,0},$$

so it does not survive to the E_3–page. Hence $E_\infty^{3,0}=E_3^{3,0}=0$.

Since $\mathrm{H}^4(\mathbb{Z}_{n^2},\mathrm{U}(1))=0$, the groups $E_2^{1,3}=\mathbb{Z}_n$ and $E_2^{3,1}=\mathbb{Z}_n$ are removed on some page by differentials. On the other hand, because $\mathrm{H}^3(\mathbb{Z}_{n^2},\mathrm{U}(1))=\mathbb{Z}_{n^2}$ and $E_2^{0,3}=E_2^{2,1}=\mathbb{Z}_n$, and since $E_2^{3,0}=\mathbb{Z}_n$ is eliminated by the differential $d_2^{1,1}:E_2^{1,1}\to E_2^{3,0}$, the groups $E_2^{0,3}=\mathbb{Z}_n$ and $E_2^{2,1}=\mathbb{Z}_n$ survive to the E_∞–page.

Therefore $E_2^{3,1}=\mathbb{Z}_n$ cannot be the target of any nonzero differential; instead it supports the differential

$$d_2^{3,1}:E_2^{3,1}\longrightarrow E_2^{5,0},$$

and so does not persist to E_∞. Hence $E_\infty^{5,0}=E_3^{5,0}=0$.

Similarly, $E_2^{1,3}=E_3^{1,3}=\mathbb{Z}_n$ cannot be the target of any nonzero differential; it is the source of the differential

$$d_3:E_3^{1,3}\longrightarrow E_3^{4,1},$$

and therefore $E_\infty^{4,1}=E_4^{4,1}=0$.

Since $\mathrm{H}^6(\mathbb{Z}_{n^2},\mathrm{U}(1))=0$, the group $E_2^{5,1}=\mathbb{Z}_n$ is eliminated by some differential. On the other hand, because $\mathrm{H}^5(\mathbb{Z}_{n^2},\mathrm{U}(1))=\mathbb{Z}_{n^2}$ and $E_2^{0,5}=E_2^{2,3}=\mathbb{Z}_n$, while $E_2^{4,1}=\mathbb{Z}_n$ and $E_2^{5,0}$ are removed by the differentials d_3 and d_2 respectively, the groups $E_2^{0,5}=\mathbb{Z}_n$ and $E_2^{2,3}=\mathbb{Z}_n$ survive to E_∞. Hence $E_2^{5,1}=\mathbb{Z}_n$ cannot be the target of a nonzero differential; instead it supports

$$d_2^{5,1}:E_2^{5,1}\longrightarrow E_2^{7,0},$$

and therefore $E_\infty^{7,0}=E_3^{7,0}=0$.

In fact, one can show in general that $E_2^{d,0}$ is eliminated by the differential

$$d_2^{d-2,1}:E_2^{d-2,1}\longrightarrow E_2^{d,0},$$

so $E_\infty^{d,0}=E_3^{d,0}=0$ for every odd $d\geqslant 3$ and any $n\geqslant 2$.

The cohomology ring $\mathrm{H}^*(\mathbb{Z}_n,\mathbb{Z}_n)$ is generated by $x\in\mathrm{H}^1(\mathbb{Z}_n,\mathbb{Z}_n)$ and $y\in\mathrm{H}^2(\mathbb{Z}_n,\mathbb{Z}_n)$ and they satisfy the relation $x^2=0$ for odd n and $x^2=\frac{n}{2}y$ for even n [54, Example 3.41]. In particular, for $n=2$, $x^2=y$ and $\mathrm{H}^*(\mathbb{Z}_2,\mathbb{Z}_2)$ is generated by $x\in\mathrm{H}^1(\mathbb{Z}_2,\mathbb{Z}_2)$. The generator of $E_2^{1,1}=\mathbb{Z}_n$ is x and the generator of $E_2^{2,1}=\mathbb{Z}_n$ is y. Since we have shown that $d_2(x)$ is non-trivial and $d_2(y)=0$, the differentials are derivations, and $\smile y:\mathrm{H}^m(\mathbb{Z}_n,\mathrm{U}(1))\to\mathrm{H}^{m+2}(\mathbb{Z}_n,\mathrm{U}(1))$ is an isomorphism for odd m[4], for $n=2$, $d_2(x^k)=kd_2(x)x^{k-1}$ is non-trivial for odd k, and for $n>2$, $d_2(xy^k)=d_2(x)y^k$ is non-trivial for all $k\geqslant 0$. Hence $E_\infty^{d,0}=E_3^{d,0}=0$ for odd $d\geqslant 3$ and any $n\geqslant 2$.

[4]This is because $\mathrm{H}^m(\mathbb{Z}_n,\mathrm{U}(1))=\mathrm{H}^{m+1}(\mathbb{Z}_n,\mathbb{Z})$, while $\mathrm{H}^*(\mathbb{Z}_n,\mathbb{Z})$ is periodic of period 2, and $\mathrm{H}^2(\mathbb{Z}_n,\mathbb{Z})=\mathrm{H}^2(\mathbb{Z}_n,\mathbb{Z}_n)$ by the universal coefficient theorem.

For any group extension

$$1 \to K \to H \to G = \mathbb{Z}_n \to 1, \tag{63}$$

we have a similar Serre spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\mathbb{Z}_n, \mathrm{H}^q(K, \mathrm{U}(1))) \Rightarrow \mathrm{H}^{p+q}(H, \mathrm{U}(1)). \tag{64}$$

For degree reasons, there are no differentials from or to $E_2^{1,0} = \mathrm{H}^1(\mathbb{Z}_n, \mathrm{U}(1))$, so $E_2^{1,0} = \mathrm{H}^1(\mathbb{Z}_n, \mathrm{U}(1))$ survives to the E_∞ page. Therefore, $A_{\mathbb{Z}_n} \in \mathrm{H}^1(\mathbb{Z}_n, \mathbb{Z}_n)$ can not be trivialized by any group extension, hence $A_{\mathbb{Z}_n} p_1$ can not be trivialized by any group extension.

B Explicit $(d-1)$-cochain $\tilde{\beta}_{d-1}$ that splits the d-cocycle $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$ as a coboundary in $\mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for any odd $d \geqslant 3$ and any $n \geqslant 2$

In this appendix, we find an explicit $(d-1)$-cochain $\tilde{\beta}_{d-1}$ that splits the d-cocycle $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$ as a coboundary in $\mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for any odd $d \geqslant 3$ and any $n \geqslant 2$. Our strategy follows Ref. [10]'s symmetry-extension approach to trivialize a cocycle in terms of coboundary, especially in Ref. [10]'s Appendices.

B.1 $d = 3$ and any $n \geqslant 2$: Find $\tilde{\beta}_2$ such that $\tilde{\alpha}_3 = \delta\tilde{\beta}_2$

Explicitly, any 3-cocycle $\alpha_3 \in \mathrm{H}^3(\mathbb{Z}_n, \mathrm{U}(1))$ has the form [55]

$$\alpha_3(g_1, g_2, g_3) = \zeta_n^{g_1[\frac{g_2+g_3}{n}]} \tag{65}$$

where $g_i \in \mathbb{Z}_n$ for $i = 1, 2, 3$, ζ_n is an n-th root of unity (for example, $\zeta_n = \exp(\frac{2\pi \mathrm{i}}{n})$), and $[\frac{p}{q}]$ denotes the integer part of $\frac{p}{q}$.

We can find an explict 2-cochain $\tilde{\beta}_2 \in C^2(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ such that $\tilde{\alpha}_3 = \delta\tilde{\beta}_2$ where $\tilde{\alpha}_3 = f^*\alpha_3 \in \mathrm{H}^3(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ and f is the map in the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \xrightarrow{f} \mathbb{Z}_n \to 1$. Namely,

$$\tilde{\alpha}_3(h_1, h_2, h_3) = \delta\tilde{\beta}_2(h_1, h_2, h_3) = \frac{\tilde{\beta}_2(h_2, h_3)\tilde{\beta}_2(h_1, h_2h_3)}{\tilde{\beta}_2(h_1h_2, h_3)\tilde{\beta}_2(h_1, h_2)} \tag{66}$$

where $h_i = (g_i, k_i) \in \mathbb{Z}_{n^2}$ for $i = 1, 2, 3$ and

$$(g_1, k_1) \cdot (g_2, k_2) = (g_1 + g_2, k_1 + k_2 + [\frac{g_1 + g_2}{n}]). \tag{67}$$

Here, $\phi(g_1, g_2) = [\frac{g_1+g_2}{n}]$ is the 2-cocycle in $\mathrm{H}^2(\mathbb{Z}_n, \mathbb{Z}_n)$ classifying the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$. Note that for $n = 2$, $[\frac{g_1+g_2}{2}] = g_1g_2 \mod 2$, so the composition law (67) agrees with that in [10] for $n = 2$.

Explicitly, the 2-cochain $\tilde{\beta}_2 \in C^2(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ is given by

$$\tilde{\beta}_2(h_1, h_2) = \zeta_n^{g_1k_2}. \tag{68}$$

Then

$$\delta\tilde{\beta}_2(h_1, h_2, h_3) = \zeta_n^{g_2k_3+g_1(k_2+k_3+[\frac{g_2+g_3}{n}])-(g_1+g_2)k_3-g_1k_2} = \zeta_n^{g_1[\frac{g_2+g_3}{n}]} = \alpha_3(g_1, g_2, g_3) = \tilde{\alpha}_3(h_1, h_2, h_3). \tag{69}$$

Therefore, $\tilde{\alpha}_3 = \delta\tilde{\beta}_2$.

B.2 $d = 5$ and any $n \geqslant 2$: Find $\tilde{\beta}_4$ such that $\tilde{\alpha}_5 = \delta\tilde{\beta}_4$

Explicitly, any 5-cocycle $\alpha_5 \in \mathrm{H}^5(\mathbb{Z}_n, \mathrm{U}(1))$ has the form [55]

$$\alpha_5(g_1, g_2, g_3, g_4, g_5) = \zeta_n^{g_1[\frac{g_2+g_3}{n}][\frac{g_4+g_5}{n}]} \tag{70}$$

where $g_i \in \mathbb{Z}_n$ for $i = 1, 2, 3, 4, 5$, ζ_n is an n-th root of unity, and $[\frac{p}{q}]$ denotes the integer part of $\frac{p}{q}$.

We can find an explict 4-cochain $\tilde{\beta}_4 \in C^4(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ such that $\tilde{\alpha}_5 = \delta\tilde{\beta}_4$ where $\tilde{\alpha}_5 = f^*\alpha_5 \in \mathrm{H}^5(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ and f is the map in the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \xrightarrow{f} \mathbb{Z}_n \to 1$. Namely,

$$\tilde{\alpha}_5(h_1, h_2, h_3, h_4, h_5) = \delta\tilde{\beta}_4(h_1, h_2, h_3, h_4, h_5) = \frac{\tilde{\beta}_4(h_2, h_3, h_4, h_5)\tilde{\beta}_4(h_1, h_2h_3, h_4, h_5)\tilde{\beta}_4(h_1, h_2, h_3, h_4h_5)}{\tilde{\beta}_4(h_1h_2, h_3, h_4, h_5)\tilde{\beta}_4(h_1, h_2, h_3h_4, h_5)\tilde{\beta}_4(h_1, h_2, h_3, h_4)} \tag{71}$$

where $h_i = (g_i, k_i) \in \mathbb{Z}_{n^2}$ for $i = 1, 2, 3, 4, 5$ and

$$(g_1, k_1) \cdot (g_2, k_2) = (g_1 + g_2, k_1 + k_2 + [\frac{g_1 + g_2}{n}]). \tag{72}$$

Here,

$$\phi(g_1, g_2) = [\frac{g_1 + g_2}{n}] \tag{73}$$

is the 2-cocycle in $\mathrm{H}^2(\mathbb{Z}_n, \mathbb{Z}_n)$ classifying the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$. Note that for $n = 2$, $[\frac{g_1+g_2}{2}] = g_1g_2 \mod 2$.

Explicitly, the 4-cochain $\tilde{\beta}_4 \in C^4(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ is given by

$$\tilde{\beta}_4(h_1, h_2, h_3, h_4) = \zeta_n^{g_1k_2[\frac{g_3+g_4}{n}]}. \tag{74}$$

In fact, if we write $\tilde{\beta}_4 = \zeta_n^{\tilde{\gamma}_4}$ and $\tilde{\beta}_2 = \zeta_n^{\tilde{\gamma}_2}$, then $\tilde{\gamma}_4 = \tilde{\gamma}_2 \smile \phi$. Then

$$\delta\tilde{\beta}_4(h_1, h_2, h_3, h_4, h_5) = \zeta_n^X. \tag{75}$$

We check that

$$\begin{aligned} X &= g_2k_3[\frac{g_4+g_5}{n}] + g_1(k_2 + k_3 + [\frac{g_2+g_3}{n}])[\frac{g_4+g_5}{n}] + g_1k_2[\frac{g_3 + (g_4+g_5) \mod n}{n}] \\ &\quad -(g_1+g_2)k_3[\frac{g_4+g_5}{n}] - g_1k_2[\frac{(g_3+g_4) \mod n + g_5}{n}] - g_1k_2[\frac{g_3+g_4}{n}] \\ &= g_1[\frac{g_2+g_3}{n}][\frac{g_4+g_5}{n}]. \end{aligned} \tag{76}$$

Here, we have used the fact that

$$[\frac{g_4+g_5}{n}] + [\frac{g_3 + (g_4+g_5) \mod n}{n}] = [\frac{g_3+g_4+g_5}{n}] = [\frac{(g_3+g_4) \mod n + g_5}{n}] + [\frac{g_3+g_4}{n}]. \tag{77}$$

This is in fact the cocycle condition for ϕ. Hence

$$\delta\tilde{\beta}_4(h_1, h_2, h_3, h_4, h_5) = \zeta_n^X = \zeta_n^{g_1[\frac{g_2+g_3}{n}][\frac{g_4+g_5}{n}]} = \alpha_5(g_1, g_2, g_3, g_4, g_5) = \tilde{\alpha}_5(h_1, h_2, h_3, h_4, h_5). \tag{78}$$

Therefore, $\tilde{\alpha}_5 = \delta\tilde{\beta}_4$.

B.3 Any odd $d \geqslant 3$ and any $n \geqslant 2$: Find $\tilde{\beta}_{d-1}$ such that $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$

Explicitly, for odd $d \geqslant 3$, any d-cocycle $\alpha_d \in \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1))$ has the form [55]

$$\alpha_d(g_1, g_2, \ldots, g_d) = \zeta_n^{g_1[\frac{g_2+g_3}{n}][\frac{g_4+g_5}{n}]\cdots[\frac{g_{d-1}+g_d}{n}]} \tag{79}$$

where $g_i \in \mathbb{Z}_n$ for $i = 1, 2, \ldots, d$, ζ_n is an n-th root of unity, and $[\frac{p}{q}]$ denotes the integer part of $\frac{p}{q}$.

We can find an explict $(d-1)$-cochain $\tilde{\beta}_{d-1} \in C^{d-1}(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ such that $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$ where $\tilde{\alpha}_d = f^*\alpha_d \in \mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ and f is the map in the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \xrightarrow{f} \mathbb{Z}_n \to 1$. Namely,

$$\begin{aligned}\tilde{\alpha}_d(h_1, h_2, \ldots, h_d) &= \delta\tilde{\beta}_{d-1}(h_1, h_2, \ldots, h_d)\\ &= \frac{\tilde{\beta}_{d-1}(h_2, h_3, \ldots, h_d)\tilde{\beta}_{d-1}(h_1, h_2h_3, h_4, \ldots, h_d)\cdots\tilde{\beta}_{d-1}(h_1, h_2, \ldots, h_{d-2}, h_{d-1}h_d)}{\tilde{\beta}_{d-1}(h_1h_2, h_3, h_4, \ldots, h_d)\tilde{\beta}_{d-1}(h_1, h_2, h_3h_4, h_5, \ldots, h_d)\cdots\tilde{\beta}_{d-1}(h_1, h_2, \ldots, h_{d-1})}\end{aligned} \tag{80}$$

where $h_i = (g_i, k_i) \in \mathbb{Z}_{n^2}$ for $i = 1, 2, \ldots, d$ and

$$(g_1, k_1) \cdot (g_2, k_2) = (g_1 + g_2, k_1 + k_2 + [\frac{g_1 + g_2}{n}]). \tag{81}$$

Here, $\phi(g_1, g_2) = [\frac{g_1+g_2}{n}]$ is the 2-cocycle in $\mathrm{H}^2(\mathbb{Z}_n, \mathbb{Z}_n)$ classifying the extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$. Note that for $n = 2$, $[\frac{g_1+g_2}{2}] = g_1g_2 \mod 2$.

Explicitly, the $(d-1)$-cochain $\tilde{\beta}_{d-1} \in C^{d-1}(\mathbb{Z}_{n^2}, \mathrm{U}(1))$ is given by

$$\tilde{\beta}_{d-1}(h_1, h_2, \ldots, h_{d-1}) = \zeta_n^{g_1k_2[\frac{g_3+g_4}{n}]\cdots[\frac{g_{d-2}+g_{d-1}}{n}]}. \tag{82}$$

If we write $\tilde{\beta}_{d-1} = \zeta_n^{\tilde{\gamma}_{d-1}}$ and $\tilde{\beta}_2 = \zeta_n^{\tilde{\gamma}_2}$, then

$$\tilde{\gamma}_{d-1} = \tilde{\gamma}_2 \smile \phi^{\frac{d-3}{2}}. \tag{83}$$

If we write $\alpha_d = \zeta_n^{\epsilon_d}$ and $\alpha_3 = \zeta_n^{\epsilon_3}$, then

$$\epsilon_d = \epsilon_3 \smile \phi^{\frac{d-3}{2}}. \tag{84}$$

We have shown that $\delta\tilde{\gamma}_2(h_1, h_2, h_3) = \epsilon_3(g_1, g_2, g_3)$ and we have $\delta\phi = 0$. Hence

$$\begin{aligned}\delta\tilde{\beta}_{d-1}(h_1, h_2, \ldots, h_d) &= \zeta_n^{\delta\tilde{\gamma}_{d-1}}(h_1, h_2, \ldots, h_d)\\ &= \zeta_n^{\delta(\tilde{\gamma}_2\smile\phi^{\frac{d-3}{2}})}(h_1, h_2, \ldots, h_d)\\ &= \zeta_n^{\epsilon_3\smile\phi^{\frac{d-3}{2}}}(g_1, g_2, \ldots, g_d)\\ &= \zeta_n^{\epsilon_d}(g_1, g_2, \ldots, g_d)\\ &= \alpha_d(g_1, g_2, \ldots, g_d)\\ &= \tilde{\alpha}_d(h_1, h_2, \ldots, h_d).\end{aligned} \tag{85}$$

Therefore, $\tilde{\alpha}_d = \delta\tilde{\beta}_{d-1}$.

C dd-bulk/$(d-1)$d-boundary coupled invertible topological field theory/symmetric anomalous gapped TQFT by the symmetry extension $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ for any odd $d \geqslant 3$ and any $n \geqslant 2$

In Appendix A, we prove that the group cohomology class ($\mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1)) \cong \mathbb{Z}_n$) can be canceled by anomalous G-symmetric $K = \mathbb{Z}_n$-gauge $(d-1)$d TQFTs for odd $d \geqslant 3$ and any $n \geqslant 2$, via the appropriate symmetry-extension construction [10] of

$$1 \to K \to G_{\mathrm{Tot}} \to G \to 1. \tag{86}$$

as

$$1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1. \tag{87}$$

More generally and mathematically, in this work, for odd $d \geqslant 3$ and any $n \geqslant 2$, we prove that any group cohomology cocycle

$$\alpha_d \in \mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1)) \cong \mathbb{Z}_n \tag{88}$$

is trivialized by the group extension as eq. (87)'s $1 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 1$ [10].

In Appendix B, we find an explicit $(d-1)$-cochain β_{d-1} that splits the d-cocycle α_d by that extension for odd $d \geqslant 3$ and any $n \geqslant 2$. Namely, $\alpha_d = \delta\beta_{d-1}$ holds when pulling back the quotient $\mathbb{Z}_n$ to the total $\mathbb{Z}_{n^2}$ group, from the cocycle α_d in $\mathrm{H}^d(\mathbb{Z}_n, \mathrm{U}(1))$ to the coboundary

$$\alpha_d = \delta\beta_{d-1} \text{ in } \mathrm{H}^d(\mathbb{Z}_{n^2}, \mathrm{U}(1)). \tag{89}$$

More explicitly, for the α_d given by

$$\alpha_d = \exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^d}(A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}})\big), \tag{90}$$

we have β_{d-1} with $\alpha_d = \delta\beta_{d-1}$, obtained in Appendix B, which suggests a construction of the 5d iTFT on the bulk d-manifold M^d and the $(d-1)$d noninvertible TQFT on the $(d-1)$d boundary $M^{d-1} = \partial M^d$ with dynamical 1-cochain gauge field $a_{\mathbb{Z}_n} \in C^1(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n)$ and $(d-3)$-cochain (dual) gauge field $b_{\mathbb{Z}_n} \in C^{d-3}(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n)$, such that the full 5d/4d coupled path integral is given by

$$\begin{aligned}
&\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^d}(A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}})\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_n}\in C^1(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)\\ b_{\mathbb{Z}_n}\in C^{d-3}(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)}}\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^{d-1}=\partial M^d}(b_{\mathbb{Z}_n}\mathrm{d}a_{\mathbb{Z}_n} - b_{\mathbb{Z}_n}\beta_{(n,n)}A_{\mathbb{Z}_n} - a_{\mathbb{Z}_n}A_{\mathbb{Z}_n}(\beta_{(3,3)}A_{\mathbb{Z}_3})^{\frac{d-3}{2}})\big)\\
=\;&\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^d}(A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}})\big)\cdot\\
&\cdot\sum_{\substack{a_{\mathbb{Z}_n}\in C^1(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)\\ b_{\mathbb{Z}_n}\in C^{d-3}(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)}}\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^{d-1}=\partial M^d}(a_{\mathbb{Z}_n}(\mathrm{d}b_{\mathbb{Z}_n} - A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}}) - b_{\mathbb{Z}_n}\beta_{(n,n)}A_{\mathbb{Z}_n})\big).
\end{aligned} \tag{91}$$

This dd/$(d-1)$d coupled path integral analogously matches the discrete cocycle forms or cochain forms (e.g., [40]) derived in Appendix B, as the d-cocycle

$$\alpha_d(g_1, g_2, \dots, g_d) = \zeta_n^{g_1[\frac{g_2+g_3}{n}]\cdots[\frac{g_{d-1}+g_d}{n}]}$$

and the $(d-1)$-cochain

$$\beta_4(h_1, h_2, \dots, h_{d-1}) = \zeta_n^{g_1 k_2[\frac{g_3+g_4}{n}]\cdots[\frac{g_{d-2}+g_{d-1}}{n}]},$$

where ζ_n is an n-th root of unity such as $\zeta_n = \exp(\frac{2\pi\,\mathrm{i}}{n})$, with variables $g \in \mathbb{Z}_n$ and $k \in \mathbb{Z}_n$.

The dd bulk partition function on a dd manifold with a $(d-1)$d boundary is not gauge-invariant, but the full dd/$(d-1)$d coupled path integral eq. (91) is gauge-invariant under

$$\begin{aligned}
A_{\mathbb{Z}_n} &\mapsto A_{\mathbb{Z}_n} + \mathrm{d}\lambda_{0,\mathbb{Z}_n},\\
a_{\mathbb{Z}_n} &\mapsto a_{\mathbb{Z}_n} + \mathrm{d}\mu_{0,\mathbb{Z}_n},
\end{aligned}$$

$$
\begin{aligned}
&b_{\mathbb{Z}_n} \mapsto b_{\mathbb{Z}_n} + \lambda_{0,\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}} + \mathrm{d}\mu_{d-4,\mathbb{Z}_n},\\
&A_{\mathbb{Z}_n} \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n) = \mathbb{Z}_n,\\
&a_{\mathbb{Z}_n} \in C^1(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n),\\
&b_{\mathbb{Z}_n} \in C^{d-3}(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n),\\
&\lambda_{0,\mathbb{Z}_n} \in C^0(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n),\\
&\mu_{0,\mathbb{Z}_n} \in C^0(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n),\\
&\mu_{d-4,\mathbb{Z}_n} \in C^{d-4}(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n).
\end{aligned} \tag{92}
$$

Below we check that (91) is gauge-invariant under (92). Because $\beta_{(n,n)}\mathrm{d}\lambda_{0,\mathbb{Z}_n} = 0$, $\beta_{(n,n)}A_{\mathbb{Z}_n}$ and $\mathrm{d}b_{\mathbb{Z}_n} - A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}}$ are gauge-invariant under the gauge transformation (92), hence (91) transforms under (92) as

$$
\begin{aligned}
&\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^d}(A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}})\big)\cdot\\
&\cdot \sum_{\substack{a_{\mathbb{Z}_n}\in C^1(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)\\ b_{\mathbb{Z}_n}\in C^{d-3}(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)}} \exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^{d-1}=\partial M^d}(a_{\mathbb{Z}_n}(\mathrm{d}b_{\mathbb{Z}_n} - A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}}) - b_{\mathbb{Z}_n}\beta_{(n,n)}A_{\mathbb{Z}_n})\big)\\
\mapsto\ &\exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^d}((A_{\mathbb{Z}_n}+\mathrm{d}\lambda_{0,\mathbb{Z}_n})(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}})\big)\cdot\\
&\cdot \sum_{\substack{a_{\mathbb{Z}_n}\in C^1(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)\\ b_{\mathbb{Z}_n}\in C^{d-3}(\mathrm{B}\mathbb{Z}_n,\mathbb{Z}_n)}} \exp\big(\mathrm{i}\frac{2\pi}{n}\int_{M^{d-1}=\partial M^d}((a_{\mathbb{Z}_n}+\mathrm{d}\mu_{0,\mathbb{Z}_n})(\mathrm{d}b_{\mathbb{Z}_n} - A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}})\\
&-(b_{\mathbb{Z}_n}+\lambda_{0,\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}}+\mathrm{d}\mu_{d-4,\mathbb{Z}_n})\beta_{(n,n)}A_{\mathbb{Z}_n})\big).
\end{aligned} \tag{93}
$$

Since by the Stokes theorem, we have

$$
\int_{M^{d-1}=\partial M^d}(\mathrm{d}\mu_{0,\mathbb{Z}_n})(\mathrm{d}b_{\mathbb{Z}_n} - A_{\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-3}{2}}) = 0, \tag{94}
$$

$$
\int_{M^{d-1}=\partial M^d}(\mathrm{d}\mu_{d-4,\mathbb{Z}_n})\beta_{(n,n)}A_{\mathbb{Z}_n} = 0, \tag{95}
$$

and

$$
\int_{M^{d-1}=\partial M^d}\lambda_{0,\mathbb{Z}_n}(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}} = \int_{M^d}(\mathrm{d}\lambda_{0,\mathbb{Z}_n})(\beta_{(n,n)}A_{\mathbb{Z}_n})^{\frac{d-1}{2}}, \tag{96}
$$

(91) is gauge-invariant under (92).

D 3+1d Nonperturbative Global Anomaly in $\mathrm{Spin}\times\mathbb{Z}_n$ for integer n with $2\nmid n$ and $3\nmid n$

In this appendix, we explore the 3+1d nonperturbative global anomaly for a Weyl fermion in $\mathrm{Spin}\times\mathbb{Z}_n$ for integer n with $2\nmid n$ and $3\nmid n$, compared with the fact that

$$
\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_n) \cong \tilde{\Omega}_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_n) \cong \mathbb{Z}_n\oplus\mathbb{Z}_n, \quad 2\nmid n, \quad 3\nmid n. \tag{97}
$$

Here $\tilde{\Omega}_5^{\mathrm{SO}}(\mathrm{B}G) := \Omega_5^{\mathrm{SO}}(\mathrm{B}G)/\Omega_5^{\mathrm{SO}}$ is the reduced bordism group, modding out the $\Omega_5^{\mathrm{SO}} = \Omega_5^{\mathrm{SO}}(pt)$.

The perturbative local anomaly of U(1) charge $q = 1$ left-handed Weyl fermion of $\mathrm{Spin} \times \mathrm{U}(1)$ symmetry in 3+1d or 4d is captured by a 5d invertible field theory (iTFT) with the anomaly index $k = 1$:

$$\exp(\mathrm{i}k \int_{M^5} A\frac{c_1^2}{6} - A\frac{p_1}{24}). \tag{98}$$

Now we redefine the U(1) gauge field A as a $\mathbb{Z}_n$ gauge field $A_{\mathbb{Z}_n} \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_n, \mathbb{Z}_n) = \mathbb{Z}_n$ with the following replacement:

$$\begin{aligned} A &\mapsto \frac{2\pi}{n} A_{\mathbb{Z}_n}. \\ c_1 = \frac{\mathrm{d}A}{2\pi} &\mapsto \frac{\mathrm{d}A_{\mathbb{Z}_n}}{n} \equiv \beta_{(n,n)} A_{\mathbb{Z}_n}. \end{aligned} \tag{99}$$

The $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \mapsto \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n \xrightarrow{\cdot m} \mathbb{Z}_{nm} \to \mathbb{Z}_m$. Thus we get the 5d topological invariant of the $\mathrm{Spin} \times \mathbb{Z}_n$ as:

$$\exp\big(\mathrm{i}2\pi k \int_{M^5} (\frac{1}{6n} A_{\mathbb{Z}_n} (\beta_{(n,n)} A_{\mathbb{Z}_n})(\beta_{(n,n)} A_{\mathbb{Z}_n}) - \frac{1}{24n} A_{\mathbb{Z}_n} p_1)\big). \tag{100}$$

Since the anomaly of 4d Weyl fermion with symmetry $\mathrm{Spin} \times \mathbb{Z}_n$ does not contain 2-torsion and 3-torsion for $2 \nmid n$ and $3 \nmid n$ [12, 35–39], we can regard $6 = 2 \cdot 3$ and $24 = 2^3 \cdot 3$ as invertible in $\mathbb{Z}_n$ for $2 \nmid n$ and $3 \nmid n$.

In fact, for $2 \nmid n$ and $3 \nmid n$, there exists an integer x_n such that $x_n = 1 \mod n$ and $24 | x_n$. Then we can rewrite (100) as

$$\exp\big(\mathrm{i}\frac{2\pi k}{n} \int_{M^5} (\frac{x_n}{6} A_{\mathbb{Z}_n} (\beta_{(n,n)} A_{\mathbb{Z}_n})(\beta_{(n,n)} A_{\mathbb{Z}_n}) - \frac{x_n}{24} A_{\mathbb{Z}_n} p_1)\big). \tag{101}$$

Since $\gcd(n, x_n) = 1$, $\gcd(n, \frac{x_n}{6}) = 1$, and $\gcd(n, \frac{x_n}{24}) = 1$, the first term and the second term individually in eq. (101), as well as the combined two terms in eq. (101), all generate a $\mathbb{Z}_n$ class. We expect eq. (101) as a schematic way to write one of the two cobordism invariant generators of $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_n) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$, with $2 \nmid n$ and $3 \nmid n$.

E 3+1d Nonperturbative Global Anomaly in $\mathrm{Spin} \times \mathbb{Z}_{3^r} = \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2 \cdot 3^r}^{\mathrm{F}}$

In this appendix, we explore the 3+1d nonperturbative global anomaly for a Weyl fermion in $\mathrm{Spin} \times \mathbb{Z}_{3^r} = \mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2 \cdot 3^r}^{\mathrm{F}}$, compared with the fact that

$$\begin{aligned} &\Omega_5^{\mathrm{Spin} \times \mathbb{Z}_{3^r}} \cong \Omega_5^{\mathrm{Spin} \times_{\mathbb{Z}_2^{\mathrm{F}}} \mathbb{Z}_{2 \cdot 3^r}} \cong \tilde{\Omega}_5^{\mathrm{SO}}(\mathrm{B}\mathbb{Z}_{3^r}) \\ &= \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}}. \end{aligned} \tag{102}$$

Here $\tilde{\Omega}_5^{\mathrm{SO}}(\mathrm{B}G) := \Omega_5^{\mathrm{SO}}(\mathrm{B}G) / \Omega_5^{\mathrm{SO}}$ is the reduced bordism group, modding out the $\Omega_5^{\mathrm{SO}} = \Omega_5^{\mathrm{SO}}(pt)$.

We also prove that the $k = 3^r$ anomaly of the 4d Weyl fermion with $\mathrm{Spin} \times \mathbb{Z}_{3^r}$ symmetry can be trivialized by a $\mathbb{Z}_3$ extension.

The perturbative local anomaly of U(1) charge $q = 1$ left-handed Weyl fermion of $\mathrm{Spin} \times \mathrm{U}(1)$ symmetry in 3+1d or 4d is captured by a 5d invertible field theory (iTFT) with the anomaly index $k = 1$:

$$\exp(\mathrm{i}k \int_{M^5} A\frac{c_1^2}{6} - A\frac{p_1}{24}). \tag{103}$$

Now we redefine the U(1) gauge field A as a $\mathbb{Z}_{3^r}$ gauge field $A_{\mathbb{Z}_{3^r}} \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}_{3^r}, \mathbb{Z}_{3^r}) = \mathbb{Z}_{3^r}$ with the following replacement:

$$\begin{aligned} A &\mapsto \frac{2\pi}{3^r} A_{\mathbb{Z}_{3^r}}. \\ c_1 = \frac{\mathrm{d}A}{2\pi} &\mapsto \frac{\mathrm{d}A_{\mathbb{Z}_{3^r}}}{3^r} \equiv \beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}}. \end{aligned} \tag{104}$$

The $\beta_{(n,m)} : \mathrm{H}^*(-, \mathbb{Z}_m) \mapsto \mathrm{H}^{*+1}(-, \mathbb{Z}_n)$ is the Bockstein homomorphism associated with the extension $\mathbb{Z}_n \stackrel{\cdot m}{\to} \mathbb{Z}_{nm} \to \mathbb{Z}_m$. Thus we get the 5d topological invariant of the $\mathrm{Spin} \times \mathbb{Z}_{3^r}$ as:

$$\exp\big(\mathrm{i}2\pi k \int_{M^5} (\frac{1}{2 \cdot 3^{r+1}} A_{\mathbb{Z}_{3^r}} (\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})(\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}}) - \frac{1}{8 \cdot 3^{r+1}} A_{\mathbb{Z}_{3^r}} p_1)\big). \tag{105}$$

Since the anomaly of 4d Weyl fermion with symmetry $\mathrm{Spin} \times \mathbb{Z}_{3^r}$ contains only 3-torsion [12, 35–39], we can regard 2 and 8 as invertible in $\mathbb{Z}_{3^{r+1}}$.

In fact, there exists an integer y_r such that $y_r = 1 \mod 3^{r+1}$ and $8|y_r$. Then we can rewrite (105) as

$$\exp\big(\mathrm{i}\frac{2\pi k}{3^{r+1}} \int_{M^5} (\frac{y_r}{2} A_{\mathbb{Z}_{3^r}} (\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})(\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}}) - \frac{y_r}{8} A_{\mathbb{Z}_{3^r}} p_1)\big). \tag{106}$$

Since $\gcd(3^{r+1}, y_r) = 1$, $\gcd(3^{r+1}, \frac{y_r}{2}) = 1$, and $\gcd(3^{r+1}, \frac{y_r}{8}) = 1$, the first term and second term in eq. (106) individually generate a $\mathbb{Z}_{3^r}$ class.

Since $A_{\mathbb{Z}_{3^r}} = A_{\mathbb{Z}_3} \mod 3$ and $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ [33, 34] (see Appendix F for the proof), we have $A_{\mathbb{Z}_{3^r}} p_1 = 0 \mod 3$ and we can rewrite eq. (106) as

$$\exp\big(\mathrm{i}\frac{2\pi k}{3^{r+1}} \int_{M^5} (\frac{y_r}{2} A_{\mathbb{Z}_{3^r}} (\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})(\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}}) - 3 \cdot \frac{y_r}{8} \frac{A_{\mathbb{Z}_{3^r}} p_1}{3})\big). \tag{107}$$

Since $\frac{A_{\mathbb{Z}_{3^r}} p_1}{3}$ generates a $\mathbb{Z}_{3^r}$ class and the $k = 3^r$ anomaly of the 4d Weyl fermion with $\mathrm{Spin} \times \mathbb{Z}_{3^r}$ symmetry is

$$\exp(\mathrm{i}\frac{2\pi}{3} \int_{M^5} \frac{y_r}{2} A_{\mathbb{Z}_{3^r}} (\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})(\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})). \tag{108}$$

Since $A_{\mathbb{Z}_{3^r}} (\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}})(\beta_{(3^r,3^r)} A_{\mathbb{Z}_{3^r}}) = A_{\mathbb{Z}_3} (\beta_{(3,3)} A_{\mathbb{Z}_3})(\beta_{(3,3)} A_{\mathbb{Z}_3}) \mod 3$, this term generates a $\mathbb{Z}_3$ class. Therefore, the combined two terms in eq. (107) generate a $\mathbb{Z}_{3^{r+1}}$ class. We expect eq. (107) as a schematic way to write the first one of the two cobordism invariant generators of $\Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_{3^r}) \cong \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}}$.

By the results in Appendix A, the $k = 3^r$ anomaly eq. (108) of the 4d Weyl fermion with $\mathrm{Spin} \times \mathbb{Z}_{3^r}$ symmetry, namely $k = 3^r \in \mathbb{Z}_{3^{r+1}} \subset \Omega_5^{\mathrm{Spin}}(\mathrm{B}\mathbb{Z}_{3^r}) \cong \mathbb{Z}_{3^{r+1}} \oplus \mathbb{Z}_{3^{r-1}}$, can be trivialized by a $\mathbb{Z}_3$ extension.

F Proof of $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$

In this appendix, we prove $A_{\mathbb{Z}_3} p_1 = 0 \mod 3$ [33, 34] and explain why 3 is special.

Let $P_q^r : \mathrm{H}^i(-, \mathbb{Z}_q) \to \mathrm{H}^{i+2(q-1)r}(-, \mathbb{Z}_q)$ be the mod q Steenrod reduced power where q is an odd prime.

On an oriented n-manifold M^n, by the Poincaré duality, there exists $s_q^r \in \mathrm{H}^{2(q-1)r}(M^n, \mathbb{Z}_q)$ such that

$$P_q^r(x) = s_q^r \smile x, \quad \forall x \in \mathrm{H}^{n-2(q-1)r}(M^n, \mathbb{Z}_q). \tag{109}$$

Let $P_q := \sum_{r=0}^{\infty} P_q^r$ be the mod q total Steenrod reduced power, and $s_q := \sum_{r=0}^{\infty} s_q^r$. We prove the following theorem [56, Theorem 4.3] as its proof is hard to find in the literature.

Theorem F.1. *On an oriented n-manifold M^n, we have*

$$P_q(s_q) = \sum_{j=0}^{\infty} b_{q,j} \mod q \tag{110}$$

where $\sum_{j=0}^{\infty} b_{q,j} = \prod_i (1 + x_i^{q-1})$ and $\pm x_i$ are the Chern roots of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$. In particular, for $q = 3$, $b_{3,j} = p_j$ is the j-th Pontryagin class of M^n. Equivalently, we have

$$\sum_{i+r=j} P_q^i s_q^r = b_{q,j} \mod q. \tag{111}$$

We mimic the proof for $\mathrm{Sq}(v) = w$ given in [57] to prove the above theorem. Here, Sq is the total Steenrod square, v is the total mod 2 Wu class, and w is the total Stiefel-Whitney class.

Proof. Let $b_i \in \mathrm{H}^*(M)$ be a basis, and $b_i^\sharp \in \mathrm{H}^*(M)$ the dual basis such that $\langle b_i \smile b_j^\sharp, [M] \rangle = \delta_{ij}$ where $[M]$ is the fundamental class of M.

Then for all $x \in \mathrm{H}^*(M)$, $x = \sum b_i \langle x \smile b_i^\sharp, [M] \rangle$. Apply this to $x = s_q$, then

$$s_q = \sum b_i \langle s_q \smile b_i^\sharp, [M] \rangle = \sum b_i \langle P_q(b_i^\sharp), [M] \rangle. \tag{112}$$

Therefore,

$$P_q(s_q) = \sum P_q(b_i) \langle P_q(b_i^\sharp), [M] \rangle. \tag{113}$$

Since each Chern root x_i is a degree-2 cohomology class,

$$P_q(x_i) = x_i + x_i^q = (1 + x_i^{q-1}) x_i \mod q. \tag{114}$$

So, by Cartan's formula,

$$P_q(\prod_i x_i) = \prod_i (1 + x_i^{q-1}) \prod_i x_i \mod q. \tag{115}$$

Therefore,

$$P_q(e) = (\sum_{j=0}^{\infty} b_{q,j}) \smile e \mod q \tag{116}$$

where e is the Euler class of M. Equivalently,

$$P_q^j(e) = b_{q,j} \smile e \mod q. \tag{117}$$

Note that $e = \Delta^*(U)$ where $\Delta : M \to M \times M$ is the diagonal map and the diagonal cohomology class

$$U = \sum (-1)^{\dim b_i} b_i \times b_i^\sharp \tag{118}$$

such that

$$U/[M] = \sum (-1)^{\dim b_i} b_i \langle b_i^\sharp, [M] \rangle = 1. \tag{119}$$

This is obtained by applying $x = \sum b_i \langle x \smile b_i^\sharp, [M] \rangle$ to $x = 1$ and noting that the only nonvanishing $\langle b_i^\sharp, [M] \rangle$ occurs when b_i has degree 0, so the sign $(-1)^{\dim b_i}$ disappears. Here, the slant product is defined as $(a \times b)/[M] := a \langle b, [M] \rangle$.

By Cartan's formula[5], we have

$$P_q(U) = \sum (-1)^{\dim b_i} P_q(b_i) \times P_q(b_i^\sharp) \tag{121}$$

and

$$P_q(U)/[M] = \sum (-1)^{\dim b_i} P_q(b_i) \times P_q(b_i^\sharp)/[M] = \sum (-1)^{\dim b_i} P_q(b_i) \langle P_q(b_i^\sharp), [M] \rangle. \tag{122}$$

Since the Steenrod reduced power P_q increases the degree by an even integer, the only nonvanishing $\langle P_q(b_i^\sharp), [M] \rangle$ occurs when $b_i^\sharp$ has even codimension, i.e. b_i has even degree. So the sign $(-1)^{\dim b_i}$ disappears and

$$P_q(U)/[M] = \sum P_q(b_i) \langle P_q(b_i^\sharp), [M] \rangle. \tag{123}$$

On the other hand, since $P_q^j(e) = b_{q,j} \smile e \mod q$ and $e = \Delta^*(U)$, by [57, Theorem 11.3 and Lemma 11.5], we have

$$P_q^j(U) = (b_{q,j} \times 1) \smile U \mod q, \tag{124}$$

hence

$$P_q(U) = ((\sum_{j=0}^{\infty} b_{q,j}) \times 1) \smile U \mod q. \tag{125}$$

Therefore,

$$P_q(U)/[M] = (((\sum_{j=0}^{\infty} b_{q,j}) \times 1) \smile U)/[M] = (\sum_{j=0}^{\infty} b_{q,j}) \smile (U/[M]) = (\sum_{j=0}^{\infty} b_{q,j}) \smile 1 = \sum_{j=0}^{\infty} b_{q,j} \mod q. \tag{126}$$

So

$$P_q(s_q) = \sum P_q(b_i) \langle P_q(b_i^\sharp), [M] \rangle = \sum_{j=0}^{\infty} b_{q,j} \mod q. \tag{127}$$

□

By the above theorem, we have $p_1 = s_3^1 \mod 3$, hence

$$A_{\mathbb{Z}_3} p_1 = A_{\mathbb{Z}_3} s_3^1 = P_3^1(A_{\mathbb{Z}_3}) = 0 \mod 3 \tag{128}$$

since $\deg(A_{\mathbb{Z}_3}) = 1$ and $P_3^r(x) = 0$ for $\deg(x) < 2r$.

For odd prime $q > 3$, $\deg(s_q^1) = 2(q-1) > 4 = \deg(p_1)$, so there is no similar result for odd prime $q > 3$.

[5]Cartan's formula usually applies to the cup product. The cross product $a \times b$ is defined as $\pi_1^* a \smile \pi_2^* b$ where π_i is the projection from $M \times M$ onto its i-th factor. Then

$$P_q(a \times b) = P_q(\pi_1^* a \smile \pi_2^* b) = P_q(\pi_1^* a) \smile P_q(\pi_2^* b) = \pi_1^* P_q(a) \smile \pi_2^* P_q(b) = P_q(a) \times P_q(b). \tag{120}$$

References

[1] H. Harari, *Three generations of quarks and leptons*, in *Presented at 5th Intern. Conf. on Exptl. Meson Spectry*, pp. 29–30, July, 1977.

[2] J. Wang, *Family Puzzle, Framing Topology, $c_- = 24$ and 3(E8)$_1$ Conformal Field Theories: 48/16 = 45/15 = 24/8 =3*, arXiv:2312.14928.

[3] J. Wang, *Topological Quantum Dark Matter via Global Anomaly Cancellation*, arXiv:2502.21319.

[4] X.-G. Wen, *Zoo of quantum-topological phases of matter*, Rev. Mod. Phys. **89** (2017) 041004, [arXiv:1610.03911].

[5] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121** (1989) 351–399.

[6] M. Atiyah, *On framings of 3-manifolds*, *Topology* **29** (1990) 1–7.

[7] J. McNamara and C. Vafa, *Cobordism Classes and the Swampland*, arXiv:1909.10355.

[8] S. Koren, *A Note on Proton Stability in the Standard Model*, Universe **8** (2022) 308, [arXiv:2204.01741].

[9] J. Wang, Z. Wan and Y.-Z. You, *Proton stability: From the standard model to beyond grand unification*, Phys. Rev. D **106** (2022) 025016, [arXiv:2204.08393].

[10] J. Wang, X.-G. Wen and E. Witten, *Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions*, Phys. Rev. **X8** (2018) 031048, [arXiv:1705.06728].

[11] D. Tong, *Line Operators in the Standard Model*, JHEP **07** (2017) 104, [arXiv:1705.01853].

[12] I. Garcia-Etxebarria and M. Montero, *Dai-Freed anomalies in particle physics*, JHEP **08** (2019) 003, [arXiv:1808.00009].

[13] B. A. Dobrescu and E. Poppitz, *Number of fermion generations derived from anomaly cancellation*, Phys. Rev. Lett. **87** (2001) 031801, [arXiv:hep-ph/0102010].

[14] J. Davighi and N. Lohitsiri, *Omega vs. pi, and 6d anomaly cancellation*, JHEP **05** (2021) 267, [arXiv:2012.11693].

[15] Z. Wan, J. Wang and et al., *Higher anomalies, higher symmetries, and cobordisms IV: Beyond Standard Models, Grand Unification, and Ultra Unification (202n)*, arXiv:2xxx.xxxxx.

[16] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, Geom. Topol. **25** (2021) 1165–1330, [arXiv:1604.06527].

[17] Z. Wan and J. Wang, *Anomalous (3+1)d Fermionic Topological Quantum Field Theories via Symmetry Extension*, arXiv:2512.25038.

[18] T. Eguchi and P. G. O. Freund, *Quantum Gravity and World Topology*, Phys. Rev. Lett. **37** (1976) 1251.

[19] L. Alvarez-Gaume and E. Witten, *Gravitational Anomalies*, Nucl. Phys. **B234** (1984) 269.

[20] L. Alvarez-Gaume and P. H. Ginsparg, *The Structure of Gauge and Gravitational Anomalies*, Annals Phys. **161** (1985) 423.

[21] J. Wang, *Anomaly and Cobordism Constraints Beyond the Standard Model: Topological Force*, arXiv:2006.16996.

[22] J. Wang, *Anomaly and Cobordism Constraints Beyond Grand Unification: Energy Hierarchy*, arXiv:2008.06499.

[23] J. Wang, *Ultra Unification*, *Phys. Rev. D* **103** (2021) 105024, [arXiv:2012.15860].

[24] J. Wang, Z. Wan and Y.-Z. You, *Cobordism and deformation class of the standard model*, *Phys. Rev. D* **106** (2022) L041701, [arXiv:2112.14765].

[25] P. Putrov and J. Wang, *Categorical Symmetry of the Standard Model from Gravitational Anomaly*, *Phys. Rev. D* **110** (2024) 125028, [arXiv:2302.14862].

[26] J. Wang, *Topological Leptogenesis*, 12, 2024, arXiv:2501.00607.

[27] S. L. Adler, *Axial vector vertex in spinor electrodynamics*, *Phys. Rev.* **177** (1969) 2426–2438.

[28] J. S. Bell and R. Jackiw, *A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ model*, *Nuovo Cim.* **A60** (1969) 47–61.

[29] A. Belavin, A. M. Polyakov, A. Schwartz and Y. Tyupkin, *Pseudoparticle Solutions of the Yang-Mills Equations*, *Phys. Lett. B* **59** (1975) 85–87.

[30] G. 't Hooft, *Symmetry Breaking Through Bell-Jackiw Anomalies*, *Phys. Rev. Lett.* **37** (1976) 8–11.

[31] R. Jackiw and C. Rebbi, *Vacuum Periodicity in a Yang-Mills Quantum Theory*, *Phys. Rev. Lett.* **37** (1976) 172–175.

[32] C. G. Callan, Jr., R. F. Dashen and D. J. Gross, *The Structure of the Gauge Theory Vacuum*, *Phys. Lett. B* **63** (1976) 334–340.

[33] Y. Tomonaga, *Mod 3 pontryagin class and indecomposability of differentiable manifolds*, *Tohoku Mathematical Journal, Second Series* **16** (1964) 252–256.

[34] Y. Tomonaga, *Some properties of pontryagin classes mod 3*, *Transactions of the American Mathematical Society* **117** (1965) 37–48.

[35] A. Kapustin, R. Thorngren, A. Turzillo and Z. Wang, *Fermionic Symmetry Protected Topological Phases and Cobordisms*, *JHEP* **12** (2015) 052, [arXiv:1406.7329].

[36] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, *arXiv e-prints* (Apr, 2016) arXiv:1604.06527, [arXiv:1604.06527].

[37] C.-T. Hsieh, *Discrete gauge anomalies revisited*, arXiv:1808.02881.

[38] M. Guo, K. Ohmori, P. Putrov, Z. Wan and J. Wang, *Fermionic Finite-Group Gauge Theories and Interacting Symmetric/Crystalline Orders via Cobordisms*, *Commun. Math. Phys.* **376** (2020) 1073, [arXiv:1812.11959].

[39] Z. Wan, *Anomaly of 4d Weyl fermion with discrete symmetries*, arXiv:2506.19710.

[40] J. C. Wang, Z.-C. Gu and X.-G. Wen, *Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond*, *Phys. Rev. Lett.* **114** (2015) 031601, [arXiv:1405.7689].

[41] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo et al., *Tunneling Topological Vacua via Extended Operators: (Spin-)TQFT Spectra and Boundary Deconfinement in Various Dimensions*, *PTEP* **2018** (2018) 053A01, [arXiv:1801.05416].

[42] M. Cheng, J. Wang and X. Yang, *(3+1)d boundary topological order of (4+1)d fermionic SPT state*, arXiv:2411.05786.

[43] X. Yang and M. Cheng, *Gapped boundary of (4+1)d beyond-cohomology bosonic SPT phase*, *Phys. Rev. B* **110** (2024) 045137, [arXiv:2303.00719].

[44] T. D. Décoppet and M. Yu, *The Classification of 3+1d Symmetry Enriched Topological Order*, arXiv:2509.10603.

[45] A. Debray, W. Ye and M. Yu, *How to Build Anomalous (3+1)d Topological Quantum Field Theories*, arXiv:2510.24834.

[46] C. Cordova and K. Ohmori, *Anomaly Constraints on Gapped Phases with Discrete Chiral Symmetry*, *Phys. Rev. D* **102** (2020) 025011, [arXiv:1912.13069].

[47] T. D. Brennan and K. Intriligator, *Anomalies of 4d $Spin_G$ theories*, *JHEP* **07** (2024) 157, [arXiv:2312.04756].

[48] J. Davighi, B. Gripaios and N. Lohitsiri, *Global anomalies in the Standard Model(s) and Beyond*, *JHEP* **07** (2020) 232, [arXiv:1910.11277].

[49] Z. Wan and J. Wang, *Beyond Standard Models and Grand Unifications: Anomalies, Topological Terms, and Dynamical Constraints via Cobordisms*, *JHEP* **07** (2020) 062, [arXiv:1910.14668].

[50] O. Bar and U. J. Wiese, *Can one see the number of colors?*, *Nucl. Phys. B* **609** (2001) 225–246, [arXiv:hep-ph/0105258].

[51] R. Shrock, *Implications of anomaly constraints in the N(c) extended standard model*, *Phys. Rev. D* **53** (1996) 6465–6476, [arXiv:hep-ph/9512430].

[52] E. Witten, *An SU(2) Anomaly*, *Phys. Lett.* **117B** (1982) 324–328.

[53] M. Montero, C. Vafa and I. Valenzuela, *Neutrinos, B-L Symmetry and the Dark Dimension*, arXiv:2512.09052.

[54] A. Hatcher, *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[55] H.-L. Huang, Z. Wan and Y. Ye, *Explicit cocycle formulas on finite abelian groups with applications to braided linear Gr-categories and Dijkgraaf–Witten invariants*, *Proc. Roy. Soc. Edinburgh A* **150** (2020) 1937–1964, [arXiv:1703.03266].

[56] F. Hirzebruch, "On steenrod reduced powers in oriented manifolds." Princeton University, 1953.

[57] J. W. Milnor and J. D. Stasheff, *Characteristic classes.* No. 76. Princeton university press, 1974.