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PROPERTIES OF THE RECURSIVE DIVISOR FUNCTION AND THE NUMBER OF ORDERED FACTORIZATIONS

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ABSTRACT. We recently introduced the recursive divisor function $\kappa_x(n)$, a recursive analogue of the usual divisor function. Here we calculate its Dirichlet series, which is $\zeta(s-x)/(2-\zeta(s))$. We show that $\kappa_x(n)$ is related to the ordinary divisor function by $\kappa_x * \sigma_y = \kappa_y * \sigma_x$, where * denotes the Dirichlet convolution. Using this, we derive several identities relating κ_x and some standard arithmetic functions. We also clarify the relation between κ_0 and the much-studied number of ordered factorizations $K(n)$, namely, $\kappa_0 = \mathbf{1} * K$.

Several arithmetic functions, such as the divisor function and the Euler totient function, play a fundamental role in our understanding of the theory of numbers. They are explicitly defined, in the sense that values of the function are not defined in terms of prior values of the function. However, it is possible to write down meaningful recursive arithmetic functions. We consider two of them in this paper, and show that they are intimately related to a number of standard arithmetic functions.

We recently introduced and studied the recursive divisor function [1, 2]:

$$\kappa_x(n) = n^x + \sum_{d \lfloor n} \kappa_x(d), \tag{1}$$

where $m \lfloor n$ means $m|n$ and $m < n$. It is the recursive analogue of the usual divisor function,

$$\sigma_x(n) = \sum_{d|n} d^x.$$

For example, $\kappa_0(4) = 1+\kappa_0(1)+\kappa_0(2) = 4$ and $\kappa_1(6) = 6+\kappa_1(1)+\kappa_1(2)+\kappa_1(3) = 14$. The first 12 values of κ_0 (A067824 [3]) and κ_1 (A330575 [3]) are shown is Table 1.

While the function $\kappa_x(n)$ has received little attention [1, 2], a related but simpler recursive arithmetic function has been studied for 90 years, namely, the number $K(n)$ of ordered factorizations into integers greater than 1 [4, 5, 6, 7, 8, 9]. It is defined as

$$K(n) = \varepsilon(n) + \sum_{d \lfloor n} K(d), \tag{2}$$

where $\varepsilon(n)$ is 1 for $n = 1$ but zero otherwise. For example, $K(8) = 4$ because 8 is the product of integers greater than one in four ways: $8 = 4 \cdot 2 = 2 \cdot 4 = 2 \cdot 2 \cdot 2$. The first 12 values of K (A074206 [3]) are shown is Table 1.

In what follows, we denote the Dirichlet convolution of two arithmetic functions f and g as $f * g$, and we denote the Dirichlet series of f as $\widetilde{f}$. The various arithmetic functions that we use in this paper are summarized in Table 1.

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1. Statement of results

We prove the following three theorems.

Theorem 1. *The Dirichlet series $\widetilde{\kappa}_x$ for the recursive divisor function $\kappa_x(n)$ is*

$$\widetilde{\kappa}_x = \sum_{n=1}^{\infty} \frac{\kappa_x(n)}{n^s} = \frac{\zeta(s-x)}{2-\zeta(s)},$$

where ζ is the Riemann zeta function.

Theorem 2. *The recursive divisor function κ_x satisfies the following:*

(3)	$\kappa_x * \sigma_y = \kappa_y * \sigma_x$	κ-σ exchange symmetry
(4)	$\kappa_x = (\mathrm{id}_x + \mathbf{1} * \kappa_x)/2$	Definition of κ_x
(5)	$\kappa_x = \dfrac{\mathrm{id}_x}{2} + \dfrac{\mathbf{1} * \mathrm{id}_x}{2^2} + \dfrac{\mathbf{1} * \mathbf{1} * \mathrm{id}_x}{2^3} + \dots$	Series representation of κ_x
(6)	$\kappa_x = J_x * \kappa_0$	Relation between κ_x and κ_0
(7)	$\kappa_x^{-1} = J_x^{-1} * (2\,\mu - \varepsilon)$	Inverse of κ_x
(8)	$\sigma_x = \kappa_x * (2\,\mathbf{1} - d)$	Relation between κ_x and σ_x.

Note the special cases: $\kappa_1 = \phi * \kappa_0$; $\kappa_0^{-1} = 2\mu - \varepsilon$; and $\kappa_0 = \frac{\mathbf{1}}{2} + \frac{\mathbf{1}*\mathbf{1}}{2^2} + \frac{\mathbf{1}*\mathbf{1}*\mathbf{1}}{2^3} + \dots$.

Theorem 3. *The number of recursive divisors is related to the number of ordered factorizations by $\kappa_0 = \mathbf{1} * K$, that is,*

$$\kappa_0(n) = \sum_{d|n} K(d). \tag{9}$$

Furthermore, K satisfies the following:

(10)	$K = (\varepsilon + \mathbf{1} * K)/2$	Definition of K
(11)	$K = \dfrac{\varepsilon}{2} + \dfrac{\mathbf{1}}{2^2} + \dfrac{\mathbf{1} * \mathbf{1}}{2^3} + \dots$	Series representation of K
(12)	$\kappa_x = \mathrm{id}_x * K$	Relation between K and κ_x
(13)	$K^{-1} = 2\,\varepsilon - \mathbf{1}$	Inverse of K.

2. Proof of Theorem 1

It is convenient to rewrite (1) as

$$2\,\kappa_x(n) = n^x + \sum_{d|n} \kappa_x(d). \tag{14}$$

Dividing by n^s and summing over n,

$$2\,\widetilde{\kappa}_x = 2 \sum_{n=1}^{\infty} \frac{\kappa_x(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n^x}{n^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \kappa_x(d).$$

Swapping the order of summation,

$$2\,\widetilde{\kappa}_x = \zeta(s-x) + \sum_{d=1}^{\infty} \kappa_x(d) \sum_{n:\, d|n} \frac{1}{n^s}$$

$$
\begin{aligned}
&= \zeta(s-x) + \sum_{d=1}^{\infty} \kappa_x(d) \sum_{n=1}^{\infty} \frac{1}{(dn)^s} \\
&= \zeta(s-x) + \sum_{d=1}^{\infty} \frac{\kappa_x(d)}{d^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \zeta(s-x) + \widetilde{\kappa}_x \, \zeta(s).
\end{aligned}
$$

From this, we arrive at Theorem 1:

$$\widetilde{\kappa}_x = \frac{\zeta(s-x)}{2-\zeta(s)}. \quad \square$$

3. Proof of Theorem 2

In what follows, we use the standard identities $\mathbf{1} * \mu = \varepsilon$, $\mathbf{1} * J_x = \mathrm{id}_x$ and $\mathbf{1} * \mathrm{id}_x = \sigma_x$.

Since $\widetilde{\sigma}_x / \widetilde{\kappa}_x = \zeta(s)(2-\zeta(s))$ is independent of x, $\widetilde{\kappa}_x \, \widetilde{\sigma}_y = \widetilde{\kappa}_y \, \widetilde{\sigma}_x$. The Dirichlet convolutions of κ_x and σ_x must follow the analogous relation, so we arrive at (3):

$$\kappa_x * \sigma_y = \kappa_y * \sigma_x. \quad \square$$

We can immediately rewrite (14) in the form of (4),

$$\kappa_x = (\mathrm{id}_x + 1 * \kappa_x)/2. \quad \square$$

		Dirichlet series	*Values from $n=1$ to $n=12$*											
ε	Identity $\lfloor 1/n \rfloor$	1	1	0	0	0	0	0	0	0	0	0	0	0
μ	Möbius function	$1/\zeta(s)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0
$\mathbf{1}$	Constant function $1 \equiv \mathrm{id}_0$	$\zeta(s)$	1	1	1	1	1	1	1	1	1	1	1	1
id_1	1st power function n	$\zeta(s-1)$	1	2	3	4	5	6	7	8	9	10	11	12
ϕ	Euler totient function $\equiv J_1$	$\dfrac{\zeta(s-1)}{\zeta(s)}$	1	1	2	2	4	2	6	4	6	4	10	4
d	No. of divisors $\equiv \sigma_0$	$\zeta^2(s)$	1	2	2	3	2	4	2	4	3	4	2	6
σ	Sum of divisors $\equiv \sigma_1$	$\zeta(s)\zeta(s-1)$	1	3	4	7	6	12	8	15	13	18	12	28
κ_0	No. of recursive divisors	$\dfrac{\zeta(s)}{2-\zeta(s)}$	1	2	2	4	2	6	2	8	4	6	2	16
κ_1	Sum of recursive divisors	$\dfrac{\zeta(s-1)}{2-\zeta(s)}$	1	3	4	8	6	14	8	20	14	20	12	42
κ_0^{-1}	Inverse of κ_0	$\dfrac{2-\zeta(s)}{\zeta(s)}$	1	-2	-2	0	-2	2	-2	0	0	2	-2	0
K	No. of ordered factorizations	$1/(2-\zeta(s))$	1	1	1	2	1	3	1	4	2	3	1	8
K^{-1}	Inverse of K	$2-\zeta(s)$	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
σ_x	Sum of xth power of divisors	$\zeta(s)\zeta(s-x)$	$1,\ 2^x+1,\ 3^x+1,\ 4^x+2^x+1, \ldots$											
id_x	xth power function n^x	$\zeta(s-x)$	$1,\ 2^x,\ 3^x,\ 4^x, \ldots$											
J_x	Jordan's totient function	$\dfrac{\zeta(s-x)}{\zeta(s)}$	$1,\ 2^x-1,\ 3^x-1, 4^x-2^x, \ldots$											
κ_x	Sum of xth power of recursive divisors	$\dfrac{\zeta(s-x)}{2-\zeta(s)}$	$1,\ 2^x+1,\ 3^x+1,\ 4^x+2^x+2, \ldots$											

TABLE 1. For each of the arithmetic functions used in this paper, we give its Dirichlet series and the first 12 terms of its sequence (four terms when x is not specified).

Iterating this recursive definition gives

$$\kappa_x = \frac{\mathrm{id}_x}{2} + \frac{\mathbf{1} * \mathrm{id}_x}{4} + \frac{\mathbf{1} * \mathbf{1} * \kappa_x}{4},$$

and so on, leading to the infinite series (5):

$$\kappa_x = \frac{\mathrm{id}_x}{2} + \frac{\mathbf{1} * \mathrm{id}_x}{2^2} + \frac{\mathbf{1} * \mathbf{1} * \mathrm{id}_x}{2^3} + \dots . \quad \square$$

Since $\sigma_x = \mathbf{1} * \mathbf{1} * J_x$, from (3) we have $\kappa_x * J_y = \kappa_y * J_x$. Setting $y = 0$, and since $J_0 = \varepsilon$, we have (6),

$$\kappa_x = J_x * \kappa_0. \quad \square$$

Convolving of (4) by κ_x^{-1} and solving for κ_x^{-1}, we have $\kappa_x^{-1} = \mathrm{id}_x^{-1} * (2\varepsilon - \mathbf{1})$. Since $\mathrm{id}_x = \mathbf{1} * J_x$, we have $\mathrm{id}_x^{-1} = \mu * J_x^{-1}$, and we arrive at (7),

$$\kappa_x^{-1} = J_x^{-1} * (2\mu - \varepsilon). \quad \square$$

From (7), $J_x = \kappa_x * (2\mu - \varepsilon)$. Convolving with $\mathbf{1} * \mathbf{1}$, we find (8),

$$\sigma_x = \kappa_x * (2\,\mathbf{1} - d). \quad \square$$

4. Proof of Theorem 3

Rewriting (2) as

$$2K(n) = \varepsilon(n) + \sum_{d|n} K(d),$$

we immediately arrive at (10),

$$K = (\varepsilon + \mathbf{1} * K)/2. \quad \square$$

Convolving (4) with $\mathbf{1}$ yields $\mathbf{1} * K = (\mathbf{1} + \mathbf{1} * \mathbf{1} * K)/2$. Replacing $1 * K$ with κ_0, we recover the definition of κ_0: $\kappa_0 = (\mathbf{1} + \mathbf{1} * \kappa_0)/2$. Thus we have established (9),

$$\kappa_0 = \mathbf{1} * K. \quad \square$$

Setting $x = 0$ in (5), and convolving with μ, we have (11),

$$K = \frac{\varepsilon}{2} + \frac{\mathbf{1}}{2^2} + \frac{\mathbf{1} * \mathbf{1}}{2^3} + \dots . \quad \square$$

Substituting $\kappa_0 = \mathbf{1} * K$ into (6), we have (12),

$$\kappa_x = \mathrm{id}_x * K. \quad \square$$

Inverting $K = \mu * \kappa_0$ gives $K^{-1} = \mathbf{1} * \kappa_0^{-1}$. Inserting into this $\kappa_0^{-1} = 2\mu - \varepsilon$ from (7) gives (13),

$$K^{-1} = 2\varepsilon - \mathbf{1}. \quad \square$$

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