Eur. Phys. J. Special Topics **225**, 2017–2023 (2016)

DOI: 10.1140/epjst/e2016-60001-7

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Mitigating cascades in sandpile models: an immunization strategy for systemic risk?

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Received 2 January 2016 / Received in final form 14 March 2016
Published online 26 October 2016

Abstract. We use a simple model of distress propagation (the sandpile model) to show how financial systems are naturally subject to the risk of systemic failures. Taking into account possible network structures among financial institutions, we investigate if simple policies can limit financial distress propagation to avoid system-wide crises, i.e. to dampen systemic risk. We therefore compare different immunization policies (i.e. targeted helps to financial institutions) and find that the information coming from the network topology allows to mitigate systemic cascades by targeting just few institutions.

1 Introduction

Cascade and contagion processes on networks are of central importance in a globalized financial world [1]. The main focus is on systemic risk, i.e. on the risk that a large portion of the financial system fails. Understanding the importance of nodes respect to systemic risk has been a productive topic in the recent years [2,3] since it is crucial in order to implement immunization policies that avoid or at least mitigate large financial crisis. To measure systemic risk with an accuracy sufficient for studying and implementing such policies, requires an accurate characterization not only of the financial network structure, but also a detailed knowledge of the balance sheet of financial institution; on the other hand, even partial information allows to reconstruct the topology a financial network [4].

While most detailed models look at extreme events like bailouts, in the spirit of statistical mechanics model we will look at the Bak, Tang and Wiesenfeld (BTW) model [5] as a simplified description of the propagation of the financial distress level [6]. The BTW model has a long history [7] to understand self-organized critical states of sandpile automaton models. The mean field theory [8] yelds the same critical exponents of the Bethe lattice [9]; in general, mean field theories for such models

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are akin to the theory of branching processes [10]. The first studies of the dynamics of BTW cascades on a random graph preceeds the blossoming of complex networks' studies and the consequent understanding of the dynamics on scale-free networks [11]. The BTW model has been proposed to explain self-organized criticality in a network of economic agents with finite consumption [12], to describe aggregate gluctuations from independent sectoral shocks [13], to underly economic fluctuations [14] and also the dynamics of market economies [15] or to explain the power law distribution of price drops of real stocks [16].

While the influence of the graph topology on the distress dynamics has already been studied in [6], in this paper we look at the possibility of mitigating distress cascades via financial immunization policies. In Sect. 2 we explain the model studied; in Sect. 3 we study our model via numerical simulation and finally in Sect. 4 we discuss the results and the possible extensions of our work.

2 Model

We model financial institutions as nodes of a network of liabilities, i.e. a network in which there is a link among two nodes a and b whenever institution a owns contractual obligations to deliver cash or similar to institution b. If a financial institution a fails, all the institutions to whom a owns suffer losses; such losses can eventually cause other failures and propagate, sometimes even causing system wide cascades. Such situation is referred to as financial contagion [17] and algorithms to assess the systemic risk of a financial network have been developed [2]. Such metrics of systemic risk are very important to perform stress-tests that allow to understand what happens in a financial system as a consequence of a shock. On the other hand, we want to capture the dynamics in absence of major stress; we will show that even with a simplified view of the financial institution behaviour, the system is driven toward self-organized criticality.

While in case of major failures there are algorithms to ensure a single clearing mechanism [18], financial institution normally release distress BEFORE defaulting. In fact, distress comes from taking risk and is hence endogenous in financial systems. Obviously, each institution has its own level of acceptable distress; when the distress exceeds such threshold, it can be released on neighbour (linked) institutions via their liabilities. Hence, in a zero$^{\text{th}}$ order model we can imagine that the distress threshold is proportional to the institution size that we will also assume to be proportional to the number of links. Furthermore, we will assume that when an institution distress rises above its threshold, it gets uniformly distributed among all its neighbours. We will mimic such process on a network via a BTW model.

2.1 BTW Model

We consider a network $G = (V, E)$ where the nodes $i \in V$ represent financial institutions and the edges $(i, j) \in E = V \times V$ are present among linked institutions. We suppose G to be connected and indicate with k_i the degree of node i, i.e. the number of neighbours (adjacent nodes) of i. In the BTW model, to each node corresponds a threshold that we will assume to be equal to its degree k_i. Also, to each node it is associated a dynamical variable s_i (the distress field) that takes integer values. The sites where $s_i \geq k_i$ are called critical sites, and the dynamics assumes that two strictly separated time-scales exists: on the slow time-scale, distress is added at random to the system until some sites are critical; on the fast time-scale, distress

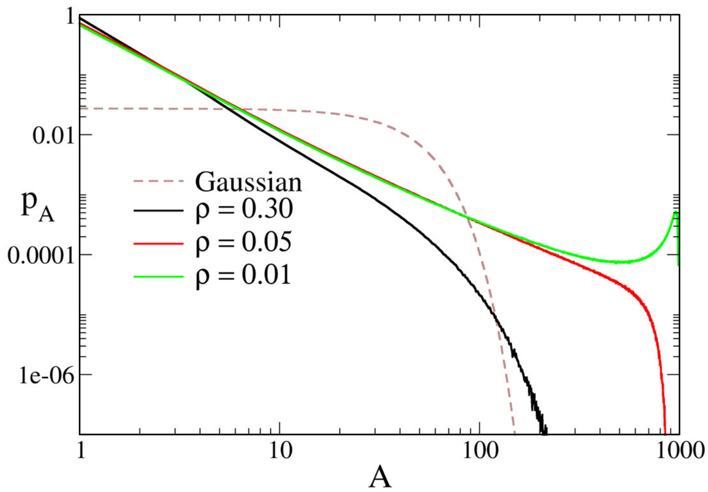

Fig. 1. Cascades will have an approximate power law distribution in a finite system. As an example, we plot the pdf p_A of the cascade area A (i.e. the number of nodes involved in a cascade) as a function of the dissipation ρ (i.e. the probability that distress is dissipated during a cascade) for a network of $|V| = 1000$ nodes. When dissipation is very high (see $\rho = 0.30$), the number of nodes involved in a cascade is much lower than the system size. Lowering the dissipation, the maximum cascade size grows up to the system size and the pdf is approximately power law (see $\rho = 0.05$). For too small dissipation, a large portion of the cascades becomes system wide generating a peak in the pdf at $A = |V|$ (curve with $\rho = 0.01$). This is due to the fact that to each value of the dissipation ρ corresponds a cascade cutoff size ξ; obviously, when considering a system of size $|V| \gg \xi(\rho)$, the peak in the cascade's pdf at $A = |V|$ disappears.

is released by distributing one unit of distress to each of the critical sites' neighbour. In the classical BTW model on a lattice, distress is absorbed when it reaches the lattice borders; since on a network there is no natural boundary, we will consider two separate cases: in the quenched case, a fraction ρ of nodes are assigned to the boundary (i.e. they absorb distress); in the *annealed* case, at each step each unit of distress that gets redistributed from critical sites has a probability ρ of being absorbed.

Hence, the BTW model is an out of equilibrium model at stationarity: the slow injection of distress is interspersed with fast relaxation events that starts when some sites become critical, consist of cascades redistributing distress and end when there are no more critical sites. By measuring either the size S of such cascades (i.e. the number of distress units redistributed during an avalanche) or the area A of a cascade (i.e. the number of nodes involved in a cascade), one finds that for vanishing dissipation ρ the probability distribution function (pdf) of either A or S become power law like in a system at the critical state (Fig. 1); historically, this is the reason why such systems have been said to show "self organised criticality". Notice that, according to the value of the dissipation ρ, the pdf of the avalanches for a finite system can be very different from the critical one.

Since we are dealing with finite size system, it is very important to understand how the pdf of the cascades scales with the dissipation ρ. The first observation is that dissipation introduces a natural cut-off to the cascades; in the following, we will concentrate on the cascade size S (i.e. the number of sites that have become critical during a cascade) and define ξ to be the limiting cut-off size of the cascades. In other word, cascades of size $S > \xi$ have a vanishing probability to be observed. Simulation studies show that cascades' pdf has a characteristic scaling form

$$p_S = S^{-\tau} f\left(S/\xi\right) \tag{1}$$

where $f(x) \approx 1$ for $x << 1$ and $f(x) \to 0$ sharply for $x > 1$. In mean field approximation, the power law exponent can be calculated to be $\tau = 3/2$ and the cut-off function f to be the exponential function. In the case of the BTW model on a random graph, it is possible to perform analytical calculations under the approximation that the system is locally tree-like [19]; in such a case, mean-field results are recovered. For a random graph, also the cut-off size can be calculated to be [19]

$$\xi \propto \rho^{-2} \tag{2}$$

in the case of annealed dissipation.

3 Results

We simulate the cascading processes on Erdos-Renyi random graphs of size $|V| =$ 10000 nodes; to check that fluctuations in the network structure do not influence our results, we have performed such simulations on 10 statistically independent network realization. The algorithm describing the BTW model on a graph is described in alg. 1.

Algorithm 1 Pseudo-code description of the BTW algorithm.

```
loop
  {slow part of the dynamics:}
  while no site is critical do
    add a unit of distress on a randomly chosen site
  end while
  {fast part of the dynamics:}
  repeat
    distribute the distress of critical sites on their neighbours
    adsorb distress on boundaries
  until no site is critical
end loop
```

We first check that cascades evolve in a tree-like fashion; if this is true, we expect to observe $A \approx S$ since if the number of loops are irrelevant then there is a vanishing probability that a site become critical more than once during a cascade. Analogously, we expect also that the the quantity of distress G distributed in a cascade to be proportional to S. We observe such behaviour in Fig. 2.

The functional form Eq. (1) for the cascades' size pdf for an *annealed* BTW model on a random graph implies that $S^{3/2} p_S$ is a decaying exponential in S/ξ; in Fig. 3 we show that this is the case for several values of the dissipation ρ. Hence, by fitting such exponentials, we can recover the observed dependence of ξ from ρ.

We compare such behaviour with *quenched* cases in which dissipation occurs on a fraction ρ of fixed lattice sites, i.e. distress gets absorbed when reaching such sites. In particular, we consider the case where sites are chosen at random and the case where sites are chosen in decreasing order of degree. In both cases we still observe that $S^{3/2} p_S$ has the form a decaying exponential in S/ξ; hence, we can use exponential fitting to estimate the dependence of ξ from ρ also in the quenched cases.

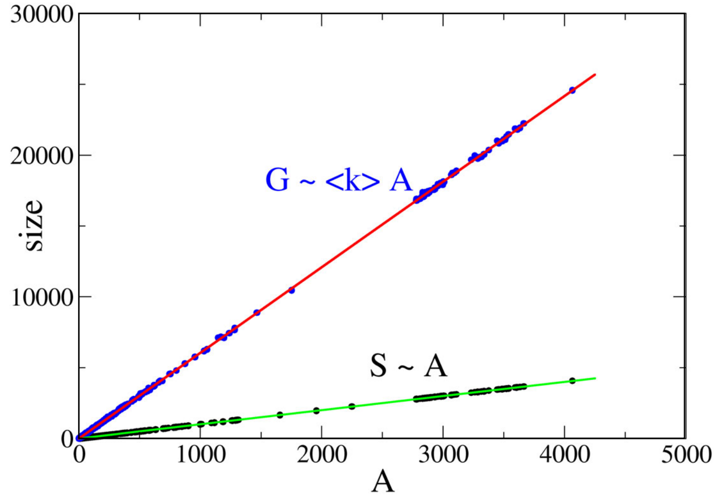

Fig. 2. Numerical checks of the tree-like nature of cascades on random graphs. The line $S \approx A$ shows that the number of sites A affected by the cascade approximatively equals the number of sites S that become critical during a cascade, i.e. each sites becomes critical just once. On the same footing, the amount of distress G distributed by critical sites is approximated by the area A times the average degree $\langle k \rangle$ of the network.

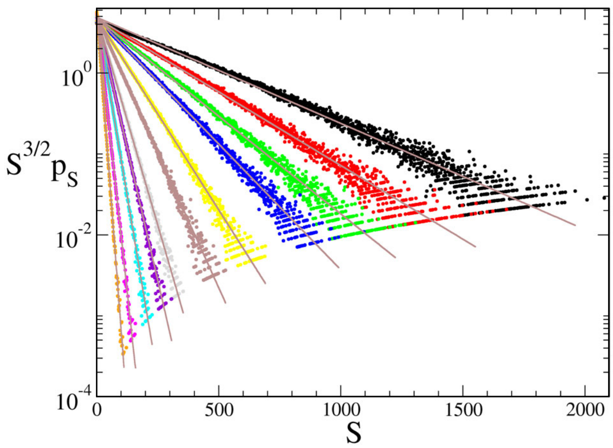

Fig. 3. Cascades on a random graph have a functional form $p_S \propto S^{-3/2} \exp(-S/\xi)$. Hence, plotting $S^{3/2}\, p_S$ vs S allows to estimate the cutoff ξ via exponential fitting.

In Fig. 4 we show $\xi(\rho)$ for the three cases of annealed dissipation ρ, of a fraction ρ of randomly chosen dissipating sites and of centrality (degree k) based dissipating sites. We find that $\xi \propto \rho^{-2}$ not only in the annealed case but in all the cases under consideration. If our purpose is to minimize cascade's impact on the system, we want to have the least possible value of ξ for a given ρ. As expected, the random quenched case is worse than the annealed case: in fact, while in the annealed case the distress has a probability ρ of being dissipated at each step, in the quenched case nothing happens until a distress site has not been reached. On the other hand, if we assume that distress redistribution resembles a random walk, sites with higher degree will be visited more frequently and hence are better choices for dissipation sites; in fact, we observe that the centrality based choice of sites leads to the lowest values of ξ.

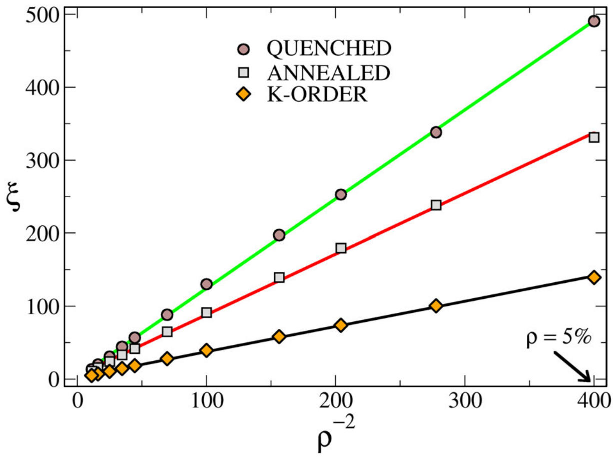

Fig. 4. Cascades' cutoff size ξ vs ρ for different immunization strategies on a random graph. Circles (quenched strategy): a fraction ρ of sites is randomly chosen to be the absorbing sites. Squares (annealed strategy): each time a unit of distress is shed from a critical site, it has a probability ρ to be absorbed. Diamonds (centrality-based strategy): a fraction ρ of sites is chosen in decreasing order of degree-centrality to be absorbing sites. The errors in the estimation of the ξ are of the order of 3%.

4 Discussion

In this paper we us the BTW sandpile model to mimic the propagation of distress on a network of financial institutions. Each institution is supposed to have both the size and the distress threshold proportional to its degree. Distress is supposed to accumulate continuously (although on a long time scale) on the network but to discharge abruptly (on very short time scale) via cascades. In absence of policies, distress is supposed to dissipate with a rate ρ during the cascades. Alternatively, financial immunization policies can be mimicked by choosing the sinks (absorbing sites) on the network; such choices correspond to "protecting" a fixed (quenched) set of financial institution. Given a policy, the amount of dissipation or the fraction of immunized institutions defines a cutoff ξ limiting cascades' size; for a given ρ, the lower the ξ, the better the policy.

We observe that a policy based on a random choice of such institutions perform worse than letting the system adjust by itself. On the other hand, targeted policies can enhance the capacity of the system to limit large cascades. In particular, we observe that centrality based policies ("too central to fail" – but in our case also "too big to fail") perform better than no policy or the random policy. The effectiveness of the centrality based policy is possibly due to the strong correlation among degree centrality and random walk centrality [20]; in fact, the path of each distress unit during a cascade is a random walk on the graph [21]; hence, sinks should be chosen according to RW-centrality to define optimal policies.

Our analysis considers random graph topologies and disregards correlations among financial institutions [22]; however, it has been observed that degree-degree correalation already enables to control the cascade size in financial networks [6]. Hence, our analysis points that controlling the topology of the financial network could strongly enhance the effectiveness of targeted immunization policies. As an example, the presence of Rich Clubs could augment the effects of targeted policies for securing the financial network. This result would represent a highly controversial point from the perspective of policy makers trying to enforce a free market and to avoid oligopolies.

Moreover, networks of networks [23] should be considered when modelling the financial system in more details; as an example, multiplex networks can be used to capture the temporal structure of the debts [24]. Of particular interest would be the possibility that coupling systems could mitigate cascades in single systems at the expense of enlarging inter-system cascades [25].

We thank CNR-PNR National Project Crisis-Lab, EU FET project DOLFINS nr 640772, EU HOME/2013/CIPS/AG/4000005013 project CI2C for support. The contents of the paper do not necessarily reflect the position or the policy of funding parties.

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