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Gravitational partial-wave absorption from scattering amplitudes

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Abstract: We study gravitational absorption effects using effective on-shell scattering amplitudes. We develop an in-in probability-based framework involving plane- and partial-wave coherent states for the incoming wave to describe the interaction of the wave with a black hole or another compact object. We connect this framework to a simplified single-quantum analysis. The basic ingredients are mass-changing three-point amplitudes, which model the leading absorption effects and a spectral-density function of the black hole. As an application, we consider a non-spinning black hole that may start spinning as a consequence of the dynamics. The corresponding amplitudes are found to correspond to covariant spin-weighted spherical harmonics, the properties of which we formulate and make use of. We perform a matching calculation to general-relativity results at the cross-section level and derive the effective absorptive three-point couplings. They are found to behave as $\mathcal{O}(G_{\text{Newton}}^{s+1})$, where s is the spin of the outgoing massive state.

Keywords: Black Holes, Effective Field Theories, Scattering Amplitudes

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1 Introduction

Gravitational-wave observations [1] have been stimulating the search of new computational methods for general relativity (GR). In addition to classical approaches to the gravitational two-body problem, which have seen constant improvement [2–12], new results have been obtained via scattering amplitudes, which encode the relevant, quantum or classical, physics in a gauge-invariant way. See [13–16] for recent reviews.

The conservative scattering of non-spinning compact bodies has been calculated up to fourth post-Minkowskian (PM) order using amplitude- [17, 18] and worldline-based methods [19–22]. For the spinning case, the conservative scattering has been evaluated at second PM order and all-order in the angular momenta [23–25] with the help of Heavy-Particle Effective Theory [26, 27]. Higher PM orders have also been obtained, though limited to lower spin orders [22, 28–31]. Progress on the spinning front has resulted in different and complementary on-shell approaches [27, 32–42]. For the interesting case of black-hole (BH) dynamics, many of these works rely on the matching [32, 33] of three-point amplitudes to the Kerr multipole expansion [43]. An all-spin understanding of the relevant four-point Compton scattering amplitude, however, is still lacking, despite recent progress in the description of massive higher-spin particles [44–46], matching to the Teukolsky-equation solutions [38, 39] through sixth order in spin, and the availability of the conservative tree-level Compton with arbitrary coefficients [47]. The quantum-field-theoretic (QFT) program of gravitational dynamics has also seen impressive advances in methods for obtaining classical observables from amplitudes, such as the Kosower-Maybee-O'Connell (KMOC) formalism [48–53], heavy-particle expansion [26, 27, 54–58], eikonal ideas [59–65], worldline QFT [28–30], boundary-to-bound map [66–69], and strong-field amplitudes [70–72].

Despite the successes in the conservative section, the progress in non-conservative effects has been slower, since those effects are naturally smaller. In particular, the absorption of mass and angular momentum is tiny, especially for non-spinning bodies, and is unlikely to be observed by ground-based detects, as shown in [73] for 5-to-50 solar masses black holes. However, for space-based detectors, the fraction of the radiated energy that is absorbed by the BHs will be around 5% [74]. This becomes especially important for rapidly rotating BHs, as shown in [75]. The change of mass and spin of a BH naturally leads to a change in the horizon by the second law of BH thermodynamics [76]. Such effects are already included in a few of the effective-one-body waveform templates [77–79] and will be needed for a future precision program.

In this paper, we initiate the study of absorption effects using modern on-shell methods for scattering amplitudes. In particular, we use mass-changing three-point amplitudes to describe leading absorption effects from a simplified single-quantum approach. We thus construct an in-in on-shell probability-based formalism for a partial wave impinging on a BH. Using this covariant effective-field-theory (EFT) description, we can match the microscopic cross-section calculation from GR literature and obtain the values of the relevant effective coupling coefficients. As a concrete application, we focus on absorption by a non-spinning BH, while leaving the more phenomenologically relevant spinning case for future work.

Absorption effects have been considered before in the literature, starting with Starobinsky, Churilov [80, 81] and Page [82, 83] and with later higher-order corrections in [84] and relatively recently in [85] by using traditional GR methods. The scattering and absorption cross-sections are obtained using a partial-wave expansion (in spin-weighted spherical harmonics) of the scattering phases and transmission factors. These factors are obtained by solving the Teukolsky equation, which describes perturbation around Kerr

BHs. Absorption of mass and angular momentum by a BH was also computed in great detail [73, 74, 86, 87] in post-Newtonian theory.

From the worldline perspective, the study of absorption is more recent. It has been considered in [88] for scalar BHs, with subsequent inclusion of spin effects [89, 90]. Furthermore, absorption has been combined with spontaneous emission to understand superradiance effects in [91]. The authors of [88–92] put EFT operators on a classical worldline to model the intricate behavior of a compact object. In particular, higher-derivative operators were included in [92] for the spinning case, which starts at 1.5 PN order, tackling the discrepancy in the literature of the horizon fluxes in the test-body limit. We propose to go further and consider the object itself as a quantum particle, but amenable to an appropriately defined classical limit. This lets us profit not just from QFT techniques, which have been available on the worldline, but also from the on-shell approach to scattering amplitudes.

Purely mass-changing absorption effects from on-shell scattering amplitudes were never studied to the best of our knowledge,[1] although similar amplitudes have appeared in the context of quantum emission [93, 94]. The basic building blocks for modeling absorption effects are three-point amplitudes of two different massive particles and a graviton, in which the initial state absorbs the graviton, changing its mass and spin. These amplitudes induce the introduction of a spectral-density function for the black holes, which goes beyond the simplest point particle approach. Even before matching, the EFT cross-section reproduces known facts about Schwarzchild BHs: *(i)* the cross-section does not depend on the magnetic quantum number m, and *(ii)* there is no absorption in the static limit $\sigma_{\text{abs}}(\omega_{\text{cl}} \to 0) = 0$.

Properly modeling the interaction of a BH with a classical wave from amplitudes requires the use of massless coherent states. For that, we describe a covariant probability-based formalism for spherical coherent states, so as to substantiate the single-quantum leading-order calculation, and to explain how one could improve the absorption description to higher orders and combine it with conservative effects.

This paper is organized as follows. In section 2 we give construct a scattering-matrix element for a compact object absorbing a partial/spherical wave in terms of mass-changing and spin-changing amplitudes, the form of which is specified in section 3. In section 4 we combine these ingredients into an absorptive cross-section with the help of the BH spectral density function, which enters as an additional effective coupling factor. In section 5 we match to the microscopic cross-section from GR to make sense of the effective couplings. Finally, in section 6, we connect the single-quantum cross-section description to the framework involving massless spherical coherent states. In this section, we also introduce a diagrammatic expansion of the T-matrix, which allows for perturbations of the BH-wave interaction that can be matched to higher orders of the cross-section. We conclude in section 7. Though we assume familiarity with the spinor-helicity formalism [95], we briefly explain it and its connection to covariant spin-weighted spherical harmonics in appendix A.

[1]In [38, 39] the authors have introduced contact terms non-analytical in spin for the Compton amplitude to match the solutions of the Teulkosky equation. These terms are then suggested to model absorption effects, despite the masses of the initial and the final particles being equal. Here what we call absorption effects are strictly inelastic changing-mass interactions.

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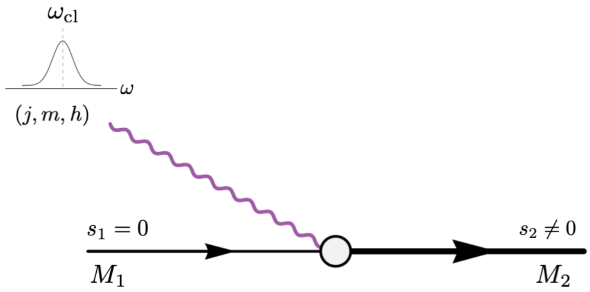

Figure 1. Wave impinging on a scalar black hole.

2 Partial-wave absorption matrix element

In this section we describe our setup for obtaining classical absorption effects from the quantum on-shell scattering amplitudes. Such a setup of course relies on EFT ideas, such as treating black holes as point particles. These concepts have been heavily used recently to provide predictions for conservative dynamics and dissipation effects.

As in most of EFTs, the knowledge of the coefficients that parametrize the theory is either provided by experimental data or by performing a matching calculation to the underlying theory. In our case, the underlying theory is Einstein's GR, or more practically, the solution to the Teukolsky equation [96–98]. Given these two sides of the EFT matching, we will sometimes be referring to the EFT side of the calculation as macroscopic, and to the solution to Teukolsky equation as microscopic. On the EFT side, we will model absorption effects using mass-changing amplitudes.

We focus on the simplest relevant process depicted in figure 1: a graviton spherical state impinging on a massive particle of mass M_1 (for simplicity taken spinless), which absorbs the graviton and changes its mass to M_2 and spin to s_2. It is natural to think of the corresponding scattering amplitude in terms of plane-wave states as described in section 3. However, GR methods give us results [80–83, 85, 99–101] for spherical waves with fixed angular-momentum quantum numbers. Therefore, we start by translating between these two pictures — with a focus on single-graviton states. In section 6 we will come back to justifying this setup further using classical coherent states, which are more appropriate for modeling classical waves.

2.1 Spherical helicity amplitude

By definition (see e.g. [102]), spherical helicity states partially diagonalize the total spin operator $\boldsymbol{J}$. They are eigenstates of $\boldsymbol{J}^2$, J_z and helicity $(\boldsymbol{J}\cdot\boldsymbol{P})/\boldsymbol{P}^2$, as well as the Hamiltonian P^0. These states are labeled by energy ω, angular-momentum quantum numbers j, $m = -j, \ldots, j$ and helicity $h = \pm 2$ (graviton) or ± 1 (photon):[2]

$$|\omega, j, m, h\rangle = a^\dagger_{j,m,h}(\omega)|0\rangle, \qquad \langle\omega', j', m', h'|\omega, j, m, h\rangle = \hat{\delta}(\omega'-\omega)\delta^j_{j'}\delta^m_{m'}\delta^h_{h'}. \tag{2.1}$$

[2]Here and below, the hat notation [48] means $\hat{d}^n p := d^n p/(2\pi)^n$ and $\hat{\delta}^n(\ldots) := (2\pi)^n\delta^n(\ldots)$. For the spherical helicity states, we also assume that masslessness: $P^2|\omega, j, m, h\rangle = 0$.

This is in contrast to the more familiar plane-wave states $|k, h\rangle$, which diagonalize the four-momentum P^μ in addition to the helicity $(\boldsymbol{J}\cdot\boldsymbol{P})/\boldsymbol{P}^2$:

$$|k,h\rangle := a_h^\dagger(k)|0\rangle, \qquad \langle k',h'|k,h\rangle = 2|\boldsymbol{k}|\hat{\delta}^3(\boldsymbol{k}'-\boldsymbol{k})\delta^h_{h'}. \tag{2.2}$$

The two bases of one-particle states may be related by [91]

$$\langle k,h'|\omega,j,m,h\rangle = \frac{4\pi}{\sqrt{2\omega}}\delta^h_{h'}\hat{\delta}(|\boldsymbol{k}|-\omega)\,{}_{-h}Y_{jm}(\hat{\boldsymbol{k}}), \tag{2.3}$$

where the spin-weighted spherical harmonics ${}_{-h}Y_{jm}(\hat{\boldsymbol{k}})$ depend on the momentum direction $\hat{\boldsymbol{k}} := \boldsymbol{k}/|\boldsymbol{k}|$ and constitute a generalization [103, 104] of the usual (scalar) spherical harmonics. The corresponding completeness relations imply that the one-particle spinning spherical state can be written as

$$|\omega,j,m,h\rangle = \frac{4\pi}{\sqrt{2\omega}}\int_k \hat{\delta}(k^0-\omega)\,{}_{-h}Y_{j,m}(\hat{\boldsymbol{k}})|k,h\rangle = \sqrt{2\omega}\int\frac{d\Omega_{\hat{\boldsymbol{k}}}}{4\pi}\,{}_{-h}Y_{j,m}(\hat{\boldsymbol{k}})|k,h\rangle\big|_{|\boldsymbol{k}|=\omega}, \tag{2.4}$$

where $d\Omega_{\hat{\boldsymbol{k}}}$ denotes the spherical-angle integration measure over the directions of $\boldsymbol{k}$. Here and below, we use a shorthand for the on-shell integration measure (for $M_k = 0$)

$$\int_p := \int\frac{d^4p}{(2\pi)^3}\Theta(p^0)\delta(p^2-M_p^2) =: \int\frac{d^4p}{(2\pi)^3}\delta^+(p^2-M_p^2). \tag{2.5}$$

In order to write the scattering matrix element for a spherical helicity state, we need to be careful with the massive particle at the origin, which, strictly speaking, cannot be a plane-wave state either. So instead we use a wavepacket

$$|\psi\rangle := \int_{p_1}\psi_\xi(p_1)|p_1\rangle : \qquad \begin{aligned} \langle\psi|\psi\rangle = 1, \qquad \langle\psi|P^\mu|\psi\rangle = p^\mu_{1,\mathrm{cl}} := (M_1,\boldsymbol{0}),\\ \langle\psi|P^\mu P^\nu|\psi\rangle = \langle\psi|P^\mu|\psi\rangle\langle\psi|P^\nu|\psi\rangle + \mathcal{O}(\xi). \end{aligned} \tag{2.6}$$

For concreteness, we may think of $\psi_\xi(p_1) \propto \exp\big(-\frac{p_1^0}{\xi M_1}\big)$. The dimensionless parameter $\xi = \ell_{\mathrm{C}}^2/\ell_{\mathrm{WP}}^2$ encodes the ratio of the Compton wavelength and the position-space spread of the wavepacket [48]. We will be focusing on the scale hierarchy $\ell_{\mathrm{C}} \ll \ell_{\mathrm{WP}} \ll 2\pi\hbar c/\omega$ relevant for classical scattering of a wave with frequency $\omega/\hbar$.

We are now ready to express the S-matrix element for a spherical helicity state in terms of the conventional plane-wave scattering amplitude:

$$\langle X|S|\psi;\omega,j,m,h\rangle = \frac{4\pi i}{\sqrt{2\omega}}\int_{p_1}\psi_\xi(p_1)\int_k\hat{\delta}(k^0-\omega)\hat{\delta}^4(p_1+k-p_X)\,{}_{-h}Y_{j,m}(\hat{\boldsymbol{k}})\mathcal{A}(X|p_1;k,h). \tag{2.7}$$

As usual, we have ignored the no-scattering term in $S = 1 + iT$. For the amplitude arguments, we choose to mimic the structure of the matrix elements and write the outgoing particles first separated from the incoming particles by a vertical line.

Unfortunately, the matrix element (2.7) by itself is too singular to handle unambiguously, which is due to the infinite norm $\langle\omega,j,m,h|\omega,j,m,h\rangle = \hat{\delta}(0)$ of the massless spherical state (2.4). So we also smear its energy with a wavefunction:

$$|\gamma\rangle := \int_0^\infty \hat{d}\omega\,\gamma_\zeta(\omega)|\omega,j,m,h\rangle : \qquad \begin{aligned} \langle\gamma'|\gamma\rangle = \delta^j_{j'}\delta^m_{m'}\delta^h_{h'}, \qquad \langle\gamma|P^0|\gamma\rangle = \omega_{\mathrm{cl}},\\ \langle\gamma|P^0P^0|\gamma\rangle = \langle\gamma|P^0|\gamma\rangle\langle\gamma|P^0|\gamma\rangle + \mathcal{O}(\zeta). \end{aligned} \tag{2.8}$$

The corresponding scattering-matrix element is finally

$$\langle X|S|\psi;\gamma\rangle = 4\pi i\int_{p_1}\psi_\xi(p_1)\int_k \frac{\gamma_\zeta(k^0)}{\sqrt{2k^0}}\hat{\delta}^4(p_1+k-p_X)\,{}_{-h}Y_{j,m}(\hat{\boldsymbol{k}})\mathcal{A}(X|p_1;k,h). \tag{2.9}$$

2.2 Covariant spherical states

Before we proceed to the absorption cross-section, it is rewarding to covariantize our spherical-helicity state setup. By covariantization we mean allowing for an arbitrary time direction u^μ, with $u^2=1$, as well a spacelike spin quantization axis n^μ, with $n^2=-1$ and $n\cdot u=0$. (In section 2.1, these were set to $(1,\mathbf{0})$ and $(0,0,0,1)$, respectively.) The corresponding angular-momentum operator is

$$J^\mu(u) := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}u_\rho \qquad \Rightarrow \qquad [J^\mu(u),J^\nu(u)] = i\epsilon^{\mu\nu\rho\sigma}u_\rho J_\sigma(u), \tag{2.10}$$

which is not to be confused with the Pauli-Lubanski spin vector $W^\mu := \epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\rho/2$. A spherical helicity state $|\omega,j,m,h\rangle$ is then an eigenstate of "energy" $E(u) := u\cdot P$ and angular-momentum combinations $-J(u)^2$, $n\cdot J(u)$ and $J(u)\cdot P = W\cdot P$. Similarly to eq. (2.4), we choose to construct them directly from the plane-wave states:

$$|\omega,j,m,h\rangle_{u,n} := \frac{4\pi}{\sqrt{2\omega}}\int \hat{d}^4k\,\hat{\delta}^+(k^2)\hat{\delta}(k\cdot u-\omega)\,{}_{-h}Y_{j,m}(k;u,n)|k,h\rangle. \tag{2.11}$$

The crucial new ingredient here is the covariant spin-weighted spherical harmonic. We define these functions in terms of spinor products as follows:

$${}_h\tilde{Y}_{j,m}(k;u,n) := \frac{1}{\langle k|u|k]^j}\left[[u_ak]^{\odot(j+h)}\odot\langle ku_a\rangle^{\odot(j-h)}\right]_{\{a\}=(\underbrace{1\ldots1}_{j-m}\underbrace{2\ldots2}_{j+m})}. \tag{2.12}$$

We have hereby followed [39, 105] in using the massive spinor-helicity formalism [95] (see appendix A for a brief review) to covariantize the spinorial construction dating back to Newman and Penrose [103]. The exposed indices $\{a\}=\{a_1,\ldots,a_{2j}\}$ correspond to the little group SU(2) of u^μ and are understood to be fully symmetrized, as indicated by the symmetric tensor-product symbol $\odot$. We adopt the spinor conjugation conventions $(|p^a\rangle_\alpha)^* = [p_a|_{\dot\alpha}$, $([p^a|_{\dot\alpha})^* = -|p_a\rangle_\alpha$ for $p^0>0$, which immediately imply

$${}_h\tilde{Y}^*_{j,m}(k;u,n) = (-1)^{2j+m-h}{}_{-h}\tilde{Y}_{j,-m}(k;u,n). \tag{2.13}$$

The properly normalized functions seen in eq. (2.11) are written without the tildes:

$${}_hY_{j,m}(k;u,n) := (-1)^m(2j)!\sqrt{\tfrac{2j+1}{4\pi(j+m)!(j-m)!(j+h)!(j-h)!}}\,{}_h\tilde{Y}_{j,m}(k;u,n), \tag{2.14}$$

with the orthonormality statement being

$$\frac{2}{\omega}\int d^4k\,\delta^+(k^2)\delta(k\cdot u-\omega)\,{}_hY^*_{j',m'}(k;u,n)\,{}_hY_{j,m}(k;u,n) = \delta_j^{j'}\delta_m^{m'}. \tag{2.15}$$

For the proof and a more detailed exposition of the harmonics (2.12), see appendix A.

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Here let us point out the new important features of these harmonics. First of all, the harmonics are by definition (2.12) insensitive to the overall scale of both k^μ and u^μ. Moreover, they are now clearly formulated in a convention-independent way — in the sense that it is covariant with respect to the two little groups:

- the massless little-group U(1) of k^μ may be used to change the phases of all spherical harmonics in a local but mutually consistent way. Namely, transforming $|k\rangle \to e^{-i\phi(k)/2}|k\rangle$, $|k] \to e^{i\phi(k)/2}|k]$ implies phase adjustments of the form ${}_hY_{j,m}(k;u,n) \to e^{ih\phi(k)}{}_hY_{j,m}(k;u,n)$, which connect between various possible definitions of spin-weighted spherical harmonics, e.g. via quaternions [106].

- the massive little group SU(2) of u^μ may be used to change the physical meaning of the magnetic quantum number m. For instance, the explicit spinor parametrizations (A.9) and (A.10) correspond to the m-quantization along $\boldsymbol{u} \neq 0$ and the conventional z-axis for $\boldsymbol{u} = 0$, respectively. However, we may just as well apply transformations $|u^a\rangle \to U^a{}_b(u)|u^b\rangle$, $|u^a] \to U^a{}_b(u)|u^b]$ to the massive spinors, and this will rotate the spin quantization axis

$$n^\mu := \frac{1}{2}(\langle u_2|\sigma^\mu|u^2] + [u_2|\bar{\sigma}^\mu|u^2\rangle) \qquad \Rightarrow \qquad n^2 = -1, \quad u\cdot n = 0. \tag{2.16}$$

  Having this relation in mind, we henceforth compress our notation to ${}_hY_{j,m}(k;u)$.

In addition, we can specify general frame transformations of the covariant spherical harmonics (2.12). Indeed, it is shown in appendix B that under the time-direction change $u^\mu \to v^\mu = L^\mu{}_\nu(v\leftarrow u)u^\nu$ the massive spinors are boosted as follows:

$$|v^a\rangle = \frac{\sqrt{\mu}}{\mu+1}|u+v|u^a], \qquad |v^a] = \frac{\sqrt{\mu}}{\mu+1}|u+v|u^a\rangle, \qquad \mu := u\cdot v + \sqrt{(u\cdot v)^2 - 1}. \tag{2.17}$$

Here we have assumed that the spin quantization axis for the resulting time direction v^μ is automatically $L^\mu{}_\nu(v\leftarrow u)n^\nu$, i.e. the boosted version of the original quantization axis n^μ. Of course, it can then be easily tweaked by an additional little-group transformation of the resulting spinors $|v^a\rangle \to U^a{}_b(v)|v^b\rangle$, $|v^a] \to U^a{}_b(v)|v^b]$.

Given this covariant formulation of the spherical states, we rewrite eq. (2.9) as

$$\begin{aligned}\langle X|S|\psi;\gamma\rangle = 4\pi i\int_0^\infty \frac{\hat{d}\omega}{\sqrt{2\omega}}\gamma_\zeta(\omega)\int \hat{d}^4p_1\hat{\delta}^+(p_1^2-M_1^2)\psi_\xi(p_1) \\ \times \int \hat{d}^4k\,\hat{\delta}^+(k^2)\hat{\delta}(k\cdot u_1-\omega)\hat{\delta}^4(p_1+k-p_X)\,{}_{-h}Y_{j,m}(k;u_1)\mathcal{A}(X|p_1;k,h).\end{aligned} \tag{2.18}$$

This is the scattering matrix-element formula that we are going to use in the absorption cross-section calculation below. For concreteness, we will employ the following Lorentz-covariant realization of the massive wavefunction ψ_ξ is [48, 107]

$$\psi_\xi(p_1) = \frac{1}{M_1}\left[\frac{8\pi^2}{\xi K_1(2/\xi)}\right]^{1/2}\exp\Big(-\frac{p_1\cdot u_1}{\xi M_1}\Big), \tag{2.19}$$

where K_1 denotes the modified Bessel function of the second kind.

So far we have not specified the form of the scattering amplitude $\mathcal{A}(X|p_1;k,h)$. In our EFT approach to non-conservative effects, it is natural to assume that the leading contribution to the absorption process comes from mass-changing three-point amplitudes, i.e. X is particle of mass M_2 and s_2. We discuss such amplitudes next.

3 Basic mass-changing amplitudes

In this section we construct the three-point amplitudes that serve as basic building blocks for modeling absorption effects. As discussed above, these amplitudes involve a massless messenger particle and two particles with different masses. They were first explored in [108] and covariantized in [95]. Here we further reorganize the latter formulation, while also using coherent-spin eigenvalues [52], which saturate the massive spin indices and thus act as a book-keeping device [44].

We continue to work in the massive spinor-helicity formalism [95], which is briefly reviewed in appendix A. In this formalism, an amplitude $\mathcal{A}_{\{b\}}{}^{\{a\}}$ involving two massive particles carries $2s_1$ symmetrized little-group SU(2) indices $a_1,\ldots,a_{2s_1}$ for the incoming particle 1, and $2s_2$ such indices $b_1,\ldots,b_{2s_2}$ for the outgoing particle 2. We choose to use the chiral basis of massive spinors (angle brackets) for positive helicities and the antichiral basis (square brackets) for negative helicities. Since $\det\{|1^a\rangle_\alpha\} = \det\{|1^a]_{\dot\alpha}\} = M_1$ and $\det\{|2^b\rangle_\beta\} = \det\{|2^b\rangle_{\dot\beta}\} = M_2$, we may proceed by stripping the spinors $|1_a\rangle$ and $|2^b\rangle$ for the positive messenger helicity and $|1_a]$ and $|2^b]$ for negative helicity. For instance, in the positive-helicity case we write

$$\mathcal{A}_{\{b\}}{}^{\{a\}}(p_2,s_2|p_1,s_1;k,h) =: A(k,h)^{\{\alpha\},\{\beta\}}(|1_a\rangle_\alpha)^{\odot 2s_1}(|2^b\rangle_\beta)^{\odot 2s_2}, \tag{3.1}$$

where $\odot$ denotes the symmetrized tensor product. In addition to the $\mathcal{A}_{\{b\}}{}^{\{a\}}$ and $A^{\{\alpha\},\{\beta\}}$ objects, a third perspective on the same amplitude is provided by contracting massive spinors with auxiliary SU(2)-spinor variables [44],

$$|\mathbf{1}\rangle := |1_a\rangle\alpha^a, \qquad |\bar{\mathbf{2}}\rangle := |2^b\rangle\tilde\beta_b, \tag{3.2}$$

which may serve as an extra handle on the spin quantization axis.[3] We write the fully contracted amplitude in boldface as a scalar in terms of the spinor-stripped one:

$$\boldsymbol{\mathcal{A}}(p_2,s_2|p_1,s_1;k,h) := A(k,h)^{\{\alpha\},\{\beta\}}(|\mathbf{1}\rangle^{\otimes 2s_1})_{\{\alpha\}}(|\bar{\mathbf{2}}\rangle^{\otimes 2s_2})_{\{\beta\}}, \tag{3.3}$$

the advantage being that the index symmetrization is now entirely automatic.

3.1 Classifying mass-changing amplitudes

Going back to the stripped amplitude $A(k,h)_{\{\alpha\},\{\beta\}}$ with two sets of symmetrized $\mathrm{SL}(2,\mathbb{C})$ indices, we may decompose it in the chiral-spinor basis of $|k\rangle$ and $|p_1|k]$. Unlike the equal-mass case, these two spinors are linearly independent (and there is no need for a helicity factor [95]), because

$$\langle k|p_1|k] = 2p_1\cdot k = M_2^2 - M_1^2 \neq 0 \tag{3.4}$$

[3]The auxiliary SU(2) spinors α^a and $\tilde\beta_b$ transform under the action of the little groups of p_1 and p_2, respectively, and in this sense have an implicit dependence on their momenta. Moreover, in the coherent-spin framework they constitute eigenvalues of the spin-annihilation operators [52, 109].

due to momentum conservation $p_2 = p_1 + k$. This equation also tells us about the possible dimensionful scales entering the three-point process from an EFT perspective, which will have to be matched later. We can either use the mass pair (M_1, M_2) or $(M_1, 2p_1 \cdot k)$, and in this work we are going to favor the latter. For instance, we may use M_1 to absorb the mass dimension of the amplitude and allow the EFT coefficients to depend on the dimensionless ratio

$$w := \frac{2p_1 \cdot k}{M_1^2}, \tag{3.5}$$

while expanding in terms of the dimensionless spinors of helicity $-1/2$ and $1/2$:

$$\lambda_\alpha := M_1^{-1/2} |k\rangle_\alpha, \qquad \mu_\alpha := M_1^{-3/2} p_{1,\alpha\dot{\beta}} |k]^{\dot{\beta}} \qquad \Rightarrow \qquad \langle \lambda\mu \rangle = w. \tag{3.6}$$

Therefore, the most general stripped amplitude involving two unequal masses and one massless positive-helicity particle is schematically given by [95, 108]

$$A(k,h)_{\{\alpha\},\{\beta\}} = M_1^{1-s_1-s_2} \sum_i c^h_{(i),s_1,s_2}(w) [\lambda^{s_1+s_2-h} \mu^{s_1+s_2+h}]^{(i)}_{\{\alpha\},\{\beta\}}. \tag{3.7}$$

Here i enumerates inequivalent tensor products with the given spinorial index structure, and their scalar coefficients $c^h_{i,s_1,s_2}(\omega)$ may depend on each spin and in the dimensionless ratio w. Before we specify the relevant spinorial structures, note that there are natural constraints that follow already from the form of eq. (3.7), such as

$$s_1 + s_2 \pm h \in \mathbb{Z}_{\geq 0} \qquad \Rightarrow \qquad s_1 + s_2 \geq |h|. \tag{3.8}$$

Moreover, there can clearly be no three-point amplitude for one or three half-integer spins — in QFT this standard fact is usually derived from the spin-statistics theorem.

We find it helpful to observe that the massless little-group dependence may be completely factored out (in the tensor-product sense). This leaves a polynomial in λ and μ, which is independent of the massless helicity:

$$\begin{aligned} [\lambda\mu \oplus \mu\lambda]^n_{\{\alpha\},\{\beta\}} := {} & c_0 (\lambda^n)_{\alpha_1 \ldots \alpha_n} (\mu^n)_{\beta_1 \ldots \beta_n} + c_1 (\lambda^{n-1}\mu)_{\alpha_1 \ldots \alpha_n} (\mu^{n-1}\lambda)_{\beta_1 \ldots \beta_n} \\ & + \ldots + c_{n-1} (\lambda\mu^{n-1})_{\alpha_1 \ldots \alpha_n} (\mu\lambda^{n-1})_{\beta_1 \ldots \beta_n} + c_n (\mu^n)_{\alpha_1 \ldots \alpha_n} (\lambda^n)_{\beta_1 \ldots \beta_n}, \end{aligned} \tag{3.9}$$

where we have also omitted the $\otimes$ sign for brevity. The exponent n depends on the total-spin quantum numbers, and in the amplitude each such term may have its own coefficient. Without loss of generality, we consider $s_2 \geq s_1$, where we have two cases:

- $s_2 - s_1 \geq h$, where we saturate the s_1 indices by the above polynomial, while the remaining s_2 indices are accounted for by the tensor product, which is unambiguously defined by the overall helicity weight. The corresponding spinorial structures belong to the following tensor power of a direct sum:

$$[\lambda^{s_1+s_2-h} \mu^{s_1+s_2+h}]^{(i)}_{\{\alpha\},\{\beta\}} \in [\lambda\mu \oplus \mu\lambda]^{2s_1}_{\{\alpha\},\{\beta\}} (\lambda^{s_2-s_1-h} \mu^{s_2-s_1+h})_{\{\beta\}}; \tag{3.10}$$

- $s_2 - s_1 < h$, where the polynomial (3.9) saturates the number of λ's, which is equal to $s_1 + s_2 - h$, while the remaining $2h$ of μ's are unambiguously distributed among the two massive particles. The spanning spinorial structure is thus

$$[\lambda^{s_1+s_2-h}\mu^{s_1+s_2+h}]^{(i)}_{\{\alpha\},\{\beta\}} \in [\lambda\mu \oplus \mu\lambda]^{s_1+s_2-h}_{\{\alpha\},\{\beta\}} (\mu^{s_1-s_2+h})_{\{\alpha\}}(\mu^{s_2-s_1+h})_{\{\beta\}}. \tag{3.11}$$

  Note that in electromagnetism this case only occurs for $s_1 = s_2$, whereas in GR both $s_2 = s_1$ and $s_2 = s_1 + 1$ are possible.

In both cases, we have the polynomial with free coefficients and the additional factor, which carries the massless helicity. This factor completes the $\mathrm{SL}(2,\mathbb{C})$ indices of either massive particle that are not accounted for by the polynomial, and of course all α's and all β's are implicitly symmetrized.

This analysis should be repeated for $s_1 \leq s_2$, and the $\mathrm{SL}(2,\mathbb{C})$ can then be contracted with the massive spinors (and auxiliary variables), for which the Dirac equations $|p_1|\mathbf{1}\rangle = M_1|\mathbf{1}]$ and $|p_2|\bar{\mathbf{2}}\rangle = M_2|\bar{\mathbf{2}}]$ hold. In this way, we arrive at

$$\boldsymbol{\mathcal{A}}(p_2, s_2|p_1, s_1; k, h) = \begin{cases} \boldsymbol{F}^h_{s_1,s_2}\, \langle\bar{\mathbf{2}}k\rangle^{s_2-s_1-h}[\bar{\mathbf{2}}k]^{s_2-s_1+h}, & s_2 - s_1 \geq |h|, \\ \boldsymbol{F}^h_{s_1,s_2}\, [\bar{\mathbf{2}}k]^{s_2-s_1+h}[k\mathbf{1}]^{s_1-s_2+h}, & |s_2 - s_1| < h, \\ \boldsymbol{F}^h_{s_1,s_2}\, \langle\bar{\mathbf{2}}k\rangle^{s_2-s_1-h}\langle k\mathbf{1}\rangle^{s_1-s_2-h}, & |s_2 - s_1| < -h, \\ \boldsymbol{F}^h_{s_1,s_2}\, \langle k\mathbf{1}\rangle^{s_1-s_2-h}[k\mathbf{1}]^{s_1-s_2+h}, & s_1 - s_2 \geq |h|. \end{cases} \tag{3.12a}$$

where the factor $\boldsymbol{F}^h_{s_1,s_2}$ contains free coefficients and can now be written as

$$\boldsymbol{F}^h_{s_1,s_2} = M_1^{1-2s_1-2s_2} \sum_{r=0}^{n} g^h_{r,s_1,s_2}(w)\, \langle\bar{\mathbf{2}}|k|\mathbf{1}]^r\, [\bar{\mathbf{2}}|k|\mathbf{1}\rangle^{n-r}. \tag{3.12b}$$

These coefficients $g^h_{r,s_1,s_2}(w)$ are a refined version of $c^h_{(i),s_1,s_2}(w)$ in eq. (3.7); the main difference between them is some degree of rescaling by M_2/M_1. The polynomial degree n above is related to the maximal number of terms:

$$n+1 \;=\; \begin{cases} 2s_1+1, & s_2 - s_1 \geq |h|, \\ s_1+s_2-|h|+1, & |s_2 - s_1| < |h|, \\ 2s_2+1, & s_1 - s_2 \geq |h|, \end{cases} \tag{3.12c}$$

This number matches the counting in [110]. For completeness, the above formulae (3.12) already include the result of the above analysis for the negative messenger helicity, in which case we used the anti-chiral basis, $|k]$ and $|p_1|k\rangle$.

Interestingly, the coupling counting (3.12c) obeys the bound

$$\#\ \text{coeffs.} \leq 2\mathrm{min}(s_1, s_2) + 1. \tag{3.13}$$

For instance, there is only one term for the case of the scalar massive incoming state $s_1 = 0$. Indeed, the constraint (3.8) immediately implies $s_2 > |h|$, so we get a trivial polynomial of degree $n(0, s_2, h) = 0$. In that case, the amplitude takes the form

$$\boldsymbol{\mathcal{A}}(p_2, s_2|p_1, s_1 = 0; k, h) = g^{|h|}_{0,0,s_2}(w) M_1^{1-2s_2} \langle\bar{\mathbf{2}}k\rangle^{s_2-h}[\bar{\mathbf{2}}k]^{s_2+h}. \tag{3.14}$$

Note that we have now assumed parity and thus conflated the dimensionless coupling coefficients $g^{\pm h}_{0,0,s_2}(w)$ into the single coupling $g^{\pm|h|}_{0,0,s_2}(w)$, which still depends on the absolute helicity value of the messenger particle.[4]

3.2 Minimal mass-changing amplitudes

As a minor digression, let us note that, for non-zero initial spin, the proliferation of possible effective couplings in the mass-changing three-point amplitude (3.12) may be reduced if we come up with some notion of minimality. Indeed, in a similar situation in the equal-mass case, $M_1 = M_2$, Arkani-Hamed, Huang and Huang [95] managed to single out the so-called "minimal" amplitudes by considering its massless limit. For positive helicity, these minimal amplitudes include, for instance,

$$\mathcal{A}(p_2, s|p_1, s; k, h) = g_0^h (p_1\cdot \varepsilon_k^+)^h \langle \bar{\mathbf{2}}\mathbf{1}\rangle^{2s}, \tag{3.15}$$

where for simplicity we have assumed $s_1 = s_2 = s$. In other words, the stripped amplitude is proportional to the tensor product of $\mathrm{SL}(2,\mathbb{C})$ Levi-Civita tensors $(\epsilon^{2s})_{\{\alpha\},\{\beta\}}$.

To expose a similar unique structure in the unequal-mass case, where the couplings correspond to the terms in the polynomial (3.9), we may change the basis inside of it to the antisymmetric and symmetric combinations of the basis spinors:

$$[\lambda\mu \oplus \mu\lambda]^n_{\{\alpha\},\{\beta\}} = [\epsilon\oplus\sigma]^n_{\{\alpha\},\{\beta\}}, \qquad \epsilon_{\alpha\beta} = \frac{\lambda_\alpha\mu_\beta - \mu_\alpha\lambda_\beta}{\langle\lambda\mu\rangle}, \qquad \sigma_{\alpha\beta} := \lambda_\alpha\mu_\beta + \mu_\alpha\lambda_\beta. \tag{3.16}$$

Since of course $\langle \mathbf{1}|^\alpha \langle \bar{\mathbf{2}}|^\beta \epsilon_{\alpha\beta} = \langle \mathbf{1}\bar{\mathbf{2}}\rangle$ and the symmetric combination leads to

$$\langle \mathbf{1}|^\alpha \langle \bar{\mathbf{2}}|^\beta \sigma_{\alpha\beta} = \frac{M_2^2 + M_1^2}{M_1^2}\langle \mathbf{1}\bar{\mathbf{2}}\rangle + \frac{2M_2}{M_1}[\mathbf{1}\bar{\mathbf{2}}], \tag{3.17}$$

the main amplitude factor can simply be expanded in the angle and square brackets:

$$\boldsymbol{F}^h_{s_1,s_2} = M_1^{1-2s_1-2s_2+n} \sum_{r=0}^{n} \tilde{g}^h_{r,s_1,s_2}(w)\, \langle \bar{\mathbf{2}}\mathbf{1}\rangle^{n-r} [\bar{\mathbf{2}}\mathbf{1}]^r. \tag{3.18}$$

So we propose to define the minimal mass-changing stripped amplitudes as those with highest power in $\epsilon_{\alpha\beta}$, or, equivalently,

$$\begin{aligned} &\mathcal{A}_{\min}(p_2, s_2|p_1, s_1; k, h) \qquad\qquad (3.19)\\ &= \tilde{g}^h_{0,s_1,s_2}(w) \begin{cases} M_1^{1-2s_2} \langle\bar{\mathbf{2}}\mathbf{1}\rangle^{2s_1} \langle \bar{\mathbf{2}}k\rangle^{s_2-s_1-h} [\bar{\mathbf{2}}k]^{s_2-s_1+h}, & s_2 - s_1 \geq h \geq 0,\\ M_1^{1-s_1-s_2-h} \langle\bar{\mathbf{2}}\mathbf{1}\rangle^{s_1+s_2-h} [\bar{\mathbf{2}}k]^{s_2-s_1+h} [k\mathbf{1}]^{s_1-s_2+h}, & |s_2 - s_1| < h,\\ M_1^{1-2s_2} \langle\bar{\mathbf{2}}\mathbf{1}\rangle^{2s_2} \langle k\mathbf{1}\rangle^{s_1-s_2-h} [k\mathbf{1}]^{s_1-s_2+h}, & s_1 - s_2 \geq h \geq 0. \end{cases} \end{aligned}$$

It is clear that for $s_1 = 0$ and $s_2 > |h|$, the minimal-coupling amplitude coincides with the previously defined amplitude (3.14). Moreover, let us note in passing that these amplitudes satisfy the double-copy prescription explored in the presence of massive spinning states in [111, 112].

[4] In the worldline formalism, the parity assumption is called "electric-magnetic" duality [88, 89].

We hope to explore these amplitudes in more detail elsewhere, whereas in the rest of this paper for the sake of simplicity we focus on the mass-changing amplitudes (3.14) with the non-spinning initial state, which we use to model the radiation absorption by a Schwarzschild black hole. In this context, it is important to note that if we assume locality of the EFT Lagrangian that implies the above amplitudes, the dimensionless coupling constants $g^h_{0,s_1,s_2}(w)$ may then be constrained to only have non-negative powers of w. Unfortunately, a rigorous proof of this statement may be to technical and require dealing with all sorts of field redefinitions. So for the purposes of this paper, let us simply impose that $g^h_{0,0,s_2}(w)$ have no poles in w:

$$g^h_{0,0,s_2}(w) = \mathcal{O}(w^0) \qquad \Rightarrow \qquad \boldsymbol{\mathcal{A}}(p_2, s_2|p_1, s_1 = 0; k, h) = \mathcal{O}(w^{s_2}), \qquad w \to 0, \tag{3.20}$$

which constitutes is a non-trivial EFT modeling assumption.

4 Absorption from mass-changing amplitudes

In this section we combine the ingredients from the previous two sections: the partial-wave absorption setup leading to the matrix element (2.18), and the mass-changing three-point amplitudes (3.14). The goal of this section will be to derive the absorptive cross-section as a function of the effective coupling coefficients $g^{|h|}_{0,0,s_2}(w)$.

4.1 Mass-changing amplitudes as harmonics

Focusing on the mass-changing amplitudes (3.14), it is rewarding to notice that they are simply proportional to the spin-weighted spherical harmonics (2.12), namely

$$\mathcal{A}_{\underbrace{1\ldots1}_{s_2-m}\underbrace{2\ldots2}_{s_2+m}}(p_2, s_2|p_1; k, h) = M_1 g^{|h|}_{0,0,s_2}(w)(-1)^{s_2-h} w^{s_2}\, {}_h\tilde{Y}_{s_2,m}(k; u_2) =: \mathcal{A}^h_{s_2,m}(p_2|p_1; k). \tag{4.1}$$

However, the harmonics are defined with respect to u_2^μ, which is counterproductive for plugging these amplitudes in the partial-wave absorption formula (2.18), where u_1^μ is fixed but u_2^μ changes along with the integration variable k^μ. So we wish to make the transition between the two velocity vectors, which are related by the boost

$$u_2^\rho = L^\rho_\sigma(u_2 \leftarrow u_1) u_1^\sigma = \exp\Big(\tfrac{i \log(u_1\cdot u_2 + \sqrt{(u_1\cdot u_2)^2-1})}{\sqrt{(u_1\cdot u_2)^2-1}} u_1^\mu u_2^\nu \Sigma_{\mu\nu}\Big)^\rho{}_\sigma\, u_1^\sigma. \tag{4.2}$$

The corresponding spinor transformations, given by eq. (2.17), may be rewritten as

$$|u_2^a\rangle = \frac{\sqrt{M_1}}{\sqrt{M_2}}\Big(|u_1^a\rangle + \frac{|k|u_1^a]}{M_1+M_2}\Big), \qquad |u_2^a] = \frac{\sqrt{M_1}}{\sqrt{M_2}}\Big(|u_1^a] + \frac{|k|u^a\rangle}{M_1+M_2}\Big), \tag{4.3}$$

where we have used $\mu := u_1 \cdot u_2 + \sqrt{(u_1 \cdot u_2)^2 - 1} = M_2/M_1$. The net effect of this is that the projection of the massive spinors onto the directions $|k\rangle$ and $|k]$ is invariant under this boost, so the spherical harmonics are related simply by

$$\langle 2^a k\rangle = \langle 1^a k\rangle, \qquad [2^a k] = [1^a k] \qquad \Rightarrow \qquad {}_h\tilde{Y}_{s_2,m}(k; u_2) = {}_h\tilde{Y}_{s_2,m}(k; u_1). \tag{4.4}$$

(This is because we switch between rest frames of p_1 and $p_2 = p_1 + k$ inside the harmonics in the same direction k.) The caveat here is that the spin of particle 2 is now quantized along $L^\mu{}_\nu(u_2 \leftarrow u_1)n_1^\sigma$, i.e. the boost of the spin quantization axis of particle 1, which may be arbitrary but has to be the same for every $p_2 = p_1 + k$. With this restriction in mind, we may rewrite the three-point amplitude as

$$\mathcal{A}^h_{s_2,m}(p_2|p_1;k) = M_1\, g^{|h|}_{0,0,s_2}(w)(-1)^{s_2-h}w^{s_2}\,{}_h\tilde{Y}_{s_2,m}(k;u_1). \tag{4.5}$$

Let us now introduce the spherical scattering amplitude[5]

$$\mathcal{A}_{\{b\}}(p_2,s_2|p_1;\omega,j,m,h) := \frac{4\pi}{\sqrt{2\omega}}\int_k \hat{\delta}(k\cdot u_1-\omega)\,{}_{-h}Y_{j,m}(k;u_1)\mathcal{A}_{\{b\}}(p_2,s_2|p_1;k,h) \tag{4.6}$$

in an analogous manner to eq. (2.11). Using the conjugation and orthogonality properties (2.13) and (2.15), we find

$$\begin{aligned}
&\mathcal{A}_{\underbrace{1\ldots1}_{s_2-m'}\underbrace{2\ldots2}_{s_2+m'}}(p_2,s_2|p_1;\omega,j,m,h)\\
&= \frac{(-1)^{-2j+m+h}}{\pi\sqrt{2\omega}}\int d^4k\,\delta^+(k^2)\delta(k\cdot u_1-\omega)\,{}_hY^*_{j,-m}(k;u_1)\mathcal{A}^h_{s_2,m'}(p_2|p_1;k)\\
&= (-1)^{-j}\delta^j_{s_2}\delta^{-m}_{m'}M_1^{3/2}\Big[4\pi(2j+1)\tbinom{2j}{j+m}\tbinom{2j}{j+h}\Big]^{-1/2} g^{|h|}_{0,0,j}(w)\,w^{j+1/2}\big|_{w=2\omega/M_1}.
\end{aligned} \tag{4.7}$$

This neatly expresses the angular-momentum conservation law. This simple form of the spherical scattering amplitude is valid under our assumption that the magnetic quantum number m' is defined with respect to the axis $L^\mu{}_\nu(u_2 \leftarrow u_1)n_1^\sigma$.

4.2 Leading-order absorption cross-section

We are now ready to construct the leading absorption cross-section from the three-point amplitudes discussed above. The inclusive cross-section for the spherical scattering setup described in section 2.1 is [88, 91]

$$\sigma_{\text{inc}}(\omega_{\text{cl}},j,m,h) = \frac{\pi}{\omega_{\text{cl}}^2}P_{\text{inc}}(\omega_{\text{cl}},j,m,h) = \frac{\pi}{\omega_{\text{cl}}^2}\sum_X \frac{\big|\langle X|S|\psi;\gamma\rangle\big|^2}{\langle X|X\rangle\langle\psi|\psi\rangle\langle\gamma|\gamma\rangle}, \tag{4.8}$$

It is invariant under the basis choice for the outgoing states. The leading contribution due to absorption is then given by the 3-point process:

$$P^{\text{LO}}_{\text{inc}}(\omega_{\text{cl}},j,m,h) = V\int_0^\infty dM_2^2\rho(M_2^2)\int \hat{d}^3p_2\,\frac{\big|\langle p_2|S|\psi;\gamma\rangle\big|^2}{\langle p_2|p_2\rangle\langle\psi|\psi\rangle\langle\gamma|\gamma\rangle}. \tag{4.9}$$

Here $V := \langle p_2|p_2\rangle/(2p_2^0) = \hat{\delta}^3(\mathbf{0})$ is the space volume, which immediately cancels against the normalization of the outgoing state, for which we have temporarily suppressed any

[5]Note that the definition (4.6) ignores the delta function $\hat{\delta}^4(p_1+k-p_2)$, which accompanies the scattering amplitude and imposes momentum conservation. Although it will play a role in the cross-section calculation in the next section, the above definition can still be found useful.

quantized degrees of freedom. We have also been compelled to include the spectral density $\rho(M_2^2)$, which is positive and normalized to 1:

$$\rho(q^2) \geq 0, \qquad \int_0^\infty \rho(q^2) dq^2 = 1. \tag{4.10}$$

In a conservative scenario, one may simply assume $\rho(q^2) = \delta(q^2 - M_1^2)$, and the relevant amplitude would be the same-mass three-point amplitude. More generally, it is allowed to contain suitably normalized delta-functions for the "elementary" particles and the continuous part due to multi-particle states. Since we are interested in modeling absorption effects, we are led to explore the continuous part of the spectrum for $q^2 > M_1^2$. It can be checked that without a continuous part of the spectral-density function the three-point kinematics would be overconstrained, and the cross-section integration would yield a distribution.

In view of the normalization of the initial states, $\langle\psi|\psi\rangle = \langle\lambda|\lambda\rangle = 1$, the resulting leading-order probability is given by

$$P_{\rm inc}^{\rm LO}(\omega_{\rm cl}, j, m, h) = \sum_{s_2} \int dM_2^2 \rho_{s_2}(M_2^2) \int_{p_2} \sum_{b_1,\ldots,b_{s_2}} \left|\langle p_2, s_2, \{b\}|S|\psi;\gamma\rangle\right|^2, \tag{4.11}$$

where we have now made the spin degrees of freedom of the outgoing state explicit. The integration over masses of p_2 different from M_1 is what allows the three-point amplitude to exist on real kinematics and thus makes this cross-section meaningful. As we will see, momentum conservation will later fix this mass to

$$M_2^2 = M_1^2 + 2M_1\omega_{\rm cl}. \tag{4.12}$$

After restoring $\hbar$ in front of $\omega_{\rm cl}$, it actually becomes sent back to M_1 in the classical limit, so the spectral density will only by probed in the vicinity of the original BH mass. This, however, does not negate the crucial roles that the unequal masses and the spectral density play in allowing for a non-singular construction of the cross-section from three-point amplitudes.

Coming back to the squared amplitude in the integrand of eq. (4.11), we have

$$\begin{aligned}
&\sum_{\{b\}} \left|\langle p_2, s_2, \{b\}|S|\psi;\gamma\rangle\right|^2 = \\
&8\pi^2 \int_0^\infty \frac{\hat{d}\omega \hat{d}\omega'}{\sqrt{\omega\omega'}} \gamma_\zeta^*(\omega)\gamma_\zeta(\omega') \int_{p_1,p_1',k,k'} \psi_\xi^*(p_1)\psi_\xi(p_1') \\
&\times \hat{\delta}(k\cdot u_1 - \omega)\hat{\delta}(k'\cdot u_1 - \omega')\hat{\delta}^4(p_1 + k - p_2)\hat{\delta}^4(p_1' + k' - p_2) \\
&\times {}_{-h}Y_{j,m}^*(k;u_1)\, {}_{-h}Y_{j,m}(k';u_1)\, \mathcal{A}^{*\{b\}}(p_2, s_2|p_1;k,h)\, \mathcal{A}_{\{b\}}(p_2, s_2|p_1';k',h),
\end{aligned} \tag{4.13}$$

where the summation over the little-group indices $\{b\}$ is now implicit. We may use $\hat{\delta}^4(p_1 + k - p_2)$ to perform the integration over p_2, which leaves the on-shell constraint $\hat{\delta}((p_1+k)^2 - M_2^2)$. We then change the integration variables to

$$p_{\rm a}^\mu := (p_1^\mu + p_1'^\mu)/2, \qquad q^\mu := p_1'^\mu - p_1^\mu, \tag{4.14}$$

and remove q with $\hat{\delta}^4(q+k'-k)$ originating from $\hat{\delta}^4(p_1'+k'-p_2)$. Thus we get

$$\begin{aligned}
&P^{\rm LO}_{\rm inc}(\omega_{\rm cl},j,m,h) = \\
&8\pi^2 \sum_{s_2} \int dM_2^2 \rho_{s_2}(M_2^2) \int_0^\infty \frac{\hat{d}\omega \hat{d}\omega'}{\sqrt{\omega\omega'}} \gamma^*_\zeta(\omega)\gamma_\zeta(\omega') \int_{k,k'} \hat{\delta}(k\cdot u_1-\omega) \\
&\times \hat{\delta}(k'\cdot u_1-\omega')\, {}_{-h}Y^*_{j,m}(k;u_1)\, {}_{-h}Y_{j,m}(k';u_1) \int \hat{d}^4 p_{\rm a}\, \hat{\delta}^+(p_{\rm a}^2-M_1^2-k'\cdot k/2)|\psi_\xi(p_{\rm a})|^2 \\
&\times \hat{\delta}(2p_{\rm a}\cdot k-2p_{\rm a}\cdot k')\hat{\delta}(M_1^2+2p_{\rm a}\cdot k+k'\cdot k-M_2^2)\, \mathcal{A}^{*\{b\}}\left(p_{\rm a}+\tfrac{k+k'}{2}, s_2|p_{\rm a}+\tfrac{k'-k}{2};k,h\right) \\
&\times \mathcal{A}_{\{b\}}\left(p_{\rm a}+\tfrac{k+k'}{2}, s_2|p_{\rm a}+\tfrac{k-k'}{2};k',h\right),
\end{aligned} \tag{4.15}$$

where we have also used the convenient property $\psi^*_\xi(p_{\rm a}-\frac{q}{2})\psi_\xi(p_{\rm a}+\frac{q}{2}) = |\psi_\xi(p_{\rm a})|^2$ of the momentum wavepackets (2.19).

4.3 Absorption cross-section in classical limit

So far no classical limit was taken, and eq. (4.15) still represents a quantum probability. To rectify that, we send $\xi\to 0$ and evaluate the integral over $p_{\rm a}$, which in the presence of the squared wavefunction $|\psi_\xi(p_{\rm a})|^2$ and the mass-shell delta function has the effect of setting the momentum $p_{\rm a}^\mu$ to its classical value $u_1^\mu\sqrt{M_1^2+k'\cdot k/2} =: M_{\rm a}u_1^\mu$. Subsequently, using the delta function $\hat{\delta}(2p_{\rm a}\cdot k-2p_{\rm a}\cdot k')$ becomes $\hat{\delta}(\omega-\omega')/(2M_{\rm a})$, which removes the integration over ω'. In the integral over the remaining ω, we send $\zeta\to 0$, so the squared wavefunction $|\gamma_\zeta(\omega)|^2$ localizes it at the classical value $\omega_{\rm cl}$. In this way, the above probability becomes

$$\begin{aligned}
&\lim_{\zeta\to 0}\lim_{\xi\to 0} P^{\rm LO}_{\rm inc}(\omega_{\rm cl},j,m,h) = \\
&\frac{16\pi^3}{\omega_{\rm cl}} \sum_{s_2} \int_{k,k'} \frac{1}{2M_{\rm a}} \rho_{s_2}(M_1^2+2M_{\rm a}\omega_{\rm cl}+k'\cdot k)\hat{\delta}(k\cdot u_1-\omega_{\rm cl}) \\
&\times \hat{\delta}(k'\cdot u_1-\omega_{\rm cl})\, {}_{-h}Y^*_{j,m}(k;u_1)\, {}_{-h}Y_{j,m}(k';u_1)\, \mathcal{A}^{*\{b\}}\left(p_{\rm a}+\tfrac{k+k'}{2}, s_2|p_{\rm a}+\tfrac{k'-k}{2};k,h\right) \\
&\times \mathcal{A}_{\{b\}}\left(p_{\rm a}+\tfrac{k+k'}{2}, s_2|p_{\rm a}+\tfrac{k-k'}{2};k',h\right)\Big|_{p_{\rm a}=M_{\rm a}u_1},
\end{aligned} \tag{4.16}$$

where we have also taken the integral over M_2^2 using $\hat{\delta}(M_1^2+2M_{\rm a}\omega+k'\cdot k-M_2^2)$.

Even though we have simplified the probability expression considerably, the integrals over k^μ and k'^μ are still intertwined, in particular because the spectral density and $M_{\rm a}$ both depend on $k\cdot k'$. Note, however, that the two massless momenta are constrained to have the energy projection $\omega_{\rm cl}$, so $|k\cdot k'|\le 2\omega_{\rm cl}^2$, as most easily seen in the rest frame of u_1^μ. The basic classical-limit assumption $\omega_{\rm cl}\ll M_1$ then implies

$$|k^\mu|, |k'^\mu| \;\ll\; M_1 \qquad \Rightarrow \qquad |k\cdot k'| \;\ll\; M_1 u_1\cdot k = M_1 u_1\cdot k' = M_1\omega_{\rm cl}. \tag{4.17}$$

Therefore, we may define the classical limit of the above probability as

$$\begin{aligned}
P^{\rm LO}_{\rm inc,\,cl} = \frac{8\pi^3}{M_1\omega_{\rm cl}} \sum_{s_2} \rho_{s_2}(M_1^2) \int_{k,k'} &\hat{\delta}(k\cdot u_1-\omega_{\rm cl})\, {}_{-h}Y^*_{j,m}(k;u_1)\, \mathcal{A}^{*\{b\}}(p_2,s_2|p_1;k,h) \\
&\times \hat{\delta}(k'\cdot u_1-\omega_{\rm cl})\, {}_{-h}Y_{j,m}(k';u_1)\, \mathcal{A}_{\{b\}}(p_2,s_2|p_1';k',h),
\end{aligned} \tag{4.18}$$

where for brevity we have now used the momenta

$$p_1 = M_1 u_1 + \tfrac{k'-k}{2}, \qquad p_1' = M_1 u_1 + \tfrac{k-k'}{2}, \qquad p_2 = M_1 u_1 + \tfrac{k+k'}{2} =: M_2 u_2 \tag{4.19}$$

not as independent integration variables but to denote their classical values. Note that in the expression above, we have already assumed that the outgoing states are described by a sufficiently smooth spectral-density function, which makes sense because our EFT is meant to describe absorption of classical waves of arbitrary frequency (provided it is small). Therefore, ρ_{s_2} can be expanded in $\omega_{\rm cl}/M_1$, for which $2M_1\omega_{\rm cl}$ and $k'\cdot k$ provide linear and quadratic terms, respectively, and both may be dropped, leaving only the leading term $\rho_{s_2}(M_1^2)$ in the classical limit.

Let us now deal with the momentum dependence of the amplitudes, which, as we have noticed in eq. (4.1), are proportional to the covariant spin-weighted spherical harmonics ${}_h\tilde{Y}_{s_2,m'}(k;u_2)$, while their prefactors depend on the dimensionless ratio

$$w := \frac{2p_1\cdot k}{M_1^2} \simeq \frac{2\omega_{\rm cl}}{M_1} \simeq \frac{2p_1'\cdot k'}{M_1^2} =: w'. \tag{4.20}$$

Moreover, just as we did in section 4.1, we may boost the time direction u_2^μ of either harmonic to our preferred u_1^μ, with their difference now being equal to $(k+k')^\mu/2$, but the result still being[6] ${}_h\tilde{Y}_{s_2,m'}(k;u_2) \simeq {}_h\tilde{Y}_{s_2,m'}(k;u_1)$. Therefore, the squared amplitude is

$$\begin{aligned}&\mathcal{A}^{*\{b\}}(p_2,s_2|p_1;k,h)\mathcal{A}_{\{b\}}(p_2,s_2|p_1';k',h)\\ &\simeq M_1^2\,|g_{0,0,s_2}^{|h|}(w)|^2 w^{2s_2}\sum_{m'=-s_2}^{s_2}\tbinom{2s_2}{s_2+m'}\,{}_h\tilde{Y}^*_{s_2,m'}(k;u_1)\,{}_h\tilde{Y}_{s_2,m'}(k';u_1).\end{aligned}\tag{4.21}$$

Having thus completely disentangled the integrations in k and k', we may evaluate

$$\begin{aligned}&P^{\rm LO}_{\rm inc,\,cl}(\omega_{\rm cl},j,m,h)\\ &=\frac{8\pi^3}{\omega_{\rm cl}}M_1\sum_{s_2}\rho_{s_2}(M_1^2)|g^{|h|}_{0,0,s_2}(w)|^2w^{2s_2}\\ &\quad\times\sum_{m'=-s_2}^{s_2}\binom{2s_2}{s_2+m'}\bigg|\int\hat{d}^4k\,\hat{\delta}^+(k^2)\hat{\delta}(k\cdot u_1-\omega_{\rm cl})\,{}_{-h}Y_{j,m}(k;u_1)\,{}_h\tilde{Y}_{s_2,m'}(k;u_1)\bigg|^2\\ &=\frac{M_1^2}{4(2j+1)}\binom{2j}{j+h}^{-1}\rho_j(M_1^2)\,|g^{|h|}_{0,0,j}(w)|^2\,w^{2j+1},\end{aligned}\tag{4.22}$$

[6] More explicitly, one may use the most general spinor transformations (B.6) to observe:

$$|2^b\rangle = U^b{}_a(u_2\!\leftarrow\! u_1)\sqrt{M_1}\bigg\{|u_1^a\rangle + \frac{|k+k'|u_1^a]}{2(M_1+M_2)}\bigg\} \;\Rightarrow\; \langle 2^b k\rangle = U^b{}_a(u_2\!\leftarrow\! u_1)\sqrt{M_1}\Big\{\langle u_1^a k\rangle + \mathcal{O}\big([u_1^a k']\tfrac{\langle k'k\rangle}{M_1}\big)\Big\},$$

and similarly for the anti-chiral spinors. We have thus exposed the spinorial (square-root) version of the classical hierarchy assumption (4.17). Coming back to the original three-point amplitude (3.14), one can then expose its classically meaningful term as (proportional to)

$$\langle 2^b k\rangle^{\odot(s_2-h)}\odot[2^b k]^{\odot(s_2+h)} \;\simeq\; M_1^{s_2}\big\{(U^b{}_a(u_2\!\leftarrow\! u_1)\big\}^{\odot 2s_2}\langle u_1^a k\rangle^{\odot(s_2-h)}\odot[u_1^a k]^{\odot(s_2+h)},$$

Moreover, the unitarity of the SU(2) transformation matrices $U^b{}_a(u_2\leftarrow u_1)$ ensures that they cancel in all inclusive-probability expressions, such as eq. (4.21), and hence justifies our liberal treatment of the little-group indices, also phrased as the quantization-axis choice assumption in section 4.1.

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where we have used the conjugation and orthogonality properties (2.13) and (2.15).[7] Note that its power in $\omega_{\rm cl} = M_1 w/2$ is dictated by the total angular-momentum quantum number j of the incoming wave, which also determines the absorptive three-point coupling that is being probed.

In this way, we have arrived at the partial-wave absorption cross-section

$$\begin{aligned}\sigma^{\rm LO}_{\rm inc,\,cl}(\omega_{\rm cl},j,m,h) &:= \frac{\pi}{\omega_{\rm cl}^2} P^{\rm LO}_{\rm inc,\,cl}(\omega_{\rm cl},j,m,h) \\ &= \frac{\pi}{4\omega_{\rm cl}^2}\frac{(j+h)!(j-h)!}{(2j+1)!} M_1^2 \rho_j(M_1^2)\,|g^{|h|}_{0,0,j}(w)|^2\, w^{2j+1},\end{aligned} \tag{4.23}$$

where $w = 2\omega_{\rm cl}/M_1$. We will deal with the apparent issue of w being small for $\omega_{\rm cl} \ll M_1$ in the next section.

5 Matching to microscopic calculation

The absorptive cross-section of a Kerr black hole in general relativity was originally obtained by Starobinsky, Churilov [80, 81] and Page [82, 83] for the $j = |h|$ case and recently generalized to arbitrary j in [99–101]. However, the dynamics of non-spinning BHs under small perturbations dates back to Regge and Wheeler [113], who proved linear stability of Schwarzschild BHs. From the point of view of the EFT amplitudes, which treat the BH as a particle, the GR results serve as the microscopic computation, to which the effective couplings should be matched.

5.1 Classical absorption cross-section

In the general case of wave of spin $|h|$ scattering off a spinning BH, the transmission and scattering coefficients are usually obtained by solving the Teukolsky equation [96–98]. In this work, we focus on the simpler case of non-spinning BHs. Let the Schwarzschild radius be $r_{\rm S} := 2GM_1$ and ω the frequency of the classical spin-$|h|$ wave, which obey $r_{\rm S}\omega \ll 1$. Then the absorption cross-section is given by [101][8]

$$\sigma^{\rm Schw}_{\rm abs}(\omega,j,m,h) = \frac{(-1)^h 2\pi}{\omega^2}\frac{(j+h)!(j-h)!}{(2j)!(2j+1)!}(2r_{\rm S}\omega)^{2j+1}{\rm Im}F^{\rm Schw}_{-hjh}(\omega). \tag{5.1}$$

Here $F^{\rm Schw}_{hjm}$ is the harmonic near-zone response function

$$F^{\rm Schw}_{hjm}(\omega) = i(-1)^h\, r_{\rm S}\omega\, \frac{(j+h)!(j-h)!}{(2j)!(2j+1)!}\prod_{l=1}^{j}\left[l^2 + (2r_{\rm S}\omega)^2\right], \tag{5.2}$$

[7] Alternatively, the result of eq. (4.22) may be obtained by plugging in the previously computed spherical scattering amplitudes (4.7) with classical momentum values (4.19):

$$P^{\rm LO}_{\rm inc,\,cl}(\omega_{\rm cl},j,m,h) = \frac{\pi}{M_1}\rho_{s_2}(M_1^2)\sum_{m'=-s_2}^{s_2}\binom{2s_2}{s_2+m'}\big|\mathcal{A}_{\underbrace{1\ldots1}_{s_2-m'}\underbrace{2\ldots2}_{s_2+m'}}(p_2,s_2|p_1;\omega_{\rm cl},j,m,h)\big|^2.$$

[8] We have dropped the prefactor $(2j+1)$ from the expressions in the literature, which comes from summing over $m = -j,\ldots,j$.

which does not depend on the quantum number m, since we wrote it for a non-spinning black hole. We have followed the GR literature [99, 101] in writing the cross-section (5.1) using the response function so as to point out that it is the latter that contains the expansion in ω, whereas the outside power of ω is fixed to be $2j-1$, combined from the π/ω^2 dimensionful prefactor and $2j+1$ powers of a dimensionless frequency combination. This factorization mimics the structure of the corresponding EFT cross-section (4.23), that we are going to match to next.

Our focus, however, is on the leading powers in ω for each j, which amounts to replacing the complicated product in the response function (5.2) by $(j!)^2$. We obtain

$$\sigma_{\text{abs, LO}}^{\text{Schw}}(\omega, j, m, h) = 4\pi r_{\text{S}}^2 \left[\frac{j!(j+h)!(j-h)!}{(2j)!(2j+1)!}\right]^2 (2r_{\text{S}}\omega)^{2j}, \tag{5.3}$$

where of course $|m|, |h| \leq j$, and otherwise it vanishes.

5.2 Scales and effective couplings

In order to properly compare the classical and EFT results, it is helpful to restore $\hbar$ (while leaving $c=1$ throughout this paper). This introduces the distinction between frequencies/lengths and masses:

$$[\hbar] = L\times M, \quad [M_1] = [\omega_{\text{cl}}] = M, \quad [\omega] = L^{-1}, \quad [r_{\text{S}}] = L, \quad [G] = L\times M^{-1}, \tag{5.4}$$

where we have insisted on the new meaning of $\omega := \omega_{\text{cl}}/\hbar$ as the wave frequency. We should also multiply the right-hand side of the cross-section given from (4.23) by $\hbar^2$, so as to switch its dimensionality from M^{-2} to L^2. This gives

$$\sigma_{\text{inc, cl}}^{\text{LO}}(\omega, j, m, h) = \frac{\pi}{4\omega^2}\frac{(j+h)!(j-h)!}{(2j+1)!} M_1^2 \rho_j(M_1^2)\, |g_{0,0,j}^{|h|}(\omega)|^2 \left(\frac{2\hbar\omega}{M_1}\right)^{2j+1}. \tag{5.5}$$

Here we have left the effective couplings $g_{0,0,s_2}(\omega)$ fully dimensionless. Note, however, that in view of the presence of multiple scales, they are now allowed to depend on ω through more than just the $\hbar\omega/M_1$ ratio.

Let us discuss the two basic assumptions underlying the EFT- and GR-based computations, i.e. $\hbar\omega \ll M_1$ and $r_{\text{S}}\omega \ll 1$. The point is that the latter is a much stronger inequality than the former, as the Schwarzschild radius must of course be assumed to be many orders of magnitude larger than the Compton wavelength of the black hole:

$$\omega \ll \frac{1}{r_{\text{S}}} \ll \frac{1}{\lambda_{\text{C}}} := \frac{M_1}{2\pi\hbar}. \tag{5.6}$$

Otherwise, we would be in the realm of quantum gravity and not GR. It is then clear that in the context of comparing the classical and amplitude-based results, which both constitute frequency expansions, we should retain only the leading order in $\hbar\omega/M_1$, but classical frequency dependence may still be present in the form of $r_{\text{S}}\omega$.

Therefore, matching the leading-order cross-sections (5.3) and (5.5) directly, we obtain

$$M_1^2 \rho_j(M_1^2)\, |g_{0,0,j}^{|h|}(\omega)|^2 = \frac{8[j!]^2(j+h)!(j-h)!}{[(2j)!]^2(2j+1)!}\left(\frac{M_1 r_{\text{S}}}{\hbar}\right)^{2j+1} r_{\text{S}}\omega. \tag{5.7}$$

It is perhaps more aesthetically pleasing to rephrase this relationship in terms of the classical response function:

$$M_1^2\rho_j(M_1^2)\,|g_{0,0,j}^{|h|}(\omega)|^2 = \frac{8(-1)^h}{(2j)!}\left(\frac{M_1 r_\mathrm{S}}{\hbar}\right)^{2j+1}\mathrm{Im}F^{\mathrm{Schw}}_{-hjh,\,\mathrm{LO}}(\omega). \tag{5.8}$$

In other words, we have related the j-th effective absorptive coupling squared to the imaginary part of the response function, resembling a dispersion relation. It might seem awkward to keep $\hbar$ in the now classically meaningful cross-section expression (5.5), as well as eqs. (5.7) and (5.8). However, the effective couplings are a priori arbitrary, and we are free to make convenient modeling assumptions about them, so nothing prevents us from absorbing the Planck constants into them as[9]

$$\bar{g}_{0,0,s_2}^{|h|}(\omega) := \hbar^{s_2+1/2} g_{0,0,s_2}^{|h|}(\omega). \tag{5.9}$$

Comparing the macroscopic and microscopic formulae (5.1) and (5.5), there are a number of things to observe.

- Both cross-sections are consistent in that neither depends on the magnetic quantum number m of the spherical wave.
- The EFT cross-section (5.5) reproduces the static limit $\sigma^{\mathrm{LO}}_{\mathrm{inc,cl}}(\omega=0,j,m,h) = 0$ for electromagnetism and gravity ($|h| = 1$ and 2, respectively) because of the locality assumption (3.20) that the Wilson coefficients have no negative powers of ω. This can be considered as an EFT prediction, i.e. it holds prior to the matching of the three-point couplings.
- As previously mentioned, the growth of the superficial leading power in ω with j is the same in both cross-sections, where by superficial we mean excluding the ω dependence in the response function and the three-point couplings. In other words, the matching (5.7) contains that same leading power of ω for any j, and the cleaner matching (5.8) between the response functions and the three-point couplings does not involve ω at all.
- In the EFT cross-section (5.5), every three-point coupling $|g_{0,0,s_2}^{|h|}(\omega)|^2$ comes accompanied by the dimensionless combination $M_1^2\rho_{s_2}(M_1^2)$ involving the spectral density. Its appearance is very sensible from the QFT point of view, as the probability that a massive particle absorbs a lower-energy massless boson is necessarily proportional to the number of possible resulting states with nearly identical mass. However, since it always accompanies the couplings, one may regard the complete expression $M_1\sqrt{\rho_{s_2}(M_1^2)}\,g_{0,0,s_2}^{|h|}(\omega)$ as a kind of effective coupling. Alternatively, if one's focus is on modeling classical effects that are guaranteed to be insensitive to the difference between spectral densities for different masses and spins, one could consider disregarding the normalization constraint (4.10) altogether and make a modeling assumption $\rho_{s_2}(M_1^2) = 1/M_1^2$.

[9] Recalling the form of the three-point amplitude (3.14), we see that the effective-coupling rescaling (5.9) amounts to replacing massless momenta k^μ with wavevectors $\bar{k}^\mu := k^\mu/\hbar$, which is commonplace in the KMOC formalism [48], plus an additional overall $\hbar^{-1/2}$.

- Perhaps most importantly, we observe that the matching (5.7) means that

$$g^{|h|}_{0,0,s_2}(\omega) = \mathcal{O}(G^{s_2+1}), \tag{5.10}$$

in the post-Minkowskian expansion, since $r_\mathrm{S} = GM$. In other words, the amplitude that the scalar particle which models a Schwarzschild black hole absorbs a spherical wave with total angular momentum j is a $(j+1)$-PM object.

- For gravity ($|h| = 2$), the PM behavior (5.10) means that the Wilson coefficient starts at $s_2 = 2$ and scales as $\mathcal{O}(G^3)$, whereas the resulting leading absorption cross-section is at 6PM for a $j = 2$ spherical wave, and higher harmonics are suppressed in the PM expansion.

In view of the classical cross-section (5.1) being a polynomial in ω spanned by $\{\omega^{2j}, \ldots, \omega^{4j}\}$, one might hope that higher orders in $r_\mathrm{S}\omega$ could be retained, as long as they are captured by the response function (5.2) in a perturbation scheme [101] that is consistent classically. Unfortunately, this is not the case in the present three-point setup, because going to higher orders requires a more subtle matching. Indeed, the higher orders in $r_\mathrm{S}\omega$ in the EFT cross-sections (5.5) are subject to interference from higher-multiplicity amplitudes. More specifically, the next order in the cross-section is $\mathcal{O}(G^{2j+4})$, for which the EFT treatment must, for instance, include amplitudes with two additional conservative couplings to the graviton, each $\mathcal{O}(\sqrt{G})$. Furthermore, double-graviton absorption or even the mass-changing contact terms contribution to the Compton amplitude might contribute to this matching. We will discuss these matters further below in sections 6.4 and 6.5.

Generalizing this result to spinning objects is another story. In the non-spinning case, the coupling constant G only enters in the Schwarzschild radius r_S, whereas in the Kerr case where the dimensionless spin ratio $a_* = a/GM$ also contains negative powers in G. This shows that for Schwarzschild black holes, the first contribution to such amplitudes is at 6PM (as can be reproduced by off-shell EFT methods [88, 89]), while it comes at a lower order for Kerr black holes due to the negative power of G in a_*. For instance, the authors of [92] consider four-point contact interactions, where such effects come at spin-5 in $\mathcal{O}(G)$ amplitudes. Nevertheless, the general formalism presented in this paper does allow to go to higher orders in spin, and we leave this for future work.

In this purely on-shell approach, we have modeled the absorption effects by allowing a changing-mass amplitude from $s_1 = 0$ to a spinning degree of freedom and the leading order corresponds to a $s_2 = 2$ particle. We have observed some similarities with the worldline EFT approach [88, 89, 114], where the point-particle action coupled to the Weyl tensor is not enough to model absorption. One then has to introduce electric and magnetic composite operators Q^E_{ab} and Q^B_{ab} representing new degrees of freedom, which carry two indices and couple to electric and magnetic components of the Weyl tensor E^{ab} and B^{ab}, respectively. While in our approach higher orders require considering $s_2 \geq 2$ particles and higher-multiplicity amplitudes, on the worldline higher-derivative operators acting on the Weyl tensor and multi-index composite operators are needed to improve the calculation beyond ω^4, which is explored e.g. in [92].

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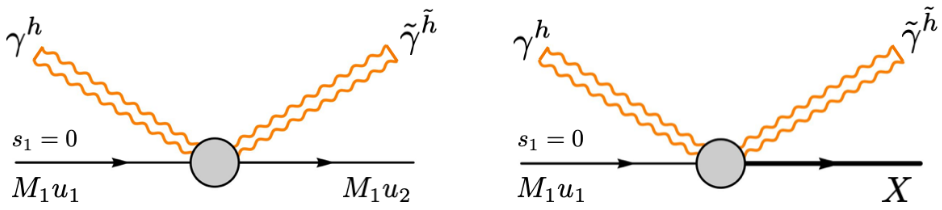

Figure 2. Gravitational diagrams in a non-spinning black-hole-wave interaction.

6 Coherent-state cross-section

A proper description of the interaction between a gravitational wave and a compact object using scattering amplitudes requires the use of a coherent-state formalism to model the incoming and outgoing wave [51, 91, 115]. In section 2, we have circumvented it by using a single-graviton state with a wavefunction peaked at the classical frequency $\omega_{\rm cl}$. The point of this section is two-fold:

- substantiate the leading-order calculation via the coherent-state framework,
- explain how higher-order calculations may be done in a similar fashion.

We focus on a coherent cross-section-based (or probability-based) formalism instead of an observable-based one [48]. We start with a quantum description and make gradual assumptions relevant to the classical limit.

6.1 Elastic-inelastic separation

The initial state for our absorption process consists of a heavy non-spinning particle $|\psi_1\rangle$ and a wave of helicity h modeled by a massless coherent state $|\gamma^h\rangle$. We write

$$|\text{in}\rangle := |\psi_1; \gamma^h\rangle = \int_{p_1} \psi_\xi(p_1) e^{ib\cdot p_1/\hbar} |p_1; \gamma^h\rangle, \tag{6.1}$$

where the relativistic momentum-space wavefunction $\psi_\xi(p_1)$ peaks at the classical momenta $p^\mu_{1,\rm cl} = M_1 u_1^\mu$, as discussed in section 2. We have also allowed for an impact parameter. For the final state, we should distinguish two cases:

(c) a different coherent state $|\tilde{\gamma}^{\tilde{h}}\rangle$, but the heavy particle's mass is preserved;

(nc) a different coherent state $|\tilde{\gamma}^{\tilde{h}}\rangle$ and an unspecified particle $|X\rangle$ with $M_2 \neq M_1$.

The two cases are depicted in figure 2, and we need to integrate over the possible final states. Despite these assumptions, the formalism easily allows for initial spinning states, and we delay the specification of the massless coherent-state type (plane-wave or partial-wave) to later on. It is also worth commenting that even though case (c) has the same mass as the initial state, intermediate mass transitions are allowed (e.g. Compton scattering with different masses in the factorization channels).

The need to separate these two cases on the quantum side comes from the discontinuous nature of basic scattering-amplitude building blocks at $M_2 = M_1$, as discussed in section 3, and on the classical side from the usual separation between conservative and non-conservative effects. The total probability will then include the following mass-preserving and mass-changing probabilities

$$P_{\gamma\to\tilde{\gamma}} = P^{(\mathrm{c})}_{\gamma\to\tilde{\gamma}} + P^{(\mathrm{nc})}_{\gamma\to\tilde{\gamma}}. \tag{6.2}$$

For the first one, we may write

$$P^{(\mathrm{c})}_{\gamma\to\tilde{\gamma}} = \sum_{2s_2=0}^{\infty} \sum_{b_1,\dots,b_{2s_2}=1,2} \int_{p_2} \langle \mathrm{in}|S^\dagger|p_2, s_2, \{b\}; \tilde{\gamma}^{\tilde{h}}\rangle\langle p_2, s_2, \{b\}; \tilde{\gamma}^{\tilde{h}}|S|\mathrm{in}\rangle. \tag{6.3}$$

In the second case, we are interested in the probability of all different configurations X involving a heavy particle of mass $M_2 \neq M_1$:

$$P^{(\mathrm{nc})}_{\gamma\to\tilde{\gamma}} = \sum_{X\ni M_2\neq M_1} |\langle X; \tilde{\gamma}^{\tilde{h}}|S|\mathrm{in}\rangle|^2 = \sum_{X\ni M_2\neq M_1} \langle \mathrm{in}|S^\dagger|X; \tilde{\gamma}^{\tilde{h}}\rangle\langle X; \tilde{\gamma}^{\tilde{h}}|S|\mathrm{in}\rangle. \tag{6.4}$$

The crucial point now is to determine what part of the Hilbert space contributes to the problem at hand. We are going to assume that all relevant configurations contain only one heavy particle; in other words, in the classical limit no new black holes are created in this S-matrix evolution. Let us also exclude decay of the heavy particle, i.e. black-hole evaporation, from current consideration. In other words, we assume that the spectral density[10] of the heavy-particle states has a non-trivial continuous part only for $M_2 > M_1$ (alongside the delta-function responsible for case (c)):

$$1^{(\mathrm{nc})} = \sum_{X_{\mathrm{rad}}} \sum_{s_2} \sum_{\{b\}} \int_{M_1^2}^{\infty} dM_2^2 \rho_{s_2}(M_2^2) \int_{p_2} |p_2, s_2, \{b\}; X_{\mathrm{rad}}\rangle\langle p_2, s_2, \{b\}; X_{\mathrm{rad}}|. \tag{6.5}$$

The above "completeness" relation should normally also include a sum over possible emitted radiation

$$|X_{\mathrm{rad}}\rangle\langle X_{\mathrm{rad}}| = \sum_{n=0}^{\infty} \sum_{h_1,\cdots,h_n} \int_{k_1,\cdots,k_n} |k_1^{h_1}; \cdots; k_n^{h_n}\rangle\langle k_1^{h_1}; \cdots; k_n^{h_n}|. \tag{6.6}$$

However, we choose to make another assumption that all the outgoing radiation belongs coherently to the wave $\tilde{\gamma}$, and there is no extra scattered photons/gravitons. In other words, the final state is given by $|p_2, \beta; \tilde{\gamma}^{\tilde{h}}\rangle$ and not $|p_2, \beta; \tilde{\gamma}^{\tilde{h}}; k_1^{h_1}; k_2^{h_2}; \cdots\rangle$, which was also assumed for the mass-preserving case (6.3). This assumption relies on the expectation

[10] In the coherent-spin approach to the classical limit [52], the SU(2) spinors β_b that saturate the little-group indices of the amplitude determine the resulting classical angular momentum of the compact object. So one could trade the s_2-dependence of the spectral density in eq. (6.5) for the perhaps more appropriate dependence on $\hbar\|\beta\|^2 = 2\sqrt{-S_{\mathrm{cl}}^2}$ and use the coherent-spin final-state integration. All formulae in this section may be adjusted accordingly, starting with eq. (6.3) which becomes

$$P^{(\mathrm{c})}_{\gamma\to\tilde{\gamma}} = \int_{p_2} \int \frac{d^4\beta}{\pi^2} \langle \mathrm{in}|S^\dagger|p_2, \beta; \tilde{\gamma}^{\tilde{h}}\rangle\langle p_2, \beta; \tilde{\gamma}^{\tilde{h}}|S|\mathrm{in}\rangle.$$

Note that as long as the integration over the SU(2) spinors β_b appears in the final-state summation, one may regard and use it as a shorthand for the bulkier spin sum.

that radiated quanta are not classically significant unless they belong to a classical wave modeled by a coherent state, see e.g. [53]. Therefore, remembering the meaning of the incoming state, we can write the absorption probability as

$$\begin{aligned} P^{(\mathrm{nc})}_{\gamma\to\tilde{\gamma}} = &\int_{p_1,p_1'} \psi^*_\xi(p_1)\psi_\xi(p_1')e^{ib\cdot(p_1'-p_1)} \\ &\times \sum_{s_2}\int_{M_1^2}^{\infty} dM_2^2\rho_{s_2}(M_2^2)\int_{p_2}\sum_{\{b\}}\langle p_1;\gamma^h|S^\dagger|p_2,s_2,\{b\};\tilde{\gamma}^{\tilde{h}}\rangle\langle p_2,s_2,\{b\};\tilde{\gamma}^{\tilde{h}}|S|p_1';\gamma^h\rangle. \end{aligned} \tag{6.7}$$

The building block $\langle p_2,s_2,\{b\};\tilde{\gamma}^{\tilde{h}}|S|p_1;\gamma^h\rangle$ involves a transition of a scalar heavy state into a possibly spinning one along with the incoming and outgoing massless coherent states. Since the latter states contain an infinite number of photons/gravitons, the matrix elements of $S=1+iT$ should be expanded in perturbation theory.

6.2 T-matrix perturbative expansion

The massless coherent states (plane or spherical) are sensitive to all orders in perturbation theory, and their matrix elements are non-trivial [51]. However, we can expand operators in terms of annihilation and creation operators, plane or spherical. We are going to perform the T-matrix expansion in the following way:[11]

$$\begin{aligned} T = \sum_{m,n=0}^{\infty}\left(T^{(\mathrm{c})}_{(m|n)}+T^{(\mathrm{nc})}_{(m|n)}\right) &= T^{(\mathrm{nc})}_{(0|1)}+T^{(\mathrm{nc})}_{(1|0)} \\ &+T^{(\mathrm{c})}_{(1|1)}+T^{(\mathrm{c})}_{(0|2)}+T^{(\mathrm{c})}_{(2|0)}+T^{(\mathrm{nc})}_{(1|1)}+T^{(\mathrm{nc})}_{(0|2)}+T^{(\mathrm{nc})}_{(2|0)}+\cdots, \end{aligned} \tag{6.8}$$

where the superscripts (c) and (nc) represent mass-preserving and mass-changing elements, respectively, while the subscript $(m|n)$ corresponds to n incoming and m outgoing photons/gravitons, and each T-matrix element will generate an $(m+n+2)$-point amplitude. In the first line of eq. (6.8), we have isolated the leading non-conservative effects due to absorption, $T^{(\mathrm{nc})}_{(0|1)}$, and emission, $T^{(\mathrm{nc})}_{(1|0)}$. Both terms are mass-changing three-point amplitudes and non-zero even on real kinematics, while the mass-preserving counterparts vanish, $T^{(\mathrm{c})}_{(1|0)}=T^{(\mathrm{c})}_{(0|1)}=0$.[12] In this paper, we have been studying the leading-order in absorption term $T^{(\mathrm{nc})}_{(0|1)}$, but the above expansion allows to also systematically understand higher orders.

In the second line, we have four-point terms that lead to the usual conservative Compton amplitude $T^{(\mathrm{c})}_{(1|1)}$ and its non-conservative counterpart $T^{(\mathrm{nc})}_{(1|1)}$. The former has been vastly studied recently, but the latter has been unexplored to the best of our knowledge. Furthermore, we have double-emission $(2|0)$ and double-absorption $(0|2)$ both on the conservative and non-conservative sides. Together with the non-conservative Compton, double-absorption would give the naive next-to-leading order (NLO) terms to our leading-order analysis.

[11]We thank Donal O'Connell for valuable discussions on the expansion (6.8).

[12]See [116] for a discussion of large gauge effects, where such amplitudes do contribute.

The T-matrix elements can be written in terms of scattering amplitudes:

$$\begin{aligned} T_{(m|n)} = \sum_{2s_1,2s_2=0}^{\infty} \int_{p_1,p_2} \sum_{\substack{h_1,\dots,h_n \\ \tilde{h}_1,\dots,\tilde{h}_m}} \int_{\substack{k_1,\dots,k_n \\ \tilde{k}_1,\dots,\tilde{k}_m}} & \hat{\delta}^4(p_1 + \textstyle\sum_i k_i - p_2 - \sum_i \tilde{k}_i) \\ & \times \mathcal{A}_{\{b\}}{}^{\{a\}}(p_2, s_2; \tilde{k}_1, \tilde{h}_1; \dots; \tilde{k}_m, \tilde{h}_m | p_1, s_1; k_1, h_1; \dots; k_n, h_n) \\ & \times \Big[a^{\dagger\{b\}}(p_2, s_2) a^{\dagger}_{\tilde{h}_1}(\tilde{k}_1) \cdots a^{\dagger}_{\tilde{h}_m}(\tilde{k}_m)\Big]\Big[a_{\{a\}}(p_1, s_1) a_{h_1}(k_1) \cdots a_{h_n}(k_n)\Big], \end{aligned} \tag{6.9}$$

where for brevity we have left the summation over the symmetrized massive little-group indices $a_1, \dots, a_{2s_1}$ and $b_1, \dots, b_{2s_2}$ implicit. The integration measure over p_2 contains either $\delta^+(p_2^2 - M_1^2)$ or $\delta^+(p_2^2 - M_2^2)$, depending on the conservative or non-conservative sector considered. The corresponding two sets of T-matrix elements span the space of one heavy particles and all possible photon/graviton radiation being emitted and absorbed. Of course, this is not the whole T-matrix space, since we could have more heavy particles and a mixed combination of photons and gravitons, but here we restrict to only one messenger particle.

For simplicity, we have used plane-wave massless creation/annihilation operators, which return the waveshape $\gamma(k)$ when applied to plane-wave coherent states:

$$a_h(k)|\gamma^{h'}\rangle = \gamma(k)\delta_h^{h'}|\gamma^h\rangle, \tag{6.10}$$

see e.g. [51]. Aiming for the leading-order absorption effects, we can evaluate the contribution of the $T^{(\mathrm{nc})}_{(0|1)}$ matrix element to the mass-changing probability as

$$\begin{aligned} \langle p_2, s_2, \{b\}; \tilde{\gamma}^{\tilde{h}}|S|p_1; \gamma^h\rangle &\simeq i\langle p_2, s_2, \{b\}; \tilde{\gamma}^{\tilde{h}}|T^{(\mathrm{nc})}_{(0,1)}|p_1; \gamma^h\rangle \\ &= i\int_k \hat{\delta}^4(p_1 + k - p_2)\mathcal{A}_{\{b\}}(p_2, s_2|p_1; k, h')\langle\tilde{\gamma}^{\tilde{h}}|a_{h'}(k)|\gamma^h\rangle \\ &= i\delta^h_{\tilde{h}}\langle\tilde{\gamma}^h|\gamma^h\rangle\int_k \gamma(k)\hat{\delta}^4(p_1 + k - p_2)\mathcal{A}_{\{b\}}(p_2, s_2|p_1; k, h). \end{aligned} \tag{6.11}$$

6.3 Partial-wave coherent states

We are interested in the scattering of a partial wave with a black hole, with the wave modeled by a covariant spherical coherent state. Such states are defined as eigenstates of the spherical annihilation operators:

$$a_{j,m,h}(\omega)|\gamma^{h'}\rangle = \gamma_{j,m}(\omega)\delta_h^{h'}|\gamma^h\rangle, \quad |\gamma^h\rangle = \mathcal{N}_\gamma \exp\bigg[\sum_{j,m}\int_0^\infty \hat{d}\omega\, \gamma_{j,m}(\omega) a^{\dagger}_{j,m,h}(\omega)\bigg]|0\rangle. \tag{6.12}$$

Setting $\langle\gamma^h|\gamma^h\rangle = 1$ gives the normalization prefactor as

$$\mathcal{N}_\gamma = \exp\bigg[-\frac{1}{2}\sum_{j,m}\int_0^\infty \hat{d}\omega\, |\gamma_{j,m}(\omega)|^2\bigg]. \tag{6.13}$$

The waveshape $\gamma_{j,m}(\omega)$ of these coherent states describes the contribution of each (j, m) component to the total wave, and we expect that in the classical limit $\gamma_{j,m}(\omega)$ is peaked at the frequency $\omega_{\rm cl}$. We can simplify the problem further by studying the incoming wave $|\gamma^h_{j,m}\rangle$ with just a particular (j,m) component, in which case the spherical waveshape reduces to $\gamma_{j',m'}(\omega) = \delta^j_{j'}\delta^m_{m'}\gamma(\omega)$, such that

$$a_{j,m,h}(\omega)|\gamma^{h'}_{j',m'}\rangle = \gamma(\omega)\delta^{j'}_j\delta^{m'}_m\delta^{h'}_h|\gamma^h_{j,m}\rangle. \tag{6.14}$$

Coming back to the initial state $|\text{in}\rangle$ given in eq. (6.1), which describes a scalar black hole and a partial wave as a wavepacket superposition of $|p_1;\gamma^h_{j,m}\rangle$. The S-matrix determines the probability amplitude of its evolution into a final massive state X and another partial wave $|\tilde\gamma^{\tilde h}\rangle$ with perhaps more than one $(\tilde j,\tilde m)$ components. Let us write the leading absorption term $T^{(\rm nc)}_{(0|1)}$ to such a process, by switching the states on the left-hand side of eq. (6.11) from plane to spherical waves:

$$\langle p_2,s_2,\{b\};\tilde\gamma^{\tilde h}|S|p_1;\gamma^h_{j,m}\rangle \simeq i\int_k \hat\delta^4(p_1+k-p_2)\mathcal{A}_{\{b\}}(p_2,s_2|p_1;k,h')\langle\tilde\gamma^{\tilde h}|a_{h'}(k)|\gamma^h_{j,m}\rangle. \tag{6.15}$$

The main difference is that to evaluate the matrix element of a plane-wave annihilation operator between two spherical coherent states, we need to summon the decomposition of the plane-wave operator into partial waves:

$$a_h(k) = 4\pi\sum_{j=|h|}^{\infty}\sum_{m=-j}^{j}\int_0^\infty\frac{\hat d\omega}{\sqrt{2\omega}}\hat\delta(k\cdot u_1-\omega)\,{}_{-h}Y_{j,m}(k;u_1)a_{j,m,h}(\omega), \tag{6.16}$$

and hence

$$a_{h'}(k)|\gamma^h_{j,m}\rangle = \frac{4\pi\delta^h_{h'}}{\sqrt{2k\cdot u_1}}\gamma_{j,m}(k\cdot u_1)\,{}_{-h}Y_{j,m}(k;u_1)|\gamma^h_{j,m}\rangle. \tag{6.17}$$

Therefore, we compute the leading mass-changing matrix element as

$$\begin{aligned}\langle p_2,s_2,\{b\};\tilde\gamma^{\tilde h}|S|p_1;\gamma^h_{j,m}\rangle \simeq 4\pi i\langle\tilde\gamma^{\tilde h}|\gamma^h_{j,m}\rangle&\int_0^\infty\frac{\hat d\omega}{\sqrt{2\omega}}\gamma_{j,m}(\omega)\\ \times&\int_k\hat\delta(k\cdot u_1-\omega)\,{}_{-h}Y_{j,m}(k;u_1)\hat\delta^4(p_1+k-p_2)\mathcal{A}_{\{b\}}(p_2,s_2|p_1;k,h).\end{aligned} \tag{6.18}$$

The leading contribution to the absorption probability (6.7) is then given by

$$\begin{aligned}P^{(\rm nc)}_{\gamma\to\tilde\gamma} \simeq 8\pi^2|\langle\tilde\gamma^{\tilde h}|\gamma^h_{j,m}\rangle|^2&\sum_{s_2}\int_{M_1^2}^\infty dM_2^2\rho_{s_2}(M_2^2)\int_{p_1,p_1',k,k',p_2}\psi^*_\xi(p_1)\psi_\xi(p_1')e^{ib\cdot(p_1'-p_1)}\\ &\times\int_0^\infty\frac{\hat d\omega\hat d\omega'}{\sqrt{\omega\omega'}}\gamma^*(\omega)\gamma(\omega')\hat\delta(k\cdot u_1-\omega)\hat\delta(k'\cdot u_1-\omega')\hat\delta^4(p_1+k-p_2)\hat\delta^4(p_1'+k'-p_2)\\ &\times{}_{-h}Y^*_{j,m}(k;u_1)\,{}_{-h}Y_{j,m}(k';u_1)\,\mathcal{A}^{*\{b\}}(p_2,s_2|p_1;k,h)\,\mathcal{A}_{\{b\}}(p_2,s_2|p_1';k',h).\end{aligned} \tag{6.19}$$

Note that apart from the overlap between the two spherical coherent states and the impact-parameter exponent, we have landed exactly on the single-quantum absorption cross-section given in eqs. (4.11) and (4.13) — with the (j,m) waveshape $\gamma(\omega)$ as the single-particle energy wavefunction. In other words, we observe that the waveshape $\gamma(\omega)$ acts as a one-dimensional wavefunction, which smears the energy spectrum but is peaked at the classical

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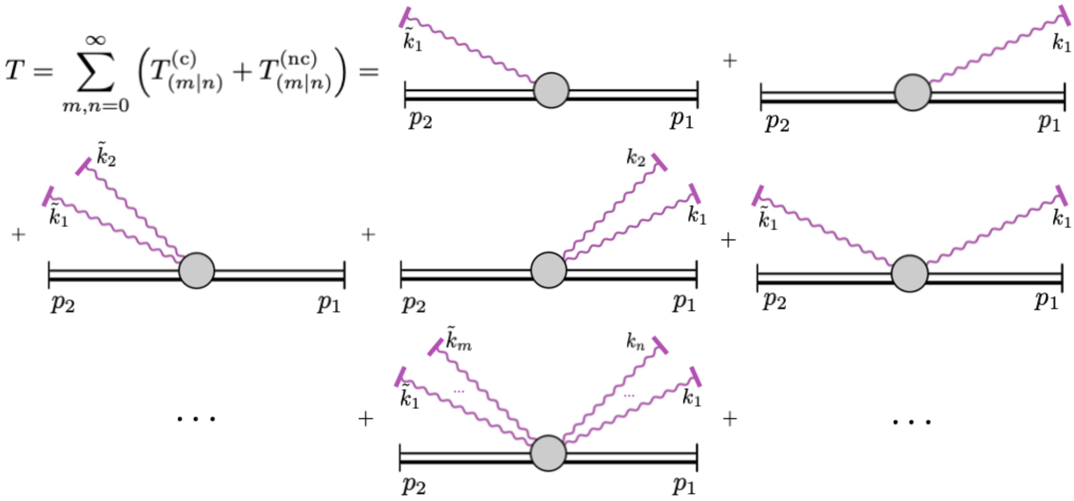

Figure 3. T-matrix operator expansion.

frequency ω_{cl}. This observation was also made in [91], where single quanta and coherent states gave the same results.

Let us discuss the seeming discrepancies between the leading-order cross-section formulae (4.11) and (6.19). For a spherical wave defined in the rest-frame of (the classical momentum of) the compact body and centered at it, the impact parameter should of course be set to zero. Moreover, eqs. (4.11) and (4.13) were written for an inclusive probability, let us rename it to $P^{(\mathrm{nc})}_{(0|1)} := P^{\mathrm{LO}}_{\mathrm{inc}}(\omega_{\mathrm{cl}}, j, m, h)$, whereas retaining the dependence on the outgoing waveshape in eq. (6.19) is actually an enhancement of the single-quantum formula:

$$P^{(\mathrm{nc})}_{\gamma\to\tilde{\gamma}} = |\langle \tilde{\gamma}^{\tilde{h}} | \gamma^{h}_{j,m} \rangle|^2 P^{(\mathrm{nc})}_{(0|1)} + \dots, \tag{6.20}$$

where the dots denote the higher orders to be briefly discussed below. In the limit where the outgoing classical wave changes very little, the above prefactor may furthermore disappear, $\langle \tilde{\gamma}^{\tilde{h}} | \gamma^{h}_{j,m} \rangle \approx 1$.

6.4 Higher-order diagrammatics

In this section, we use diagrams to help us understand all the effects relevant for BH-wave interactions. Having a diagrammatic realization of the expressions from the previous sections will guide us for the NLO corrections. However, this diagrammatic approach is general enough to be also applicable to any order in perturbation theory, as well as such processes as emitted radiation and superradiance.

Let us take a brief moment to explain the diagrammatic expansion of T-matrix in figure 3, which represents eq. (6.8). The operator nature of this diagram is represented by the "vertical line" after the wavy graviton line, and the double lines, which will "act" on a ket quantum state, e.g. the massless coherent state $|\gamma^h\rangle$ or the black-hole $|p_1\rangle$. In this diagram, we then have

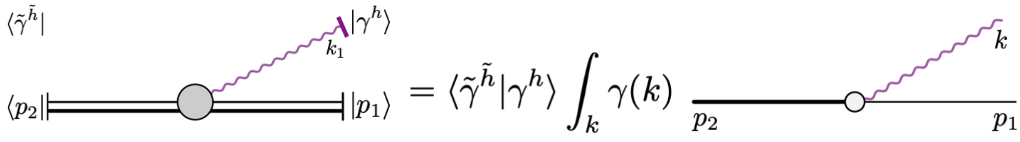

Figure 4. $T^{(\mathrm{nc})}_{(0|1)}$-matrix operator acting on the quantum states. Time flows right to left.

- n incoming graviton annihilation operators shown by wavy lines and labeled by $\{k_1, \cdots, k_n\}$;
- m outgoing graviton creation operators shown by wavy lines and labeled by $\{\tilde{k}_1, \cdots, \tilde{k}_m\}$;
- incoming and outgoing double line, labeled by p_1 and p_2. The two lines of different thickness inside of the double line represents the fact that this diagram contains mass-preserving and mass-changing transitions.
- vertical lines at the end of graviton/BH lines represent the operator nature of these diagrams. For instance, the double-line part of the operator will act on $|p_1\rangle$, while the wavy line will act on the coherent state $|\gamma^h\rangle$.
- Evaluating this operator with outgoing states on the left and incoming states on the right will result in scattering amplitudes, waveshapes, and coherent-state overlap. Due to the operator-action convention, time flows from right to left in the resulting amplitude.

Let us now apply these diagrams to the evaluation of the leading-order contribution to absorption given in eq. (6.11). We take the first term $T^{(\mathrm{nc})}_{(0|1)}$ on the right-hand side of figure 3 and take its matrix element $\langle p_2, s_2, \{b\}; \tilde{\gamma}^{\tilde{h}}|T^{(\mathrm{nc})}_{(0,1)}|p_1; \gamma^h\rangle$. The result is the overlap between the coherent states, a scattering amplitude, and the waveshape $\gamma(k)$, represented in figure 4. Note that the integrated scattering amplitude is a single-graviton amplitude smeared by the waveshape.

Similarly, figure 5 shows how this diagrammatic technique applies to the NLO non-conservative contributions. They contains double absorption and the mass-changing Compton amplitude, which both involve two photons/gravitons, now integrated with two waveshapes coming from the coherent states.

6.5 PM absorption analysis

In the previous section, we have explained how to include higher orders in multiplicity into the BH-wave interaction modeling by expanding the T-matrix. The PM expansion, however, enters into the mass-changing amplitudes in a rather intricate way. Indeed, as we have seen from eq. (5.7), even the three-point absorptive amplitudes must behave $\mathcal{O}(G^{s_2+1})$. Let us now explore the mass-changing $(m+n+2)$-point amplitude $\mathcal{A}_{\{b\}}{}^{\{a\}}(p_2, s_2; \tilde{k}_1, \tilde{h}_1; \ldots; \tilde{k}_m, \tilde{h}_m|p_1, s_1; k_1, h_1; \ldots; k_n, h_n)$ in eq. (6.9). For brevity, we compress the notation to $\mathcal{A}^{(s_2|s_1)}_{\mathrm{abs}(m|n)}$, emphasizing its distinction from the mass-conserving coun-

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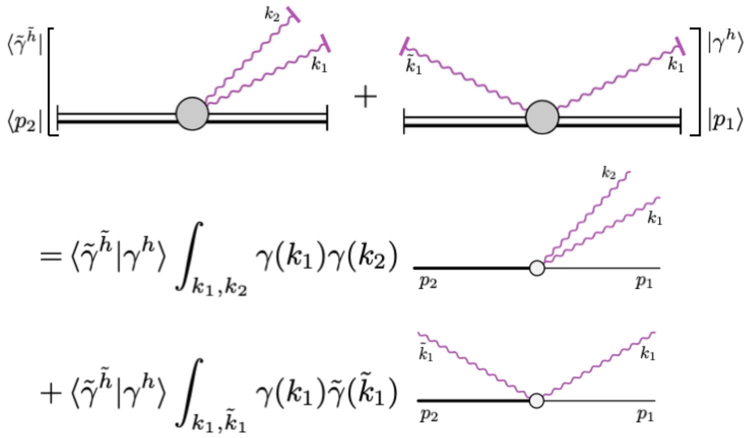

Figure 5. Next-to-leading order contributions to mass-changing absorption effects.

terparts $\mathcal{A}^{(s_2|s_1)}_{(m|n)}$. In particular, at three points we have

$$\mathcal{A}^{(s_2,0)}_{\text{abs}(0|1)} \propto G^{s_2+1}, \qquad \mathcal{A}^{(s)}_{3,\text{min}} \propto \sqrt{G}, \tag{6.21}$$

where the second one is the usual three-point same-mass amplitude [95] of the minimal form (3.15), which are known to correspond to Kerr BHs at 1PM [32, 33].

To obtain higher multiplicities, we can now naively multiply the powers of the Newton constant of these three-point amplitudes, assuming that they scale uniformly in G, and any subleading orders at three points should come from higher loop orders.[13] At four points, we have two incoming gravitons and a mass-changing heavy particle. We then have three types of contributions: a contact four-point term, two successive three-point absorptions, and one absorption together with one minimal-coupling amplitude. These terms be written respectively as

$$\mathcal{C}^{(s_2,0)}_{\text{abs}(0|2)} + \underbrace{\mathcal{A}^{(s_2,0)}_{\text{abs}(0|2)+0}}_{\propto G^{2s_2+2}} + \underbrace{\mathcal{A}^{(s_2,0)}_{\text{abs}(0|1)+1}}_{\propto G^{s_2+3/2}} =: \mathcal{A}^{(s_2,0)}_{\text{abs}(0|2)}, \tag{6.22}$$

where the subscript notation $(0|r) + n - r$ means that we have n gravitons, r out which couple via an absorptive three-point amplitude and $(n-r)$ via the mass-preserving minimal coupling. More generally, for n-graviton absorption we thus have

$$\mathcal{A}^{(s_2,0)}_{\text{abs}(0|n)} = \sum_{r=1}^{n} \mathcal{A}^{(s_2,0)}_{\text{abs}(0|r)+n-r} + \mathcal{C}^{(s_2,0)}_{\text{abs}(0|n)}, \qquad \mathcal{A}^{(s_2,0)}_{\text{abs}(0|r)+n-r} \propto G^{r(s_2+1)+(n-r)/2}. \tag{6.23}$$

In section 5, we have seen that, on the GR side, the PM expansion of the near-zone response function (5.2) suggests that the leading-order absorption cross-section scales as G^{2j+2}, whereas the NLO does as G^{2j+4}.[14] Now from squaring the amplitudes (6.22), we see that we obtain terms that scale as G^{2j+3}, $G^{3j+7/2}$ and G^{4j+4} for $s_2 = j$ (as follows from

[13] See [117] for loop corrections to Love numbers in the worldline EFT framework. For quantum corrections to Love numbers due to emission see [93], which we also ignore in the above analysis.

[14] Tail effects may modify the NLO to $\mathcal{O}(G^{2j+2})$ [92, 117], but we expect them to arise from loops.

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spin conservation seen in eq. (4.7)). Therefore, it is not possible to obtain the NLO G^{2j+4} expected on the GR side from the tree-level counting on the EFT side, unless the contact term is artificially introduced to account for this counting. However, a more natural way to obtain the expected behavior in G is from the amplitude with three incoming gravitons, which is expanded as

$$\mathcal{A}^{(s_2,0)}_{\text{abs}(0|3)} = \underbrace{\mathcal{A}^{(s_2,0)}_{\text{abs}(0|1)+2}}_{\propto G^{s_2+2}} + \underbrace{\mathcal{A}^{(s_2,0)}_{\text{abs}(0|2)+1}}_{\propto G^{2s_2+5/2}} + \underbrace{\mathcal{A}^{(s_2,0)}_{\text{abs}(0|3)+0}}_{\propto G^{3s_2+3}} + \mathcal{C}^{(s_2,0)}_{\text{abs}(0|3)}. \tag{6.24}$$

Indeed, we see that the first contribution squared induces the desired NLO G^{2j+4} correction to the absorption cross-section.

7 Summary and discussion

In this work, we have initiated the exploration of classical absorption effects for compact bodies using quantum scattering amplitudes. Central to this program are the mass-changing three-point scattering amplitudes [95, 108] that entail new degrees of freedom modeling non-conservative effects, which may change the mass and spin of the heavy particle (representing the compact object) due to the incoming wave.

We have made use of these amplitudes and their connection to covariantized spin-weighted spherical harmonics to describe leading gravitational absorption effects from a macroscopic/EFT point of view. Since this is an effective description, matching to the underlying theory was required to obtain the values of the EFT coupling coefficients. We have chosen to match at the cross-section level to the GR calculation dating back to Starobinsky, Churilov [80, 81] and Page [82, 83]. Although we have performed a leading-order match, this probability-based formalism can accommodate higher orders in the PM expansion and incoming spinning BHs and neutron stars as well. For the latter case, absorption effects were considered via tidal heating [118, 119], and it would be interesting to understand how the effective couplings g_{r,s_1,s_2} deviate from the BH values. We leave this for future work.

Having made sense of the effective couplings, we have explored how the used single-quantum framework fits into a more general and consistent description of classical waves using massless coherent states. In particular, we were able to connect the frequency wavefunction used in the former with the coherent-state waveshape, i.e. the eigenvalue of the annihilation operator. An interesting feature of this analysis is the diagrammatic approach for expanding the T-matrix and systematically introducing higher-order terms in the coherent cross-section. Crucial to this analysis was the separation of the probabilities into conservative and absorptive, which is motivated by the intrinsically distinct nature of the quantum amplitudes building blocks. Although the classical limit sends $M_2 \to M_1$, the form of the resulting cross-section follows from the amplitudes constructed on $M_2 \neq M_1$ kinematics, which are qualitatively different from their same-mass counterparts, since they belong to distinct Hilbert spaces.

The natural next step is to include spin effects for the initial black hole with the end goal of modeling a Kerr BH absorption cross-section purely from on-shell amplitudes.

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According to the microscopic calculation from the GR side, such leading-order non-spinning effects come at $\mathcal{O}(G^3)$ at the cross-section level, suggesting that the effective coupling in the amplitude should start at $\mathcal{O}(G^{3/2})$. From the EFT side, in this more general case of $s_1 \neq 0$, we have observed the proliferation of possible effective couplings in the mass-changing three-point amplitude (3.12), making the matching a harder task. However, the proposed definition of the mass-changing minimal amplitudes (3.19) might streamline the calculation and perhaps even correspond to the Kerr BH in the same way as the same-mass "minimal coupling" [95] of the form (3.15) are known to [32, 33].

Another direction that we have not explored is the study of observables from amplitudes, in particular using the KMOC formalism [48–53]. With the obtained absorption effective coefficients, many interesting local and global observables could be already be explored at leading or higher PM orders using the presented formalism. Perhaps the most interesting ones are the change in mass and spin induced by absorption, where one could naturally use such quantum operators as $\mathbb{P}^2 = \mathbb{P}^\mu \mathbb{P}_\mu$ to obtain ΔM^2 and $\mathbb{S}^2 = \mathbb{S}^\mu \mathbb{S}_\mu$ to obtain ΔS^2. Moreover, one could imagine probing the change in the area of the BH due to absorptive effects. In classical GR, the area is defined as

$$A_{\mathrm{H}} := 8\pi (GM)^2 \Big[1 + \sqrt{1-\chi^2}\Big], \qquad \chi = \frac{\mathfrak{a}}{GM}, \tag{7.1}$$

and $\mathfrak{a} = \sqrt{-S^2}/M$ is the Kerr ring radius. To obtain the change in this quantity from amplitudes, one would like to define a QFT operator for the area and try to compute ΔA_{H} in a scattering process. For that, one could substitute $(S^2, M^2) \to (\mathbb{S}^2, \mathbb{P}^2)$, which imples the following proposal for the area operator:

$$\mathbb{A}_{\mathrm{H}} = 8\pi \left[G^2\, \mathbb{P}^2 + \sqrt{(G^2\, \mathbb{P}^2)^2 + G^2\, \mathbb{S}^2}\right], \tag{7.2}$$

which mixes PM orders. The simplicity of this proposal also comes from the fact that the two operators commute $[\mathbb{S}^2, \mathbb{P}^2] = 0$. The mixing between orders in the expansion brings an interesting interplay between the $\mathbb{S}^2$ and the $\mathbb{P}^2$ calculations. We leave the exploration of such an operator for future work.

We hope that this work may open these and other avenues to include absorption effects in the on-shell amplitude approach to gravitational waves. In particular, the work [39] on matching Teukolsky-equation solutions to the gravitational Compton scattering amplitudes suggests that absorption effects could be included into them in relation to horizon effects. It is tempting to consider these effects from a purely on-shell perspective, as the four-point amplitudes are likely to be related to the leading-order absorption cross-section by dispersion relations.

Another direction is to explore in more detail the role of the spectral density function that we were forced to introduce in our formalism. For instance, it would be interesting to see if it appears in a similar way in the context of the Heavy Particle Effective Theory [26, 27], which streamlines the classical limit. We leave this also for future work.

Acknowledgments

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A Spherical harmonics and spinors

Here we discuss the spinorial construction for the spin-weighted spherical harmonics.

Spherical harmonics in 3d. The original construction due to Newman and Penrose [103] may be neatly formulated (see e.g. [120]) in terms of SU(2) spinors on the sphere $S^2 = \{\hat{\boldsymbol{k}} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)\} \subset \mathbb{R}^3$:

$$\kappa_+^a = \begin{pmatrix} e^{-\frac{i\varphi}{2}}\cos\frac{\theta}{2} \\ e^{\frac{i\varphi}{2}}\sin\frac{\theta}{2} \end{pmatrix}, \quad \kappa_-^a = \begin{pmatrix} -e^{-\frac{i\varphi}{2}}\sin\frac{\theta}{2} \\ e^{\frac{i\varphi}{2}}\cos\frac{\theta}{2} \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \hat{\boldsymbol{k}}\cdot\boldsymbol{\sigma}^a{}_b\,\kappa_\pm^b = \pm\kappa_\pm^a, \\ \hat{k}^i = -\dfrac{1}{2}\sigma^{i,a}{}_b(\kappa_+^a\kappa_{-b} + \kappa_-^a\kappa_{+b}), \end{cases} \tag{A.1}$$

where $\boldsymbol{\sigma}^a{}_b$ is the concatenation of the three standard Pauli matrices. We then define

$${}_h\tilde{Y}_{j,m}(\hat{\boldsymbol{k}}) := \underbrace{\overbrace{\kappa_+^{(1}\cdots\kappa_+^1}^{j-m}\,\kappa_+^2\cdots\kappa_+^2}_{j+h}\,\overbrace{\underbrace{\kappa_-^2\cdots\kappa_-^{2)}}_{j-h}}^{j+m\ (\text{with } \kappa_+^2\cdots\kappa_+^2)}. \tag{A.2}$$

Up to normalization, these functions are directly related to the conventional angle-dependent harmonics [104] via the spinor parametrization (A.1):

$$\begin{aligned} {}_hY_{j,m}(\theta,\varphi) &:= (-1)^m\sqrt{\tfrac{(2j+1)(j+m)!(j-m)!}{4\pi(j+h)!(j-h)!}}\,(\sin\tfrac{\theta}{2})^{2j}\sum_{r=\max(0,m-h)}^{\min(j+m,j-h)}(-1)^{j-h-r}\tbinom{j-h}{r} \\ &\quad\times\tbinom{j+h}{r+h-m}(\tan\tfrac{\theta}{2})^{m-h-2r}e^{im\varphi} \\ &= (-1)^m(2j)!\sqrt{\tfrac{2j+1}{4\pi(j+m)!(j-m)!(j+h)!(j-h)!}}\;{}_h\tilde{Y}_{j,m}(\hat{\boldsymbol{k}}). \end{aligned} \tag{A.3}$$

The latter functions obey the standard orthonormality property on S^2,

$$\int d\Omega_{\hat{\boldsymbol{k}}}\;{}_hY^*_{j',m'}(\hat{\boldsymbol{k}})\;{}_hY_{j,m}(\hat{\boldsymbol{k}}) = \delta_j^{j'}\delta_m^{m'}. \tag{A.4}$$

Note that the usual conventions (A.3) fix the (functional) U(1) freedom

$$\kappa_\pm \to e^{\pm i\phi(\hat{\boldsymbol{k}})/2}\kappa_\pm \qquad \Rightarrow \qquad {}_hY_{j,m}(\hat{\boldsymbol{k}}) \to e^{ih\phi(\hat{\boldsymbol{k}})}{}_hY_{j,m}(\hat{\boldsymbol{k}}), \tag{A.5}$$

which leaves the directional vector $\hat{\boldsymbol{k}}$ invariant and does not affect any important properties of the spherical harmonics.

Spinors in 4d. In Minkowski space, spinors carry $\mathrm{SL}(2,\mathbb{C})$ Weyl indices α or $\dot\alpha$ of negative or positive chirality, respectively. In the spinor-helicity formalism, spinors are denoted by bras and kets, written with angle or square brackets depending on their chirality. The spinors corresponding to massless vectors obey [121–126]

$$|k\rangle_\alpha [k|_{\dot\alpha} = k_{\alpha\dot\alpha} := k_\mu \sigma^\mu_{\alpha\dot\alpha}, \qquad k^2 = \det\{k_{\alpha\dot\alpha}\} = 0, \tag{A.6}$$

where $\sigma^\mu = (1, \boldsymbol{\sigma})$. Assuming $k^\mu = \omega\,(1, \cos\varphi \sin\theta, \sin\varphi\sin\theta, \cos\theta)$, we may pick

$$|k\rangle_\alpha = \sqrt{2\omega}\begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\end{pmatrix}, \qquad |k]^{\dot\alpha} = \sqrt{2\omega}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}, \tag{A.7}$$

with the understanding that little-group U(1) transformations, $|k\rangle \to e^{-i\phi(k)/2}|k\rangle$, $|k] \to e^{i\phi(k)/2}|k]$, leave eq. (A.6) invariant and are thus allowed.

Massive spinors [95][15] carry additional SU(2) little-group indices a and obey

$$|p^a\rangle_\alpha\, [p_a|_{\dot\alpha} := \epsilon_{ab}|p^a\rangle_\alpha\, [p^b|_{\dot\alpha} = p_{\alpha\dot\alpha} := p_\mu \sigma^\mu_{\alpha\dot\alpha}, \qquad p^2 = \det\{p_{\alpha\dot\alpha}\} = M^2 \neq 0. \tag{A.8}$$

For $p^\mu = (\varepsilon, \rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)$, such that $\varepsilon^2 - \rho^2 = M^2$, one may choose the massive spinors explicitly as (columns are labeled by a, rows by α or $\dot\alpha$)

$$|p^a\rangle_\alpha = \begin{pmatrix} \sqrt{\varepsilon-\rho}\cos\frac{\theta}{2} & -\sqrt{\varepsilon+\rho}\,e^{-i\varphi}\sin\frac{\theta}{2} \\ \sqrt{\varepsilon-\rho}\,e^{i\varphi}\sin\frac{\theta}{2} & \sqrt{\varepsilon+\rho}\cos\frac{\theta}{2}\end{pmatrix}, \quad [p^a|_{\dot\alpha} = \begin{pmatrix} -\sqrt{\varepsilon+\rho}\,e^{i\varphi}\sin\frac{\theta}{2} & -\sqrt{\varepsilon-\rho}\cos\frac{\theta}{2} \\ \sqrt{\varepsilon+\rho}\cos\frac{\theta}{2} & -\sqrt{\varepsilon-\rho}\,e^{-i\varphi}\sin\frac{\theta}{2}\end{pmatrix}, \tag{A.9}$$

This is ambiguous for $\boldsymbol{p} = 0$, so one may choose e.g.

$$p^\mu = (M, \mathbf{0}) = 0 \qquad \Rightarrow \qquad \langle p^a|^\alpha = \sqrt{M}\epsilon^{\alpha a}, \qquad [p^a|_{\dot\alpha} = \sqrt{M}\epsilon_{\dot\alpha a}. \tag{A.10}$$

The SU(2) little-group rotations, $|p^a\rangle \to U^a{}_b(p)|p^a\rangle$, $|p^a] \to U^a{}_b(p)|p^b]$, leave momentum p^μ invariant and correspond to choosing different spin quantization axes n^μ. (More details may be found in [52, 95, 130]). The parametrization (A.9) picks $n^\mu = (\rho, \varepsilon\cos\varphi\sin\theta, \varepsilon\sin\varphi\sin\theta, \varepsilon\cos\theta)/M$, i.e. quantization along the momentum, while eq. (A.10) chooses the conventional z-axis.

The momentum spinors serve as basic building blocks for scattering amplitudes. For massless particles, the spin is always quantized along the momentum, and is thus counted by helicity weights: $-1/2$ for each $|k\rangle$ and $+1/2$ for $|k]$. Moreover, each massive spin-s particle is represented by $2s$ symmetrized SU(2) indices. We denote the corresponding symmetrized tensor product of spinors by $\odot$, following [131].

Spherical harmonics in 4d. Returning to the spherical harmonics, we may now embed the 3d construction in 4d. Namely, we regard it as corresponding to the default choice of the time direction $u^\mu = (1, \mathbf{0})$ and the celestial sphere swept by a massless momentum $k^\mu = \omega\,(1, \hat{\boldsymbol{k}}(\theta,\varphi))$ and parametrized by the spinors $|k\rangle_\alpha = \sqrt{2\omega}\kappa_-^{a=\alpha}$ and $|k]^{\dot\alpha} = \sqrt{2\omega}\kappa_+^{a=\dot\alpha}$. Lorentz boosts change the time direction and induce Möbius transformations on the celestial sphere.

[15] For earlier iterations of the massive spinor-helicity formalism see refs. [108, 110, 127–129].

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For a general time direction u^μ (such that $u^2 = 1$ and $u^0 > 0$), we choose to parametrize the celestial sphere by the massless spinors $|k\rangle_\alpha$ and $|k]^{\dot\alpha}$. Of course, the quantum numbers of a spherical harmonic must be the same as in the rest frame of u^μ. The massive spinors $\langle u^a|^\alpha$ and $[u^a|_{\dot\alpha}$ provide a perfect transformation device between the current inertial frame and the rest frame of u^μ. This brings us to eq. (2.12), i.e.

$$
{}_h\tilde{Y}_{j,m}(k;u,n) := \frac{1}{\langle k|u|k]^j} \underbrace{\overbrace{[u_{(1}k]\cdots[u_1k]}^{j-m}\,\overbrace{[u_2k]\cdots[u_2k]}_{j+h}\,\langle ku_2\rangle\cdots\langle ku_{2)}\rangle}^{j+m}_{j-h}. \tag{A.11}
$$

Here the subscripts 1 and 2 are the explicitly symmetrized little-group indices, and the prefactor involving $\langle k|u|k] = 2k\cdot u \xrightarrow[\boldsymbol{u}\to 0]{} 2k^0$ serves to cancel out the mass dimension. Together with eq. (A.10), it guarantees the consistency with the rest-frame definition (A.2) — up to the functional U(1) transformation of the form (A.5) in view of the differences in the φ-dependence between eqs. (A.1) and (A.7). This is an example of acceptable convention discrepancies, which maybe caused by switching between different spinor parametrizations. The validity of the harmonics (A.11) as representations of the spin algebra follows from the properties of massive spinors, see e.g. [34, 52]. Note that the dependence on the spin-quantization axis n^μ enters via the choice of the massive spinors, as discussed around eq. (2.16). In other words, the SU(2) little-group transformations $|u^a\rangle \to U^a{}_b(p)|u^a\rangle$, $|u^a] \to U^a{}_b(p)|u^b]$ induce the SO(3) rotations of n^μ orthogonally to the time direction given by u^μ. Since the choice of spinors for u^μ defines n^μ, the notation may as well be compressed to ${}_hY_{j,m}(k;u)$.

Let us now discuss the orthonormality property (2.15). It is valid for the normalized versions of the covariant harmonics, rescaled from those in eq. (A.11) analogously to their non-covariant counterparts in eq. (A.3). It can be easily seen that in the rest frame of u^μ the covariant integration measure reduces to the solid-angle one:

$$
\frac{2}{\omega}\int d^4k\,\delta^+(k^2)\delta(k\cdot u-\omega) \xrightarrow[\boldsymbol{u}\to 0]{} \int d\Omega_{\hat{\boldsymbol{k}}}, \qquad k^0 = |\boldsymbol{k}| = \omega. \tag{A.12}
$$

So eq. (2.15) clearly holds for $\boldsymbol{u} = 0$, and what we need is to extend it to any u^μ.

Spinor integration. To expose the properties of the measure (A.12) in a neat way, we first rewrite it using a null basis [132]:

$$
k^\mu = t\Big(r^\mu + \gamma q^\mu + \frac{z}{2}[r|\bar\sigma^\mu|q\rangle + \frac{\bar z}{2}[q|\bar\sigma^\mu|r\rangle\Big) \;\Rightarrow\; \int d^4k = \frac{i(r+q)^4}{4}\int t^3 dt\wedge d\gamma\wedge dz\wedge d\bar z, \tag{A.13}
$$

where $\bar\sigma^\mu = (1, -\boldsymbol{\sigma})$, and the massless vectors r^μ and q^μ are not collinear but otherwise arbitrary. Adding the masslessness condition eliminates γ from the measure:

$$
\int d^4k\,\delta^+(k^2) = \frac{i(r+q)^2}{4}\int_0^\infty tdt\int dz\wedge d\bar z, \qquad k^\mu = \frac{t}{2}(\langle r| + z\langle q|)\sigma^\mu(|r] + \bar z|q]). \tag{A.14}
$$

(Here for concreteness one may assume $r^0, q^0 > 0$ so that $k^0 > 0$.) However, this massless measure may now be rewritten using spinor integration [133–135]

$$
\int d^4k\,\delta^+(k^2) = -\frac{i}{4}\int_0^\infty tdt\int_{\tilde\lambda=\bar\lambda}\langle\lambda d\lambda\rangle\wedge[\tilde\lambda d\tilde\lambda], \qquad k^\mu = \frac{t}{2}\langle\lambda|\sigma^\mu|\tilde\lambda], \tag{A.15}
$$

such that the dependence on r^μ and q^μ has entirely canceled out due to

$$(r+q)^2 dz\wedge d\bar z = -(\langle r| + z\langle q|)|q\rangle dz \wedge ([r| + \bar z[q|)|q]d\bar z = -\langle\lambda d\lambda\rangle\wedge[\tilde\lambda d\tilde\lambda]. \tag{A.16}$$

Now introducing the second delta function let us fix the energy scale of k^μ and get

$$\frac{1}{\omega}\int d^4k\,\delta^+(k^2)\delta(k\cdot u-\omega) = -i\int_{\tilde\lambda=\bar\lambda}\frac{\langle\lambda d\lambda\rangle\wedge[\tilde\lambda d\tilde\lambda]}{\langle\lambda|u|\tilde\lambda]^2}, \qquad k^\mu = \omega\frac{\langle\lambda|\sigma^\mu|\tilde\lambda]}{\langle\lambda|u|\tilde\lambda]}. \tag{A.17}$$

This measure allows us to reformulate the orthonormality property (2.15) of the spin-weighted spherical harmonics in the following way:

$$\int_{\tilde\lambda=\bar\lambda}\frac{\langle\lambda d\lambda\rangle\wedge[\tilde\lambda d\tilde\lambda]}{\langle\lambda|u|\tilde\lambda]^2}\,{}_hY^*_{j',m'}(\lambda,\tilde\lambda;u)\,{}_hY_{j,m}(\lambda,\tilde\lambda;u) = \frac{i}{2}\delta_j^{j'}\delta_m^{m'}, \tag{A.18}$$

where the notation ${}_hY_{j,m}(\lambda,\tilde\lambda;u) := {}_hY_{j,m}(k;u)$ serves to emphasize their independence of the energy scale. Then the validity of eq. (2.15) for $\boldsymbol{u}\neq 0$ follows from the fact that the entire left-hand side is independent of $\omega = k\cdot u$. Indeed, for any spinor conventions and in any frame, we can rewrite it as the same integral over the complex plane by parametrizing $|\lambda\rangle = |u^1\rangle + z|u^2\rangle$ and $|\tilde\lambda] = |u_1] + \bar z|u_2]$, so that the left-hand side of eq. (A.18) will exclusively involve the following ingredients:

$$\begin{aligned} \langle u_a\lambda\rangle &= -\delta_a^1 - \delta_a^2 z, & \langle\lambda d\lambda\rangle\wedge[\tilde\lambda d\tilde\lambda] &= -dz\wedge d\bar z := 2i\,d\Re z\wedge d\Im z,\\ [u_a\tilde\lambda] &= \epsilon_{1a}+\epsilon_{2a}\bar z, & \langle\lambda|u|\tilde\lambda] &= 1+z\bar z. \end{aligned} \tag{A.19}$$

Therefore, it only depends on the quantum numbers h, j, j', m and m', and may only produce a combinatorial result, which may as well be fixed at $u^\mu = (1,\mathbf{0})$.

B Frame transformations of harmonics

Here we derive the spinor transformations (2.17), which induce the relationship between covariant spin-weighted spherical harmonics ${}_h\tilde Y_{j,m}(k;u)$ and ${}_h\tilde Y_{j,m}(k;v)$.

These harmonics correspond to two different unit timelike vectors u^μ and v^μ, with a relative Lorentz factor

$$\gamma := u\cdot v =: \frac{1}{\sqrt{1-\nu^2}}, \qquad 0\le\nu<1. \tag{B.1}$$

These vectors can be Lorentz-transformed into each other using the minimal boost

$$L^\rho{}_\sigma(v\!\leftarrow\!u) := \delta^\rho_\sigma + 2v^\rho u_\sigma - \frac{(u+v)^\rho(u+v)_\sigma}{1+u\cdot v} = \exp\Big(\tfrac{i\log(\gamma+\sqrt{\gamma^2-1})}{\sqrt{\gamma^2-1}}u^\mu v^\nu\Sigma_{\mu\nu}\Big)^\rho{}_\sigma, \tag{B.2}$$

written in terms of the spin-1 Lorentz generators $(\Sigma^{\mu\nu})^\rho{}_\sigma := i[\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma]$. The spinors may be boosted using the corresponding $\mathrm{SL}(2,\mathbb{C})$ transformations, namely

$$S^\alpha{}_\beta(v\!\leftarrow\!u) = \exp\big(\tfrac{i\log\mu}{\gamma\nu}u^\mu v^\nu\sigma_{\mu\nu}\big)^\alpha{}_\beta, \qquad \mu := \gamma+\sqrt{\gamma^2-1}, \tag{B.3}$$

written in terms of the chiral spin-1/2 generators $\sigma^{\mu\nu} := \frac{i}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}$. Using the Clifford-algebra property $\sigma^{(\mu}\bar{\sigma}^{\nu)} = \eta^{\mu\nu}$, it is easy to derive

$$\begin{aligned} \big(\tfrac{i\log\mu}{\gamma\nu}u^\mu v^\nu \sigma_{\mu\nu}\big)^{2n}|u^a\rangle &= (\log\sqrt{\mu})^{2n}|u^a\rangle, \\ \big(\tfrac{i\log\mu}{\gamma\nu}u^\mu v^\nu \sigma_{\mu\nu}\big)^{2n+1}|u^a\rangle &= (-\log\sqrt{\mu})^{2n+1}\Big(\tfrac{1}{\nu}|u^a\rangle - \tfrac{1}{\gamma\nu}|v|u^a]\Big). \end{aligned} \tag{B.4}$$

This lets us sum the matrix exponent, whose action simplifies to

$$S^\alpha{}_\beta(v\leftarrow u)|u^a\rangle = \frac{\sqrt{\mu}}{\mu+1}\Big(|u^a\rangle + |v|u^a]\Big). \tag{B.5}$$

We thus arrive at the following massive-spinor transformations:

$$|v^b\rangle = \frac{\sqrt{\mu}}{\mu+1}U^b{}_a(v\leftarrow u)|u+v|u^a], \qquad |v^b] = \frac{\sqrt{\mu}}{\mu+1}U^b{}_a(v\leftarrow u)|u+v|u^a\rangle. \tag{B.6}$$

Here we have allowed for the SU(2) matrix $U^b{}_a(v\leftarrow u)$. Its purpose is to fix the misalignment between what we get from the minimal boost (B.2) and the desired spin quantization axis for the resulting time direction, which generically do not coincide:

$$n_v^\mu := \frac{1}{2}(\langle v_2|\sigma^\mu|v^2] + [v_2|\bar{\sigma}^\mu|v^2\rangle) \quad\neq\quad L^\mu{}_\nu(v\leftarrow u)n^\nu = n^\mu - \frac{n\cdot v}{1+u\cdot v}(u+v)^\mu. \tag{B.7}$$

In fact, unitary matrices like $U^b{}_a(v\leftarrow u)$ represent the SO(3) rotations of the spin quantization axis even in the absence of Lorentz-frame boosts. Therefore, the spinor transformations (B.6) induce the most general frame transformations of the covariant spherical harmonics.

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