

Geometric phases and cyclic isotropic cosmologies

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Abstract

In the present paper we study the evolution of the modes of a scalar field in a cyclic cosmology. In order to keep the discussion clear, we study the features of a scalar field in a toy model, a Friedman–Robertson–Walker Universe with a periodic scale factor, in which the Universe expands, contracts and bounces infinite times, in the approximation in which the dynamic features of this Universe are driven by some external factor, without the backreaction of the scalar field under study. In particular, we show that particle production exhibits features of the cyclic cosmology. Also, by studying the Berry phase of the scalar field, we show that contrary to what is commonly believed, the scalar field carries information from one bounce to another in the form of a global phase which occurs as generically non-zero. The Berry phase is then evaluated numerically in the case of the effective loop quantum cosmology closed Universe. We observe that Berry's phase is non-zero, but that in the quantum regime the particle content is non-negligible.

Keywords: scalar field, cyclic cosmology, geometric phase

(Some figures may appear in colour only in the online journal)

1. Introduction

Our current understanding of the Universe and of its history relies on several puzzling observational features. In fact, soon after the discovery of the cosmological microwave background, theoretical physicists realized that the early Universe had to follow a rapid expanding phase. Inflation was the first candidate for generating such expansion, and a more refined version of inflation is still at the basis of the current standard cosmological model.

There is compelling evidence for a rapid expanding phase, which has culminated in the recent results of the BICEP and PLANCK collaborations [1–3]. The current standard model of the Universe involves an inflation field, but current experimental results, meanwhile ruling out many proposed models, seem not to exclude several other proposals alternative to inflation [4]. Inflation has the advantage of the Occam’s razor, as it accounts for all the puzzles involved with our current understanding of the Universe, such as its flatness and acceleration, its entropy and the horizon problem in one shot.

However, an alternative and interesting proposal relies on the idea of cyclicity of the Universe, a concept almost as old as General Relativity. Cosmological bounce models have a long history, with its first proposal put forward by Tolman [5], and have been proposed in several different contexts [6]. In particular, it has been shown that oscillating cosmological Universes can provide a solution to both flatness and horizon problems, as long as maximum of the expansion in each cycle is increased in the next cycle.

Quantum gravitational phenomena play a role only in the extremely early Universe. In this scenario, quantum gravity is the force that makes the bounce occur. In fact, a big bounce has been observed in several quantum gravitational cosmological theories, as for instance loop quantum cosmology (LQC) [7–9], Epkyrotic strings [10–12], Asymptotically safe non-local gravity [13] and in General Relativity with matter-torsion interaction [14] just to name a few.

In the present paper, we ask ourselves whether such bouncing phenomena have some physical, long-term effects at the quantum level. At the classical level, we find this unlikely to occur. However, one could argue that considering the Universe conformal factor as an external parameter, in a bouncing cosmology a Berry phase might emerge at the quantum level. This is the scenario we are interested in: does a scalar field maintain a quantum memory from previous bounces? We answer this question in the modest nonphysical scenario of a scalar field in a bouncing cosmology, in which the scalar field does not lead to any back-reaction on the underlying metric; yet, this toy model features many of the stylized characteristics of bouncing cosmologies [15, 16]. The Berry phase has been already considered in the cosmological setting, by for the case of inflationary models [17].

In general, the Berry phase is not a measurable quantity if the scalar field is isolated. However, if the scalar field is entangled with other fields as considered for instance in [18], then a Berry phase is measurable. To see this, let us consider a scalar field ϕ , in the state $|\phi, n\rangle$ at the beginning of the n th cycle and let \mathcal{U}_n be the propagator which maps $|n\rangle \rightarrow |n+1\rangle$. If the scalar field’s state does not change from one cycle to the other, i.e. $\mathcal{U}_n|\phi, n\rangle = |\phi, n+1\rangle$, then the bounce produces no observable effects. However, even though the field is unchanged after one cycle, its state can pick a geometric phase $\mathcal{U}_n|\phi, n\rangle = e^{i\alpha}|\phi, n+1\rangle$ which in principle can be measured as in interferometry. For instance, for the sake of argument, we consider the simplest case of two scalar fields ϕ and ξ which are in the entangled state $|\phi_{\text{Universe}}, n\rangle = \rho_1|\phi, \xi, n\rangle + \rho_2|\tilde{\phi}, \tilde{\xi}, n\rangle$, although our argument can be extended to more complicated cases. After a cycle, because of the Berry’s phase, the state is a different one $\rho_1 e^{i\alpha}|\phi, \xi, n+1\rangle + \rho_2 e^{i\tilde{\alpha}}|\tilde{\phi}, \tilde{\xi}, n+1\rangle$. We argue that this phenomenon is poorly described by analyzing, for instance, the particle content coefficients only, in particular if these are calculated perturbatively and can vanish identically, and that a nonperturbative approach is more appropriate for calculating these phases. We will describe how to define a simple observable in section 4, and we will provide an exact formula for the operator \mathcal{U}_n . We will provide both explicit formulae for specific scale factor functions and provide general results for smooth scale factors which are periodic.

The paper is organized as follows. In section 2 we introduce the scalar field in curved spacetime, the settings and the notation, assuming that the Universe is cyclic. In section 3 we

study particle production in the toy model we introduce, both in a Kronig–Penney approximation and solving numerically the mode equation. In section 4 we study the Berry Phase. Conclusions follow.

2. Scalar field in a cyclic toy Universe

We are interested in the quantum features of a scalar field in a Friedman–Robertson–Walker space time defined by the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \quad (1)$$

which describes an evolving isotropic flatspace with a dynamic scale factor $a(t)$. In order to simplify many of the calculations, we introduce the conformal time $\eta = \int^t \frac{dt'}{a(t')}$ coordinate. In these coordinates, the metric reads:

$$ds^2 = a(\eta)^2 [d\eta^2 - (dx^2 + dy^2 + dz^2)], \quad (2)$$

where $a(\eta) \equiv a(t(\eta))$ is the scale factor in the new coordinate.

As a simplifying assumption, we consider a scale factor $a(t)$ which oscillates according to a local time and that $a(t) > 0$ as considered in many nonsingular bouncing cosmology models [7, 10, 13], and that can be expanded in Fourier series. We assume that space is locally flat, i.e. that the curvature of space is zero. In addition to this, we assume that the expansion and contracting phases of the Universe last an equal amount of conformal time, which is a scenario ruled out from the physical viewpoint.

In modeling the Universe with these features, we were inspired by the two major current theories of bouncing cosmologies: the ekpyrotic scenario [10–12] and LQC [7–9, 19, 20]. Meanwhile the former has been affected with the recent results from the Planck collaboration [1], the latter has shown to be a long-living toy model. In general, the ekpyrotic scenario relies on an asymmetric expanding and contracting phases, differently from our case, meanwhile the LQC approach is symmetric in its expanding and contracting phases. It has been argued however that the ekpyrotic scenario suffers of singularities [21]. Although violating the Tolman principle [5], one could claim that a bounce occurring in a quantum gravity scenario might require an extension of the principles of thermodynamics, similarly to what happens for black holes. Thus, in this respect, we are more interested in understanding some of the features of a quantum field when the Universe is cyclical. This simple approach has the advantage that one can carry on several calculations analytically.

The action for a scalar field on a curved background is

$$S = \int dx^4 \sqrt{-g} [g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi^* - V(g) \Phi \Phi^*], \quad (3)$$

where we consider the curvature dependent potential $V(g) = m^2 + \xi R$, being $R = 6 \frac{a''(\eta)}{a(\eta)^3}$ for the metric (2) The equations of motion for a scalar field of this form are given by:

$$\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi) + V(g) \Phi = 0. \quad (4)$$

It is convenient to work with a field re-definition useful in the conformal time coordinates, $\Phi(x) = \frac{1}{\sqrt{(2\pi)^3 a^2(\eta)}} e^{-i\vec{k}\cdot\vec{x}} \chi(\eta)$, in which the equations of motion for the scalar field are given by: $\left[\frac{d^2}{d\eta^2} + \omega_k^2(\eta) \right] \chi(\eta) = 0$, with $\omega_k^2(\eta) = k^2 + m^2 a^2(\eta) + \left(\xi - \frac{1}{6} \right) R a^2(\eta)$. If one works

with conformally coupled fields, $\xi = 1/6$, then the dependence of the curvature on the modes disappears. Moreover, we assume space isotropy, so all the quantities only depend on $k = |\vec{k}|$.

The classical field can be quantized by introducing bosonic creation and annihilation operators ($\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}}$) [22]: the expansion

$$\hat{\Phi}(\vec{x}, \eta) = \int \frac{d^3\vec{k}}{\sqrt{2(2\pi)^3 a(\eta)^2}} [e^{i\vec{k}\cdot\vec{x}} \phi_k^*(\eta) \hat{a}_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}} \phi_k(\eta) \hat{a}_{\vec{k}}^\dagger], \quad (5)$$

defines the mode operators from the quantum field $\hat{\Phi}(\vec{x}, \eta)$ and the mode functions satisfy the equation

$$\left[\frac{d^2}{d\eta^2} + \omega_k^2(\eta) \right] \phi_k(\eta) = 0. \quad (6)$$

Once the mode operators are determined, the vacuum state $|\Omega\rangle$ can be defined as the eigenstate of the annihilation operators with eigenvalue 0, i.e. $\hat{a}_{\vec{k}}|\Omega\rangle = 0$. However, different solutions of (6) define different mode operators and thus different vacua. The physical vacuum $|\Omega(\eta_0)\rangle$ at a certain conformal time η_0 corresponds to the (instantaneous) lowest energy state and it is given by the solution of (6) with initial conditions

$$\phi_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \quad \phi_k'(\eta_0) = i\omega_k(\eta_0)\phi_k(\eta_0). \quad (7)$$

From now on we call $\phi_k(\eta, \eta_0)$ the solution of (6) and (7). At another conformal time η_1 , since the vacuum satisfies different initial conditions, the theory has a different particle content described by new creation and annihilation operators $\hat{b}_{\vec{k}}^\dagger, \hat{b}_{\vec{k}}$. The mode operators at $\hat{a}_{\vec{k}}$ and $\hat{b}_{\vec{k}}$ are related by the Bogoliubov transformation

$$\hat{b}_{\vec{k}} = \alpha_k \hat{a}_{\vec{k}} + \beta_k^* \hat{a}_{\vec{k}}^\dagger, \quad (8)$$

where α_k and β_k are independent on η and defined by

$$\alpha_k = \frac{W[\phi_k(\eta, \eta_1), \phi_k(\eta, \eta_0)^*]}{2i}, \quad (9)$$

$$\beta_k^* = \frac{W[\phi_k(\eta, \eta_1), \phi_k(\eta, \eta_0)]}{2i}, \quad (10)$$

being $W[x, y] = (\partial_\eta x)y - x(\partial_\eta y)$ the Wronskian. The new vacuum is related to the old one via

$$|\Omega(\eta_1)\rangle = \prod_{\vec{k}} \frac{1}{|\alpha_k|^{1/2}} \exp\left(-\frac{\beta_k^*}{2\alpha_k} \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}}^\dagger\right) |\Omega(\eta_0)\rangle. \quad (11)$$

When $\eta_0 = n\Pi$ and $\eta_1 = (n+1)\Pi$ the above transformation defines the operator \mathcal{U}_n which maps the state of the n th cycle to the next one. In the following sections we study the vacuum in a bouncing Universe, discussing the possible creation of particles and the occurrence of a geometric phase after one or more periods.

3. Particle production

In a bouncing Universe the scale factor of the Universe oscillates. If one assumes a conformally coupled scalar field, the effect of gravity is included in the η -dependent frequency $\omega_k(\eta)$ through the mass of the scalar field, which is periodic with a certain conformal period

II. As can be expected, in general the particle content at conformal times η and $\eta + \Pi$ is the not the same. In this section we show that the low-energy vacua at different periods are not the same in general, i.e. $|\Omega(\eta + \Pi)\rangle \neq |\Omega(\eta)\rangle$, even though the functions $\phi_k(\eta, \eta_0)$ and $\phi_k(\eta, \eta_0 + \Pi)$ satisfies the same differential equation (6).

We consider the Universe in its ground state at conformal time η_0 . As discussed before, an observer at a different conformal time η_1 sees different particles which are related to the original ones by the Bogoliubov transformation (8). For instance let us call a -particles the particles created by the operators \hat{a}_k^\dagger at conformal time η_0 and b -particles the ones created by \hat{b}_k^\dagger at conformal time η_1 . The vacuum state at η_1 is the one annihilated by \hat{b}_k so, by definition, the number of b -particles in $|\Omega(\eta_1)\rangle$ is exactly zero. Nonetheless, $|\Omega(\eta_1)\rangle$ contains many pairs of a -particles and the average density of a -particles in $|\Omega(\eta_1)\rangle$ is [22]:

$$n_k = |\beta_k|^2, \quad \beta_k = -\frac{\phi_k' - i\omega_k \phi_k}{2i\sqrt{\omega_k}}, \quad (12)$$

where in the second equality we used (10). For instance, for sudden transitions between two instantaneous vacua, the Bogoliubov coefficients are given by

$$\alpha_k = \frac{1}{2} \left(\sqrt{\frac{\omega_k(\eta_1)}{\omega_k(\eta_0)}} + \sqrt{\frac{\omega_k(\eta_0)}{\omega_k(\eta_1)}} \right), \quad (13)$$

$$\beta_k = \frac{1}{2} \left(\sqrt{\frac{\omega_k(\eta_1)}{\omega_k(\eta_0)}} - \sqrt{\frac{\omega_k(\eta_0)}{\omega_k(\eta_1)}} \right). \quad (14)$$

Therefore if the Universe is cyclic and performs a sudden transition from η_0 to $\eta_1 = \eta_0 + \Pi$, then $\beta_k = 0$ so the particle content is not changed. On the other hand, when the Universe evolves smoothly the Bogoliubov coefficients are obtained by means of (10) from the solution of the differential equation (6).

The equation (6) is known in the literature as the Hill equation and has many interesting properties [23]. In particular, because of Floquet theorem, all the solutions of (6) can be written as

$$\phi_k(\eta) = e^{i\mu_k \eta} c_k^+ f_k^-(\eta) + e^{-i\mu_k \eta} c_k^- f_k^-(\eta), \quad (15)$$

where the functions $f_k^\pm(\eta)$ are periodic, $f_k^\pm(\eta + \Pi) = f_k^\pm(\eta)$, and μ_k are called the Floquet exponents. The coefficients c_k^\pm describing the mode function are readily obtained by imposing (7), whereas μ_k depends on ω_k and, in general, can be real, complex or integer. When μ_k is complex the solutions are not stable, i.e. they diverge for $\eta \rightarrow \infty$. We argue that this is not a physical occurrence so we skip the discussion of this eventuality. More interestingly, when μ_k is an integer (or a fraction) the solution (15) is periodic of period Π (or a multiple of Π). In this case, not only the geometry, but also the quantum fields display a cyclic dynamics during the evolution, so $|\Omega(\eta + \Pi)\rangle = |\Omega(\eta)\rangle$. From the mathematical point of view, $\omega_k(\eta)$ can be designed to allow for periodic solutions [24] for each value of k , but typically, for physical conformal factors, the periodic solutions appear only for specific discrete values of the parameters, e.g. for some specific values of k .

In general μ_k is a real number, making the solution equation (15), and thus the quantum fields, an aperiodic function of η . When the vacuum state is not periodic the average density of particles after T periods is evaluated from equations (12) and (15):

$$n_k(\Pi) = \frac{\sin(T\Pi\mu_k)^2}{\omega^2 |g_k^+ + g_k^-|^2}, \quad (16)$$

$$g_k^\pm = \frac{f_k^\pm(0)}{[\mu_k \mp \omega_k(0)]f_k^\pm(0) \mp if_k^{\pm'}(0)}. \quad (17)$$

The Floquet exponent μ_k can be evaluated perturbatively using the techniques reported in appendix 6 while the quantities g^\pm depend on the initial value. Therefore, $n_k(\Pi)$ is zero only when μ_k is an integer or when $(g_k^\pm)^{-1} = 0$. We will discuss below these possibilities for some particular cases.

3.1. Kronig–Penney approximation

The equation for the modes of a bouncing Universe are formally similar to those of Bloch electrons in a spatially periodical lattice, where the space dimension is the conformal time. Instead of $\sqrt{\frac{2m}{\hbar}(E - V)}$ for ω_k , we now have a relativistic energy given by $\sqrt{k^2 + m^2 a(\eta)^2}$, with the periodicity encoded within $a(\eta)$. One can thus use a similar approach to those of used for tunneling of electrons from one lattice site to the other, or for calculating the bands of a solid. In this section we will assume for simplicity a scale factor which oscillates as:

$$a(t) = a_0 + b \sin(rt). \quad (18)$$

As r is merely a redefinition of time, we consider $r = 1$ hereon. We note that this toy model requires $a_0 > b > 0$ for a regular cosmology in which the scale factor never reaches zero. For this model, we find that the conformal time $\eta = \int^t \frac{d\xi}{a(\xi)}$ is

$$\eta(t) = \frac{2 \tan^{-1} \left(\frac{a_0 \tan(\frac{t}{2}) + b}{\sqrt{a_0^2 - b^2}} \right)}{\sqrt{a_0^2 - b^2}} + \frac{2\pi \left(\theta(t - \pi) - \left\| \frac{b}{2} - \frac{t}{2\pi} \right\| \right)}{\sqrt{a_0^2 - b^2}}, \quad (19)$$

where $\|\cdot\|$ refers to the integer part, which is everywhere continuous and monotonic, allowing for $\eta(t)$ to be inverted $\forall t$. From this we obtain a closed formula for $a(\eta) \equiv a(t(\eta))$ given by:

$$a(\eta) = b \sin \left(2 \tan^{-1} \left(\frac{\Xi \tan \left(\frac{1}{2} \eta \Xi \right) - b}{a_0} \right) \right) + a_0 \quad (20)$$

with $\Xi = \sqrt{a_0^2 - b^2}$. It is easy to show that the cyclicity of this Universe depends on the parameters a_0 and b , $a(\eta) = a(\eta + \Pi)$, with $\Pi = \frac{2\pi}{\sqrt{a_0^2 - b^2}}$. From now on, we will assume $a_0 > b$.

In what follows, we will assume also that our potential can be approximated by step functions. In particular, we would like to treat the effective potential $\omega(k)$ as a two-valued function, as in the Konig–Penney approximation for electrons on a Bloch lattice. In the standard case of a scalar field in a potential with many local minima, as shown in [25, 26] the field can move through tunnelings from one local minimum to another. In our case instead, the interpretation is different, as the potential is now driven by an external factor, i.e. the bounce of the Universe. This will allow to treat the modes close to the bouncing and contracting phases of this toy Universe as a constant, and thus the local solution being a plane waves. An example of this approximation is given in figure 1. We use a full width at half maximum approach, i.e., the energy is approximated by $\omega_l(k) = \sqrt{k^2 + m^2 a^2(\eta_{\text{bounce}})}$ and

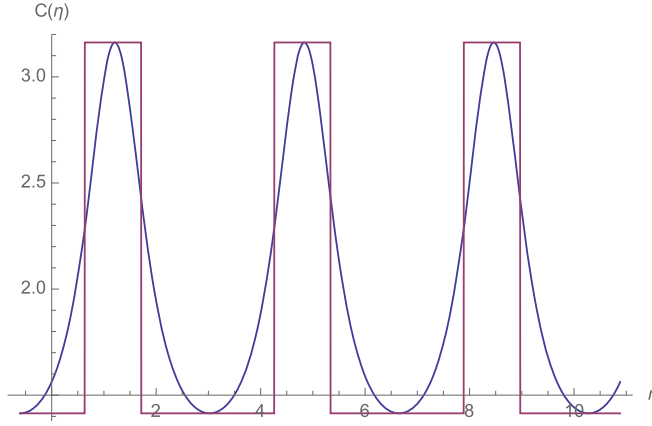


Figure 1. Kronig–Penney approximation of the energy $\omega_k(\eta)$ for $k = m = 1$. We plot both the real and approximated versions of the effective potential.

$\omega_u(k) = \sqrt{k^2 + m^2 a^2 (\eta_{\text{contr}})}$ in the low and high energy regimes, with η_{bounce} being the local bounce, meanwhile η_{contr} being the local time of the maximum expansion, and the time of the switch at half maximum, $\eta_{\text{inf}}(k)$ and $\eta_{\text{sup}}(k)$ are defined by the two (locally in each cycle) solutions of the equation for η :

$$\omega_k(\eta) = \frac{1}{2}(\omega_k(\eta_{\text{bounce}}) + \omega_k(\eta_{\text{contr}})), \quad (21)$$

in which the smaller solution is η_{inf} , and the larger given by η_{sup} , denoting the transition from the bounce to the maximum expansion and vice versa, respectively. We denote the n th recurrence of the $\eta_{\text{inf/sup}}(k)$ as $\eta_{\text{inf}}^n(k)$, as for a lattice.

For each mode k , we solve the contact equation at each transition point, from one bounce to the other one, imposing the continuity and differentiability of the solution. If ϕ_k^n is the solution within at the n th bounce, and ϕ_k^{n+1} at the $n + 1$ th bounce, one can use a transfer matrix approach for calculating the transition coefficients. If $\tilde{\phi}_k^n$ is the wavefunction in the contracting phase, one can write the contact equations between the various solutions at $\eta_{\text{inf/sup}}^n$:

$$\begin{aligned} \phi_k^n(\eta_{\text{inf}}^n) &= \tilde{\phi}_k^n(\eta_{\text{inf}}^n), \\ \partial_\eta \phi_k^n(\eta_{\text{inf}}^n) &= \partial_\eta \tilde{\phi}_k^n(\eta_{\text{inf}}^n), \\ \tilde{\phi}_k^n(\eta_{\text{sup}}^n) &= \phi_k^{n+1}(\eta_{\text{sup}}^n), \\ \partial_\eta \tilde{\phi}_k^n(\eta_{\text{sup}}^n) &= \partial_\eta \phi_k^{n+1}(\eta_{\text{sup}}^n) \end{aligned} \quad (22)$$

and if one expands the various terms in Fourier modes:

$$\begin{aligned} \phi_k^n(\eta) &= A_n e^{i\omega_l(k)\eta} + B_n e^{-i\omega_l(k)\eta}, \\ \tilde{\phi}_k^n(\eta) &= C_n e^{i\omega_u(k)\eta} + D_n e^{-i\omega_u(k)\eta} \end{aligned} \quad (23)$$

as a result of equation (22), one can write a matricial equation:

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = T \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \quad (24)$$

In general, the transfer matrix is a rotation due to Bloch theorem, $\det(T) = 1$, and in particular each element is the wronskian between the two solutions:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} W[\phi_n, \phi_{n+1}] & W[\phi_n, \phi_{n+1}^\dagger] \\ W[\phi_n^\dagger, \phi_{n+1}] & W[\phi_n^\dagger, \phi_{n+1}^\dagger] \end{pmatrix}. \quad (25)$$

Thus, the element T_{12} represents $2i$ times the Bogoliubov coefficient which represents β_k , connecting the modes from one bounce to the next. Due to the symmetry of the problem, the transfer matrix depends only on the mode k , once one has fixed the mass, and the *shape* of the function $a^2(\eta)$. From the physical point of view, we are calculating the tunneling of the modes between one bounce to the other, i.e. from one ground state to the other in this approximation.

The elements of the transfer matrix derived in the Kronig–Penney approximation of the cosmological bounce. We define $\Delta\eta = \eta_{\text{sup}} - \eta_{\text{inf}}$ and $\bar{\eta} = \eta_{\text{inf}} + \eta_{\text{sup}}$. The transfer matrix elements are given by T_{ij} , with $1 \leq i, j \leq 2$. Thus we have:

$$\begin{aligned} T_{11} &= \frac{e^{-i\omega_l(k)\Delta\eta}}{2\omega_l(k)\omega_u(k)} [2\omega_l(k)\omega_u(k)\cos(\omega_l(k)\Delta\eta) \\ &\quad + i(\omega_l(k)^2 + \omega_u(k)^2)\sin(\omega_u(k)\Delta\eta)], \\ T_{12} &= -T_{21} = im^2 \frac{a^2(\eta_{\text{contr}}) - a^2(\eta_{\text{bounce}})}{2\omega_u(k)\omega_l(k)} \\ &\quad \cdot e^{-i(\omega_u(k)+\omega_l(k))\bar{\eta}} [e^{2i\omega_u(k)\eta_{\text{inf}}} - e^{2i\omega_u(k)\eta_{\text{sup}}}], \\ &= im^2 \frac{a^2(\eta_{\text{contr}}) - a^2(\eta_{\text{bounce}})}{2\omega_u(k)\omega_l(k)} \\ &\quad \cdot e^{-i(\omega_u(k)+\omega_l(k))\bar{\eta}} e^{2i\omega_u(k)(\eta_{\text{inf}} + \frac{\Delta\eta}{2})} \sin(\omega_u(k)\Delta\eta) \\ &\propto m^2 \frac{a^2(\eta_{\text{contr}}) - a^2(\eta_{\text{bounce}})}{2\omega_u(k)\omega_l(k)} \sin(\omega_u(k)\Delta\eta), \\ T_{22} &= \frac{1}{4\omega_l(k)\omega_u(k)} e^{-i[(2\omega_u(k)+\omega_l(k))\eta_{\text{inf}} + (\omega_u(k)-\omega_l(k))\eta_{\text{sup}}]} \\ &\quad \cdot [e^{i\omega_l(k)\eta_{\text{inf}}} (e^{2i\omega_u(k)\eta_{\text{inf}}} - e^{2i\omega_u(k)\eta_{\text{sup}}}) \\ &\quad \cdot (\omega_u(k)^2 + \omega_l(k)^2) \\ &\quad + 4e^{i[(\omega_l(k)+\omega_u(k))\eta_{\text{inf}} + \omega_u(k)\eta_{\text{sup}}]} \\ &\quad \cdot \omega_u(k)\omega_l(k)\cos(\omega_u(k)\Delta\eta)], \end{aligned}$$

which one can verify to satisfy the condition of the transfer matrix: (i) $\det(T) = 1$, (ii) for $\Delta\eta \rightarrow 0$ or $\omega_l(k) \rightleftharpoons \omega_u(k)$, $T_{11} \rightarrow T_{22} \rightarrow 1$, meanwhile $T_{12} \rightarrow 0$. Since we are interested in the off-diagonal elements of this matrix, we note that for $m = 0$, the Bogoliubov coefficient goes to zero; in addition to this, it is proportional to the difference between the maximum and minimum conformal factors, $a^2(\eta_{\text{bounce}}) - a^2(\eta_{\text{contr}})$. It is also clear, that the Bogoliubov coefficient is proportional to $\sin(\omega_u(k)\Delta\eta)$, and thus the faster the Universe expands and contracts, the fewer particles are produced.

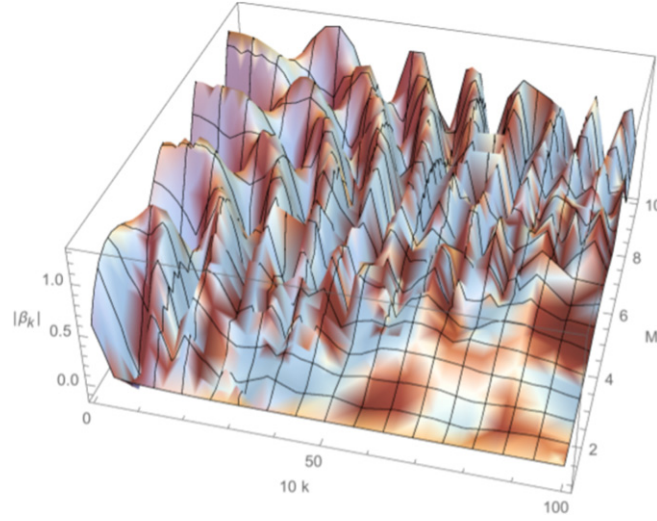


Figure 2. In this figure we plot $|\beta_k(m)|$ as a function of m and k . We observe both interference effects and thresholds related to the mass of the scalar field. These features will be also present in the other approximations.

In figure 2 we plot the Bogoliubov coefficient between one bounce and the next in the Kronig–Penney approximation for various values of k and m . As we will see, many of the phenomenological features of this approximation are retained in the exact solution of the differential equation with a smoother approximation of $a(\eta)$, as for instance interference patterns and particle production which goes to zero as $k \rightarrow \infty$. We note, however, that in this approximation the transfer matrix is independent from the bounce number n which we used to diversify one bounce from the other. One can, in fact, interpret the Kronig–Penney approximation as a two-way WKB approximation at the Bounce and at the point of maximum expansion in which one has $\partial_\eta \omega_k(\eta) = 0$. As a last comment to conclude this section, we note that due to the discontinuous approximation of the potential, several interference effects which are spurious occur. This will be solved in the next section, by means of a smoother approximation.

3.2. Mathieu fields

The Kronig–Penney approximation is a first, rough approach to calculating the Bogoliubov coefficients. In this section we work on the analytical solution of the differential equation which describes the evolution of the modes. Differently from the case discussed before, we consider a simple smooth approximation of the conformal factor given by the function:

$$a^2(\eta) = a_0 + b \cos(2\eta), \quad (26)$$

whose period is $\Pi = \pi$. Within this approximation, the differential equation (6) is the well-known Mathieu equation, whose solutions are given by (15) with the following analytical expressions

$$e^{\pm i\mu_k \eta} f_k^\pm = C \left(a_0 m^2 + k^2, -\frac{1}{2} b m^2, \eta \right) \pm i S \left(a_0 m^2 + k^2, -\frac{1}{2} b m^2, \eta \right), \quad (27)$$

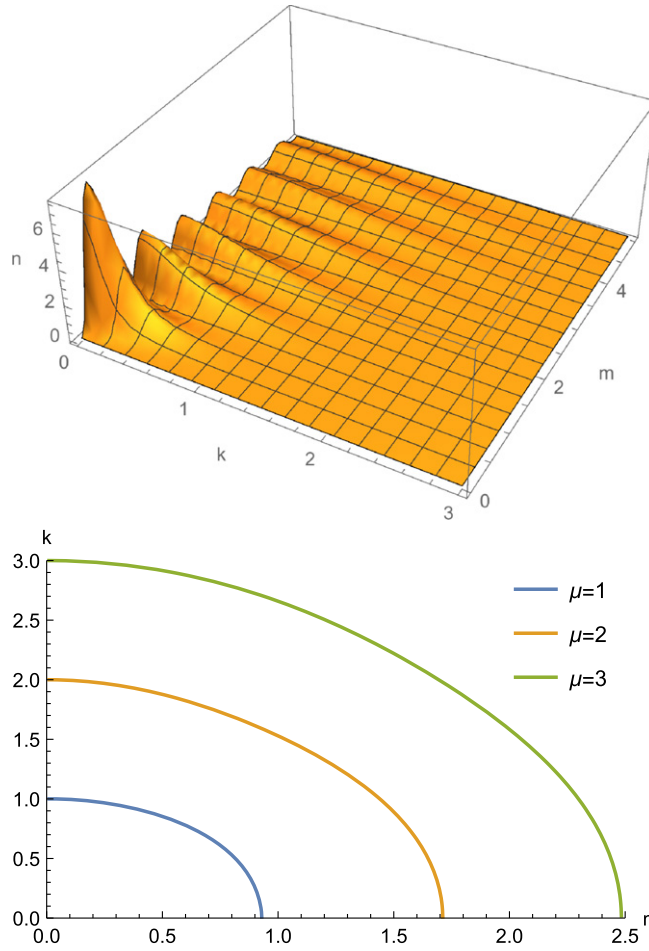


Figure 3. (top) Average number of particles n after a full oscillation ($T = 1$) in terms of momentum k and mass m , given by equation (29). (bottom) Value of (k, m) such that μ_k is integer. We used $a_0 = 2.0$, $b = -1.9$.

where $C(\cdot, \eta)$ is the even Mathieu function, $S(\cdot, \eta)$ is the odd Mathieu function and, in this case, μ_k the Mathieu exponent [27]. Mathieu functions have a long history of applications in quantum mechanics (see e.g. Slater [28]) with important applications to the theory of optical lattices [29, 30]. From (16) one finds that after T full oscillations the average number of quasi-particles is

$$n_k = \left| \frac{1 - g_k^2}{2g_k} \sin(\pi\mu_k T) \right|^2, \tag{28}$$

$$g_k = \frac{C\left(a_0 m^2 + k^2, -\frac{1}{2} b m^2, 0\right)}{S'\left(a_0 m^2 + k^2, -\frac{1}{2} b m^2, 0\right)} \omega_k(0). \tag{29}$$

For the present analysis we are interested only in the regions where μ is real but, for the sake of completeness, it is worth mentioning that there are some zones in the parameter space

where the Mathieu exponent can become complex and the solutions of equation (6) may grow exponentially as $T \rightarrow \infty$ [31]. From figure 3(a) one can see that, for small values of k , the energy gap $\Delta = \min_{\eta} \omega_k(\eta)$ is sufficiently small to allow for the creation of some particles. Similar interference effects are found also using the Kronig–Penney approximation, although emphasized due to the non-smooth approximation in of $a(\eta)$. It is interesting to see that, for fixed small k , there are some values of m where no particles are created. This can be explained by noting that for some values of (k, m) the Mathieu exponent μ_k can be an integer number, making $n_k = 0$ in (29). The locus of points (k, m) where μ_k is integer is shown in figure 3 (bottom).

On the other hand, when $a_0 \gg b, m$ we expect that the system remains in its ground state during the evolution because of the large energy gap Δ which prevents the creation of particles. In more mathematical terms, indeed we find that in this limit $g_k \approx 1$ making $n_k \simeq 0$.

An interesting question is whether the number of particles remains finite after many oscillations. Indeed, if the current lifetime of the Universe is much larger than the oscillation time, then in our Universe we can expect to observe extra particles due to the oscillating geometry. Calling $N = \sum_{\vec{k}} n_k = 4\pi \int dk k^2 n_k$ the total number of particles we find

$$N = \frac{\pi}{2} \int_0^\infty dk \left(\frac{1 - g_k^2}{g_k} \right)^2 k^2 + \mathcal{O}(T^{-1/2}), \quad (30)$$

where the $\mathcal{O}(T^{-1/2})$ correction is due to the stationary phase approximation and can be neglected for $T \gg 1$. In our numerical simulations N is finite for any reasonable value of a_0 and b . Moreover, for large energy gaps, namely when $a_0 \gg b, m$ we still find that $N \approx 0$, showing that the particle content of the Universe is cyclic.

We have discussed some limits where not only the geometry but also the quantum fields show a cyclic or almost cyclic evolution. In the next section we ask whether this peculiarity shows some quantum features in the form of a non-zero geometric phase.

4. Berry's phase

The Berry phase is a geometrical phase acquired by a quantum mechanical system during its evolution. It was introduced in [32] for adiabatic evolutions and then promptly extended to more general cases [33]. This geometrical phase is gauge-invariant and can be regarded as a generalization of the Aharonov–Bohm phase, which on the other hand appears only in specific cases when the fields are coupled to electromagnetic radiation. Aside from its fundamental implications, the Berry phase has been used also to gain further understanding of complicated phenomena such as quantum phase transitions [34, 35] and the dynamics of entanglement [37, 38], while its non-abelian generalization provides an alternative avenue for quantum computation [39].

In a cyclic Universe one can interpret the occurrence of a Berry phase as a memory of the quantum system on the previous cycle. From this perspective, we are indeed interested in showing that for the case of cyclic cosmologies, such phases could occur to be non-zero. As a matter of fact, assuming that the Universe is topologically equivalent to a sphere in which the radius for the spectator scalar field represents an external forcing undergoing a loop in the space of parameters. Thus, the conditions for the emergence of a non-trivial phase are present. Moreover, we have previously discussed that, if a Universe is cyclic, the vacuum states can display a periodic evolution or almost periodic evolution, depending on the geometry of the

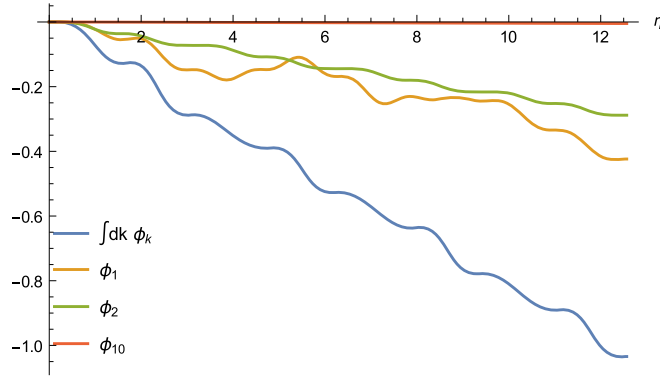


Figure 4. Integral $\int dk \phi_k$ and phases ϕ_k for different values of k as a function of η . The conformal factor is given by equation (26) with $a_0 = 2$, $b = 1$, while $m = 1$.

space–time. For cyclic evolutions, thus, a non-zero Berry phase can emerge in the ground state.

The Berry phase ϕ_B for a state $|\Omega(\eta)\rangle$ which undergoes a cyclic evolution of period Π is given by [33]

$$\phi_B = i \int_0^\Pi \langle \Omega(\eta) | \frac{\partial}{\partial \eta} \Omega(\eta) \rangle d\eta. \tag{31}$$

We specify to the case in which $|\Omega(\eta)\rangle$ is the ground state of the Universe at conformal time η , and we assume that the Universe was in its ground state at time $\eta = 0$. Exploiting the relation (11) between ground states at different conformal times one can show that the Berry connection is

$$\langle \Omega | \Omega' \rangle = \frac{1}{2} \sum_{\bar{k}} \left(-\frac{|\alpha_{\bar{k}}|'}{|\alpha_{\bar{k}}|} - \frac{\alpha_{\bar{k}}'}{\alpha_{\bar{k}}} |\beta_{\bar{k}}|^2 + \beta_{\bar{k}}^{*'} \beta_{\bar{k}} \right), \tag{32}$$

$$= \sum_{\bar{k}} \frac{1}{4} \left(\frac{\alpha_{\bar{k}}'}{\alpha_{\bar{k}}} - \frac{\alpha_{\bar{k}}^{*'}}{\alpha_{\bar{k}}^*} \right) + \frac{\beta_{\bar{k}}^{*'} \beta_{\bar{k}} - \alpha_{\bar{k}}^* \alpha_{\bar{k}}'}{2}. \tag{33}$$

The above equation can be written in a more convenient form by defining the quantity

$$\gamma_k = \frac{v_k}{v_k' + i\omega v_k}, \tag{34}$$

which satisfies the differential equation

$$\gamma_k' = 1 - 2i\omega_k \gamma_k - i\omega_k' \gamma_k^2, \tag{35}$$

with the initial condition $\gamma_k(0) = [2i\omega(0)]^{-1}$. Indeed, by substituting (10) in (33) one finds

$$\phi_B = \int_0^\Pi \sum_{\bar{k}} \frac{\omega_{\bar{k}}'(\eta)}{2} \Re[\gamma_{\bar{k}}(\eta)] d\eta, \tag{36}$$

$$= 2\pi \int_0^\Pi d\eta \int_0^\infty dk k^2 \omega_k'(\eta) \Re[\gamma_k(\eta)]. \tag{37}$$

Before evaluating the Berry phase for some relevant parameters, we study the dependence on k of the integrand $I_k = 2\pi k^2 \omega_k'(\eta) \Re[\gamma_k(\eta)]$. When $k \gg m_{\text{eff}}$, m_{eff}' it is $\omega_k \simeq k$,

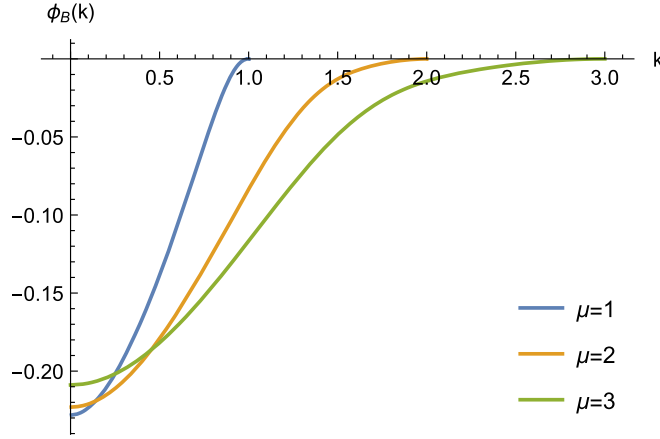


Figure 5. k -dependent Berry's phase $\phi_B(k)$ as a function of k , evaluated along the curves in the (k, m) space shown in figure 3(down).

$\omega'_k \simeq 0$ and hence $I_k \simeq 0$. Therefore, one expects that the integral over k in equation (37) can be performed up to a cut-off value k_{cut} without introducing significant errors. In particular, here we evaluate the Berry's phase numerically by solving the coupled differential equation $\phi'_k(\eta) = I_k(\eta)$ and equation (35) for several values of $k \in [0, k_{\text{cut}}]$ and then using $\phi_B = \int_k \phi_k(\Pi) dk$, where the latter integral is evaluated numerically. For instance, in figure 4, we show that $\phi_k \simeq 0$ for $k \gtrsim 10$, while on other hand the integral $\phi(\eta) = \int_k \phi_k(\eta) dk$ is non-zero and increases for increasing η .

From the form of the Berry connection one can see that, if there is no particle creation during the evolution of the Universe, i.e. $\beta_k(\eta) = 0$ for each value of η , then the Berry phase is exactly zero. In a non-trivial cyclic evolution on the other hand, some particles are created during the bounce, but then after one period the fields go back to their initial state. This observation, that will be clarified in the following, motivates our interpretation of the Berry phase as a memory of the dynamics of the Universe. However, as in equation (37) the constraint about the periodicity of the initial state is not explicit, we start our analysis by introducing the k -dependent Berry phase defined by

$$\phi_B(k) = \int_0^\Pi \frac{\omega'_k(\eta)}{2} \Re[\gamma_k(\eta)] d\eta. \quad (38)$$

Indeed, in section 3.2 we have shown that, for a given cosmological factor, there are some pairs (k, m) where there is no particle creation after a bounce. In other terms, particles are created during the evolution of the Universe but after a cycle the ground state is exactly the initial ground state. In particular, for the simple cosmological factor (26), we have shown that the locus of points where $n_k(\Pi) = 0$ define different curves (see figure 3) in the (k, m) space where the Mathieu exponent μ takes integer values. We now evaluate the k -dependent Berry phase along those curves and show that the Berry's phase is non-zero even when the ground state is perfectly periodic, thus signaling a non-trivial quantum evolution of the Universe. The results are shown in figure 5. Moreover, we observe that $\phi_B(k)$ is larger for smaller values of k where the corresponding mass in the curves figure 3 is smaller.

The results presented in figure 5 show a rigorous findings of a non-zero Berry phase acquired after one cycle by some discrete set of modes for which $n_k \equiv 0$. However, in the

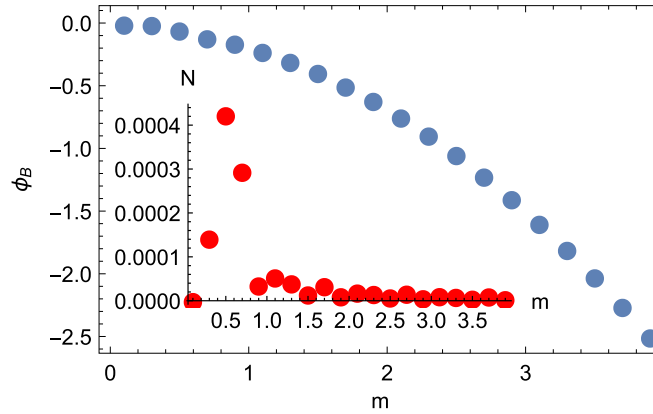


Figure 6. Berry's phase after one cycle ϕ_B as a function of the mass m . The oscillating geometry is implemented by the factor (18). Inset shows the mean density of particles in the state $|\Omega(\Pi)\rangle$.

Universe there are many modes and, as discussed in the section 3, in general the vacuum after one period $|\Omega(\eta_0 + \Pi)\rangle$ is different from $|\Omega(\eta_0)\rangle$. The closeness of these two vacua can be evaluated with the fidelity

$$\begin{aligned} \mathcal{F} &= |\langle \Omega(\eta_0) | \Omega(\eta_0 + \Pi) \rangle|^2 = \prod_{\vec{k}} |\alpha_k|^{-1} \\ &= e^{-\frac{1}{2} \sum_{\vec{k}} \log(1 + |\beta_k|^2)} \simeq e^{-N/2}, \end{aligned} \quad (39)$$

where $N = \sum_{\vec{k}} n_k = 4\pi \int dk k^2 |\beta_k|^2$ is the mean density of particles and where, in the last equality, we assumed that $|\beta_k|^2 \ll 1$. We claim that, even when $|\Omega(\eta_0 + \Pi)\rangle$ is different from $|\Omega(\eta_0)\rangle$, and thus the evolution is not perfectly cyclic, if $N \ll 1$ the two vacua are almost indistinguishable ($\mathcal{F} \simeq 1$). In that limit we can thus consider the field evolution approximately periodic and evaluate the Berry phase.

In section 3 we have shown that, as far as the conformal factor (26) is concerned, the condition $N \ll 1$ can be obtained when the gap Δ is large, i.e. when $a_0 \gg b, m$. In the inset of figure 6 we show that this holds also for the more complicated geometry of equation (18). Given that in these conditions the Universe is cyclic, in figure 6 we evaluate the resulting Berry phase of equation (37) and we find that it is non-zero. A non-trivial Berry phase seems thus a general occurrence in cyclic Universes. To motivate this further, we consider more general scale factors with higher harmonics in the Fourier expansion (26), namely we set $a^2(\eta) = \tilde{a}_{n,\alpha}^2(\eta) \equiv a_0 + \sum_{j=1}^n a_j \cos(2\eta j)$ and we choose $a_j = j^{-\alpha} / \sum_{k=1}^n k^{-\alpha}$ so that the dominant frequency is 2. As shown in figure 7, when $a(\eta) = \tilde{a}_{n,\alpha}(\eta)$ the number of particles increases with n , although $N \ll 1$ especially for $\alpha = 2$. In all the cases discussed so far, the Berry phase is non-zero after a bounce so we argue that it is a generic feature of cyclic Universes.

In order to show that the Berry phase heirs information of past evolution of the Universe in this toy model, we consider the accumulated phase in the limit $k^2 \gg m^2 a^2(\eta)$. In this case an analytical result can be obtained (see appendix 6 for a full derivation), showing that wavelengths at which Berry's phase is zero can be enumerated and given by:

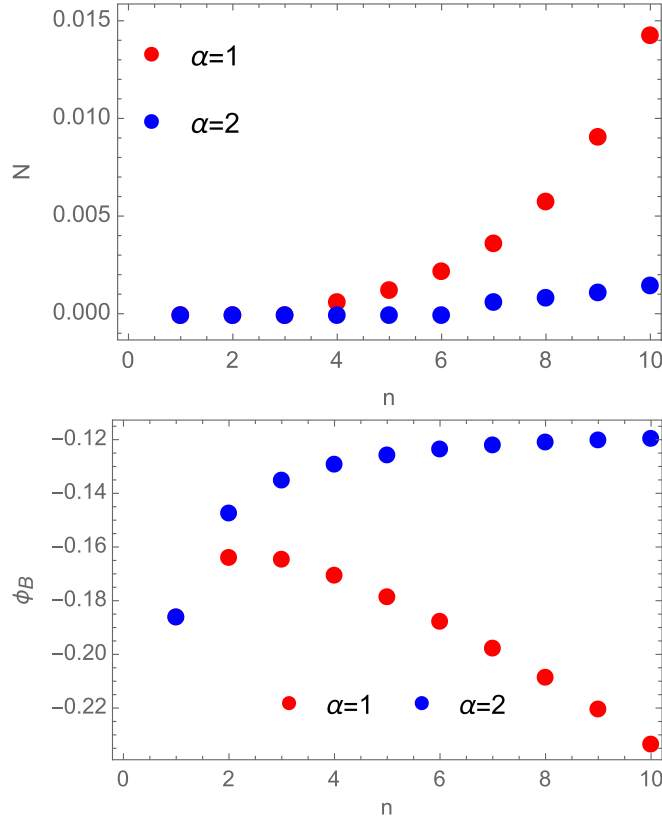


Figure 7. Total number of particles N and Berry's phase ϕ_B after the first cycle when $a(\eta) = \tilde{a}_{n,\alpha}(\eta)$ (see discussion in the text) for different values of n and α . $a_0 = 20$, $m = 3$.

$$k^*_{n+} = \Pi \left(2n - \frac{m^2 E}{4\pi n^2} \right) \tag{40}$$

where $E = \frac{1}{2\pi} \int_0^\Pi a^2(\eta) d\eta$ and Π is the length of a cycle measured in conformal time. These wavelengths contain information on the life cycle of the Universe as equation (40) shows.

Measuring the Berry's phase requires the introduction of extra degree's of freedom coupled with the field. To show this point, we use a simplified argument and we consider a further term in the potential $V(g)$ in (3) of the form $m_r^2 S^z$, where S^z is a spin-1/2 operator in the z direction. Depending on the value of the spin along the z direction, this extra coupling renormalizes the mass. Because, as we have shown, the Berry's phase depends on the mass of the field, after one cycle the state of the coupled system formed by the field and the spin is

$$\frac{1}{\sqrt{2}} \left(e^{i\phi_B^+} |\Omega_+\rangle |S^z = +\frac{1}{2}\rangle + e^{i\phi_B^-} |\Omega_-\rangle |S^z = -\frac{1}{2}\rangle \right), \tag{41}$$

namely there is a different Berry's phase depending on the state of the spin. The relative Berry's phase can then be measured, e.g., with $\langle S^x \rangle$. Although simplified, our argument shows that the Berry's phase can have measurable effects when the field is coupled to other external degrees of freedom.

5. The case of LQC

In the preceding sections we have tried to keep the discussion as general as possible. In fact, equation (40) applies to any scale factor which is periodic in conformal time. In this section, we will calculate the Berry phase using the effective LQC equations for a closed Universe. We start by introducing the standard Friedmann equations in a generic matter-dominated Universe for $k = 1$ in the dust approximation, in which $P = 0$ and $\rho(t)a(t)^3 = \text{constant}$. The first Friedmann equation becomes [40]:

$$H = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\rho_0}{a^3} - \frac{1}{a^2}, \quad (42)$$

and if for simplicity we consider the case $A = \frac{8\pi G \rho_0}{3}$, then it is a known fact that the scale factor takes the form $a(\eta) = A(1 - \cos(\eta))$. Inevitably, such equation describes a Universe in which also a Big Bang is present, as dynamically the scale factor eventually becomes zero at values of conformal time $\eta = \{0, \pi, 2\pi, \dots, n\pi\}$.

These equations describe a Universe in which there is a maximum expansion (the Big Crunch), followed by a recollapse. It is a widely accepted idea that during the recollapse phase, when the linear size of the scale factor is of the order of the Planck length, Quantum Gravity phenomena take place [19, 20]. LQC is one example in which a Bounce due to quantum corrections occurs. After the bounce, a new expansion phase is recovered, and thus it is possible to observe a cyclic dynamics of the scale factor. Thus, the machinery we have developed can be applied to the case of LQC if considering purely semiclassical Friedmann equations.

Here we consider the effective corrected equations derived in a recent paper [41], where equation (42) is corrected by a term which contains a quantum gravitational term

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \left(\frac{8\pi G}{3} \frac{\rho_0}{a^3} - \frac{1}{a^2}\right) \left(1 - \frac{\rho}{\rho_{\text{LQC}}}\right) \\ &= \left(\frac{8\pi G}{3} \frac{\rho_0}{a^3} - \frac{1}{a^2}\right) \left(1 - \frac{\rho_0}{a^3 \rho_{\text{LQC}}}\right). \end{aligned} \quad (43)$$

In general, equations of the type above strongly depend on the details of the quantization procedure, but the phenomenology described is well captured by equation (43), which differs in the exact values of ρ_{LQC} . The effective model of equation (43) agrees with other quantization procedures [42] up to higher orders in the quantization procedure, but differs from well known results in the field [20]. Another important comment to be made here is that the derivatives with respect to ‘time’, are now with respect to an external scalar field which takes the role of a clock, which is a standard procedure in quantum gravity approaches [19, 20]. Notwithstanding these important remarks and without going into the merit of the quantization procedure, the dynamical interpretation of equation (43) is clear: at large scale factor, matter forces the Universe to recollapse, meanwhile at scale factors comparable to the Planck length, the quantum regime kicks in, forcing the Universe to re-expand. This beautiful and appealing picture is the mechanism for which the singularity of the big bang is avoided, and which we considered in this paper insofar in a stylized manner. In order to proceed with our calculations, we need to recast the equation (43) into a form which is explicitly solvable as a function of the conformal time η . It is easy to use the definition of conformal time, and turn equation (43) into equation (44) below:

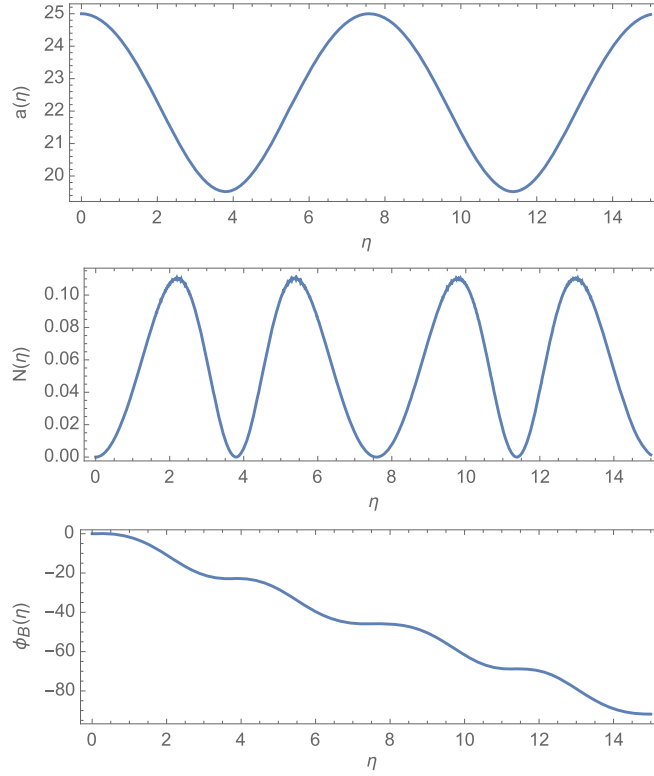


Figure 8. Conformal factor $a(\eta)$, total number of particles $N(\eta)$ and dynamical (Berry) phase $\phi_B(\eta)$ for $m = 3$, $a(0) = 25$, $\rho_{\text{LQC}} = 10^{-4}$, $\rho_0 = \rho_{\text{LQC}}/2$, $G = \frac{3}{32\pi a(0)^2 \rho_0}$.

$$a''(\eta) = a^3(\eta) \left(-\frac{3A \left(1 - \frac{2}{a(\eta)^3 \rho_{\text{LQC}}} \right)}{2a(\eta)^3} + \frac{A \left(1 - \frac{1}{a(\eta)^3 \rho_{\text{LQC}}} \right)}{a(\eta)^3} + \frac{3}{8a(\eta)^5 \rho_{\text{LQC}}} \right) \quad (44)$$

for the case $a_0 = A = 1$. One can in general obtain periodic solutions to the scale factor, as in figures 8 and 9. It is a general feature, usually discussed in the literature of LQC, that the Universe remains surprisingly close to the classical trajectories predicted by equation (44) also in the quantum regime. In this case, we can in fact use the semiclassical method described in the body of this paper, and evaluate the Berry phase numerically after each cycle.

We stress that the Berry phase can be observed only if the dynamics of quantum wave function of the scalar field is cyclic. This implies that depending on the parameters ρ_{LQC} and ρ_0 , we have to check whether the particle content of the Universe is negligible. In figure 8 we plot from the top to the bottom, the scale factor, the particle content and the Berry phase respectively obtained numerically for the case $\rho_{\text{LQC}} = 2\rho_0$, i.e. when the density of the Universe is smaller but comparable to the effective density due to LQC corrections. We observe that the particle content $\forall \eta$, $N(\eta) \ll 1$ due to the smooth Bounce. In this case, we observe a non-zero Berry phase which implies that at the quantum level the scalar field carries information from one Bounce to the other.

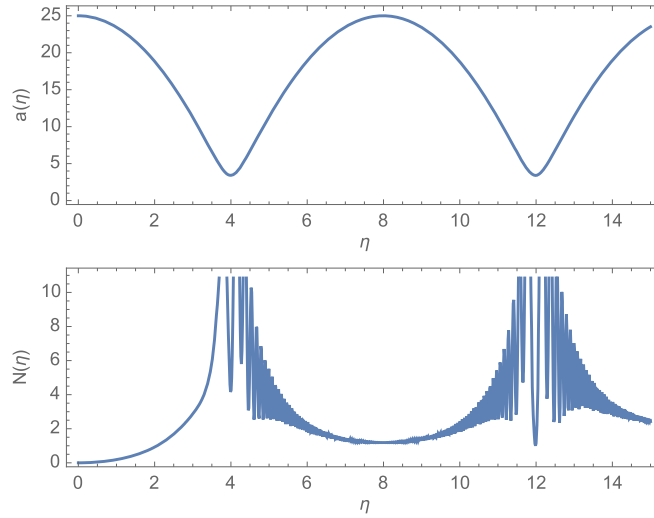


Figure 9. Conformal factor $a(\eta)$ and total number of particles $N(\eta)$ for $m = 3$, $a(0) = 25$, $\rho_{\text{LQC}} = 10^{-4}$, $\rho_0 = \rho_{\text{LQC}}/100$, $G = \frac{3}{32\pi a(0)^2 \rho_0}$.

In figure 9 we show the same plots obtained for the case $\rho_{\text{LQC}} = 100\rho_0$, and thus in the regime in which the Bounce is abrupt. In this case, we observe a high particle content, which although invalidating the calculation of the Berry phase, implies that the scalar field carries information through the Bounce in the form of particles. This picture might seem to differ however from the results obtained in [20, 43, 44], where it has been shown that the dynamics has effectively no memory from the previous process. As a first comment, it is worth to note in [45] it has been discussed that although similar, the polymeric quantization differs from the standard harmonic oscillator quantization used here. Moreover, our analysis relies on the existence of a spectator scalar field which is quantized in a standard way, and is forced (parametrically) by the dynamics of the scale factor. Thus it is not the Universe *per se* carrying the phase, but the scalar field living on it.

Before concluding this section it is worth mentioning that, although in this paper we focused on standard semi-classical methods, the rigorous origin of the Berry's phase might be further clarified with a full-quantum treatment, without any semiclassical approximation, given that the Hilbert space structure is exactly known [46]. Specifically, it may be possible to generalize recent techniques [38], where the transition between quantum to classical behavior is explicitly discussed, to the case of loop quantum gravity.

6. Conclusions

In the present paper we have studied the quantum features of scalar field both in cyclic and bouncing toy Universes, and in the case of LQC effective Friedman equations. Among the many alternatives to the inflaton field, there are many approaches in which the Universe is considered cyclic. Among these, we mentioned LQC and the Ekpyrotic String approach. Our motivation was to understand whether a scalar field does hold memory of the previous bounces in subsequent ones, thus changing the physics at each bounce. We studied the particle content after many bounces in a general manner, and observed numerically that even

when the ground state is periodic, and hence there are no particles after one cycle, the Universe keeps track of its history via a non-zero Berry phase. We argue that this is potentially a physically relevant phenomenon if the scalar field is entangled, and have provided general arguments why such phases should be expected for more general, periodic evolutions of the conformal factor of the Universe. As a result of our analysis, we have observed within various approximations, as for instance the Kronig–Penney and the case of Mathieu approximation, the presence of particles, and noted that in cases where the particle content present at each new bounce is negligible, one can reliably observe the presence of a Berry phase. In fact, after having introduced a differential equation describing the Berry’s connection as function of time, we have obtained a numerical solution of the Berry’s phase, showing that this is non-zero for our toy model, but providing a general framework, based on the Fourier expansion of the energy of the scalar field, for calculating it perturbatively. We have observed that when the ground state is cyclic, for a particular mode, one can observe a non-zero Berry’s phase. In addition to this, we have observed that values for which the Berry phase are zero, contain information on the L^2 norm of the scale factor of the Universe over the cycle. This gauge-invariant phase is of geometrical origin and provides a unified framework to explain important observable phenomena like the Aharonov–Bohm effect. Being a measurable quantity, a non-zero Berry’s phase can provide a figure of merit to demonstrate that the Universe is cyclic. This phase becomes physically relevant if the scalar field is entangled to another field which can be measured. In this case, after each bounce the physical state of the system changes, and thus the physics changes as well.

In the first part of this paper, we worked initially within the framework of a toy model and thus there are some further limitations to this approach. First of all, we have considered modes which cross the bouncing phase without any new physics to intervene, as for instance Quantum Gravity. The paper does not describe that phase, and treats the Universe as a smoothly oscillating ball. Yet, we believe that this phenomenon is interesting *per se* from the theoretical standpoint, as also for generic Fourier expansions of the scale factor a non-zero Berry’s phase was observed. In the second part of this paper, we applied these ideas to the case of LQC, where a description of the quantum phase of these oscillations is provided. Also, an extra appeal of this approach for the present paper is that in LQC the quantum description is very close to the classical one, and thus we do not require the use of a full quantum gravitational approach. In the case of LQC, we find that for the case of a specific semiclassical Friedman equations for which there are oscillations of the scale factor, one can observe two specific regimes. In the first regime, in which the effective quantum density is comparable to the density of the Universe at $a = 1$, we observed numerically small particle content and non-zero Berry phase. In the other regime, in which the quantum effective density is much higher than the density of the Universe, one observes non-negligible particle content.

Our results are general enough to be considered in other cases in details. In fact, given the standard quantization procedure of the scalar field, we have considered the case in which the only requirement is that the scale factor has a Fourier expansion. We studied some specific cases, but used known relations for the Hill equation, which underlies the evolution of the modes of the scalar field. In addition to the remarks above, nothing prevents this same analysis to be repeated in the case of particles with spin other than zero, which are far more common than scalar particles. As a final remark, we stress that there are other ways in which a Berry’s phase could be originated for non-cyclic cosmologies. This is the case, for instance, if the scalar field undergoes a quantum phase transition in the early Universe [34, 36, 47] and as shown in [17] for inflationary models as well.

Acknowledgments

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Appendix. General considerations and Floquet's theorem

The differential equation in equation (6),

$$\phi'' + \omega_k(\eta)\phi = 0, \quad (\text{A1})$$

is called Hill's equation when $\omega_k(\eta)$ is periodic, and has interesting properties [23] which can be tackled analytically. In particular, according to Floquet's theorem, such equation has a pair of solutions which can be written in the form:

$$\begin{aligned} \phi_+(\eta) &= e^{i\alpha\eta} p_+(\eta), \\ \phi_-(\eta) &= e^{-i\alpha\eta} p_-(\eta), \end{aligned} \quad (\text{A2})$$

where α is called characteristic exponent and where p_{\pm} are functions which have the same periodicity of ω_k . If α is not an integer, then one has that

$$\begin{aligned} \phi_{\pm}(\eta + \Pi_k) &= e^{\pm i\alpha(\eta + \Pi_k)} p_{\pm}(\eta + \Pi_k) \\ &= e^{\pm i\alpha\Pi_k} e^{\pm i\alpha\eta} p_{\pm}(\eta) \\ &= e^{\pm i\alpha\Pi_k} p_{\pm}(\eta) \neq \phi_{\pm}(\eta). \end{aligned} \quad (\text{A3})$$

In particular, if ω_k is an even function with period Π_k , one can expand it in Fourier series:

$$\omega_k(\eta) = \omega_{0,k}^2 + 2\theta \sum_{n=1}^{\infty} \gamma_n \cos\left(2n\frac{\eta}{\Pi_k}\right). \quad (\text{A4})$$

γ_n 's are the expansion factors, and θ is a factor which we assume can be considered small. Let us rescale the conformal time such that $\Pi_k = 1$, and rescale $\omega_{0,k}$, $\omega_{0,k} \rightarrow \tilde{\omega}_{0,k} = \omega_{0,k} \Pi_k$ and $\tilde{\theta} = \theta \Pi_k^2$. In this case, then, there is a closed formula for estimating α :

$$\begin{aligned} \alpha = \pm & \left[\tilde{\omega}_{0,k} + \frac{\tilde{\theta}^2}{4\tilde{\omega}_{0,k}} \sum_{n=1}^{\infty} \frac{\gamma_n^2}{n^2 - \tilde{\omega}_{0,k}^2} \right. \\ & + \frac{\tilde{\theta}^3}{32\tilde{\omega}_{0,k}^2} \sum_{n,m=-\infty, n \neq m}^{\infty} \frac{\gamma_{|n|} \gamma_{|m|} \gamma_{|n-m|}}{nm(n + \tilde{\omega}_{0,k})(m + \tilde{\omega}_{0,k})} \\ & \left. + O(\theta^4) \right]. \end{aligned} \quad (\text{A5})$$

For our case of interest, where $\omega_k(\eta) = \sqrt{k^2 + m^2 a^2(\eta)}$, we need to relate the coefficients γ_n to ω_k . Assuming the periodicity of a^2 , one has $a^2(\eta) = \sum_{r=0}^{\infty} b_r \cos\left(2r\frac{\eta}{\Pi}\right)$; we now rescale $k \rightarrow \Pi$ and $m \rightarrow m\Pi$, one has that the Fourier coefficients γ_n introduced above but for the periodicity fixed to be equal to π , the coefficients are given by:

$$\gamma_n = \frac{\Pi}{\pi} \int_{-\pi}^{\pi} \cos(2n\eta) \sqrt{k^2 + m^2 a^2(\eta)}. \quad (\text{A6})$$

Let us now consider the two limits: $k \gg m^2 a^2(\eta)$ and $k \ll m^2 a^2(\eta)$. In the former case, one can expand the square root, obtaining the following formula for γ_n :

$$\begin{aligned}
\frac{\gamma_n}{\Pi} &\approx \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(2n\eta) k \, d\eta \\
&+ \frac{m^2}{2k} \sum_{i,j=0}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} b_i b_j \cos(2n\eta) \cos(2i\eta) \cos(2j\eta) \, d\eta \\
&= \delta_{n,0} \frac{k}{2} \\
&+ \frac{m^2}{2\pi k} \sum_{i,j=0}^{\infty} b_i b_j \int_{-\pi}^{\pi} \cos(2n\eta) \cos(2i\eta) \cos(2j\eta) \, d\eta \\
&= \delta_{n,0} \frac{k}{2} \\
&+ \frac{m^2}{2\pi k} \sum_{i,j=0}^{\infty} b_i b_j \frac{1}{4} [\delta_{n,i-j} + \delta_{n,i+j} + \delta_{n,j-i} + \delta_{n,-i-j}].
\end{aligned}$$

Now the last delta contributes only if $n = i = j = 0$, and thus:

$$\begin{aligned}
\frac{\gamma_n}{\Pi} &= \delta_{n,0} \left(\frac{k}{2} + \frac{m^2}{8\pi k} b_0^2 \right) \\
&+ \frac{m^2}{8\pi k} \sum_{i,j=0}^{\infty} b_i b_j [\delta_{n,i-j} + \delta_{n,i+j} + \delta_{n,j-i}] \\
&= \delta_{n,0} \left(\frac{k}{2} + \frac{m^2}{8\pi k} b_0^2 \right) + \frac{m^2}{8\pi k} \left[2 \sum_{i=0}^{\infty} b_i b_{n+i} + \sum_{i=0}^{n-1} b_i b_{n-i} \right], \tag{A7}
\end{aligned}$$

so, if we define $\tilde{k} = \max_{\eta} m^2 a^2(\eta)$, we have:

$$\begin{aligned}
\frac{\gamma_0^{k \gg \tilde{k}}}{\Pi} &= \omega^2 = \frac{k}{2} + \frac{m^2}{8\pi k} \sum_{i=0}^{\infty} b_i^2 \\
\frac{\gamma_{n \geq 1}^{k \gg \tilde{k}}}{\Pi} &= \frac{m^2}{8\pi k} \left[2 \sum_{i=0}^{\infty} b_i b_{n+i} + \sum_{i=0}^{n-1} b_i b_{n-i} \right]. \tag{A8}
\end{aligned}$$

Due to the Parseval's identity, $\sum_{i=0}^{\infty} b_i^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} a^2(\eta) \, d\eta = \frac{1}{2\pi} \int_{-\pi}^{\pi} a^2(\eta) \, d\eta = E$. It is easy to see that for certain values of γ_0 's, α is ill-defined. These are all the values of k for which γ_0 takes an integer value n ; an easy calculation shows that these values are given by $k_n = \Pi \left(n \pm \sqrt{n^2 - \frac{m^2 E}{2\pi}} \right)$. Since we are in the approximation in which $k^2 \gg m^2 a^2(\eta)$, we must have: $k_{n\pm} = \Pi \left(n \pm n \sqrt{1 - \frac{m^2 E}{4\pi n^2}} \right) \approx n \Pi \left[1 \pm \left(1 - \frac{m^2 E}{4\pi n^2} \right) \right] = \Pi * \left\{ 2n - \frac{m^2 E}{4\pi n^2}, \frac{m^2 E}{4\pi n^2} \right\}$. Of these phases, we disregard k_{n-} as this violates the condition we started from. Thus, these phases contain information on the whole evolution of the Universe's scale factor $a^2(\eta)$, and are those values for which the Floquet's theory fails and the phase α becomes an integer number.

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