CRT fractionalization in first-quantized Hamiltonian theory

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Symmetry analysis is a cornerstone of modern physics, with charge- and spacetime-orientation-reversal (CRT) symmetry being a subject of particular interest. Recent research has revealed that the CRT symmetry for fermions exhibits a fractionalization distinct from the $\mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{R}} \times \mathbb{Z}_2^{\mathcal{T}}$ symmetry for scalar bosons. In fact, the CRT symmetry for fermions can be extended by internal symmetries such as fermion parity $\mathbb{Z}_2^F$, chiral symmetry $\mathbb{Z}_2^\chi$, and continuous symmetries, thereby forming a group extension of the aforementioned $\mathbb{Z}_2$ direct product, and suffices to rule out bilinear mass terms. In the conventional framework, a Majorana fermion is defined by a single Dirac fermion with trivial charge conjugation. However, this definition encounters a fundamental challenge when the spacetime dimension $d+1 = 5,\ 6,\ 7 \bmod 8$, where the real dimension of Majorana fermion ($\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,0)}$) aligns with the real dimension of Dirac fermion ($\dim_{\mathbb{R}} \psi_{\mathcal{C}\ell(d)}$), rather than being half as in other dimensions. This peculiarity necessitates the introduction of a symplectic Majorana fermion, defined by a pair of Dirac fermions with trivial charge conjugation, to account for the discrepancy. To include these two types of Majorana fermions, we embed the Majorana theory in $n_{\mathbb{R}}$ and define the Majorana fermion field as a representation of the real Clifford algebra, which exhibits an eightfold Bott periodicity. Within the Hamiltonian formalism, we identify the eightfold CRT-internal symmetry groups across general spatial dimensions. In the case of Dirac fermions, the fermion field is defined as a representation of the complex Clifford algebra, which has a twofold Bott periodicity. Interestingly, we discover that the CRT-internal symmetry groups exhibit an eightfold periodicity that is distinct from that of the complex Clifford algebra. In certain dimensions where distinct mass terms can span a mass manifold, the CRT-internal symmetries can act nontrivially upon this mass manifold. Employing domain wall reduction method, we are able to elucidate the relationships between symmetries across different dimensions.

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I. INTRODUCTION AND SUMMARY

In modern physics, the analysis of symmetry has emerged as a cornerstone and has profoundly influenced the paradigm of theoretical research, giving rise to a plethora of intriguing topics, including symmetry-protected topological (SPT) phases [1–10] and Wigner-Dyson-Altland-Zirnbauer symmetry classification [11–13]. Among these symmetries, charge conjugation ($\mathcal{C}$), mirror reflection ($\mathcal{R}$), and time-reversal ($\mathcal{T}$) symmetries [14–26] stand out as particularly well-known. Charge conjugation operates as a unitary transformation, interconverting excitations and anti-excitations. For instance, in the context of a Dirac fermion, this is encapsulated by

$$\mathcal{C}\psi\mathcal{C}^{-1} = \psi^*, \quad \mathcal{C}\psi^*\mathcal{C}^{-1} = \psi, \tag{1}$$

where ψ^* denotes the "anti-Dirac fermion" within the complex conjugate space. This transformation is an internal symmetry, manifesting itself only through an active transformation on a particle or field. In contrast, mirror reflection $\mathcal{R}$ inverts the spatial coordinate and time-reversion $\mathcal{T}$ reverses the direction of time. These transformations are passive with respect to the spacetime coordinates $(t, x) = (t, x_1, \dots, x_d)$:

$$\mathcal{R}_i(t, x)\mathcal{R}_i^{-1} = (t, x_1, \dots, -x_i, \dots, x_d),$$
$$\mathcal{T}(t, x)\mathcal{T}^{-1} = (-t, x). \tag{2}$$

By combining mirror reflections across all spatial directions, we arrive at parity $\mathcal{P}$, a concept extensively discussed in the literature. However, when the spatial dimension $d \bmod 2 = 0$, parity $\mathcal{P}$ is subsumed within the spatial rotation symmetry group $SO(d) \subset SO^+(d, 1)$, and thus is not considered an independent discrete symmetry separate from Lorentz symmetries. This insight prompts us to prioritize the consideration of CRT symmetries [27,28] over CPT in the general context of spacetime dimensions beyond the conventional 3 + 1d framework. For instance, the CRT theorem [29] in a general dimension asserts that the combined CRT symmetry is unbreakable – a proposition not universally valid for CPT. Moreover, the canonical conditions for CRT symmetries are satisfied across all dimensions, which is a crucial prerequisite for domain wall reduction. In contrast, CPT symmetries fail to exhibit canonical properties in $d = 3 \bmod 4$. These considerations have motivated our focus on CRT over CPT in the ensuing discussion.

For a scalar boson ϕ, the symmetries $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{T}$ generate a direct product symmetry group $G_\phi = \mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^R \times \mathbb{Z}_2^{\mathcal{T}}$, where each symmetry independently generates a $\mathbb{Z}_2$ group. However, theoretical advancements have unveiled that fermion excitations can engender a more intricate symmetry structure [1–8,20,29–39] such as $\mathbb{Z}_4^{\mathcal{T}F}$ and the order-8 dihedral $\mathbb{D}_8^{\mathcal{CR},\mathcal{C}}$ groups [20]. In fact, Ref. [20] discovers that 3 + 1d Dirac fermion's CPT or CRT symmetry forms an order-16 nonAbelian Pauli group which is a central product of $\mathbb{D}_8^{\mathcal{CR},\mathcal{C}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{T}F}$. The deviation from the direct product structure is referred to as symmetry fractionalization, which is associated with quantum anomalies and typically manifests at the boundaries of a system. In many-body quantum systems, CRT fractionalization exhibits even richer structural nuances [9,10,40–42]. This process of symmetry fractionalization can also be comprehended through group extensions [20,43–48]. According to Ref. [20], symmetry fractionalization implies that the matter field is not in the linear representation of the original symmetry group G, but in the projective representation of G and in the linear representation of the extended group $\tilde{G}$. The extended group $\tilde{G}$ is characterized by the short exact sequence $1 \to N \to \tilde{G} \to G \to 1$, where N represents certain internal symmetries. In Ref. [43], N is chosen to be the fermion parity $\mathbb{Z}_2^F$ which flips the sign of the fermion. However, the additional internal symmetries of the Dirac fermion render the extended group $\tilde{G}$ nonunique, suggesting a more intuitive approach is to encompass all internal symmetries within N.

In the conventional framework, a Majorana fermion is defined by a single Dirac fermion with trivial charge conjugation [49–56]. This definition is based on the Majorana fermion's real dimension being half that of the Dirac fermion:

$$\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,0)} = 1/2 \dim_{\mathbb{R}} \psi_{\mathcal{C}\ell(d)},$$

a relationship derived from the representation theory of Clifford algebras (see Secs. II A 3 and III A 3). However, this definition falters when the spacetime dimension satisfies

$d+1=5,6,7 \bmod 8$, leading to

$$\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,0)} = \dim_{\mathbb{R}} \psi_{\mathcal{C}\ell(d)}.$$

To address this, we define a symplectic Majorana fermion [54,57] as two Dirac fermions with trivial charge conjugation, as discussed in Appendixes B and C). The original quaternion representation of symplectic Majorana fermions is not always convenient for analysis, which is why it was not included in Ref. [43]. To simplify, we embed Majorana fermions in $n_{\mathbb{R}}$ where n is the real dimension of the representation. This allows us to select gamma matrices that construct the Majorana equation as a real equation, with the trivial charge conjugation constraint simplifying to $\psi = \psi^*$, yielding a real fermion field. Therefore in this article, we systematically analyze the properties of Majorana fermions in the embedded $n_{\mathbb{R}}$ space, defining them uniformly as a real Grassmannian field, acting as an irreducible representation of the eightfold real Clifford algebra.

To elucidate the Clifford algebra structure within the embedded $\mathbb{R}(n)$ space, we scrutinize the internal symmetry of the Clifford algebra generated by the corresponding gamma matrices. For instance, the Sp(1) internal symmetry of the quaternion algebra is identifiable within the embedded $\mathbb{R}(n)$ space. These internal symmetries play a pivotal role as they can rotate the mass terms, culminating in the formation of a mass manifold. Notably, the mass manifold exhibits an especially intricate structure in spacetime dimensions $d+1=5,6,7 \bmod 8$ with Sp(1) symmetries, a feature not addressed in Ref. [43].

Our novel results are multifaceted and encompass a range of scenarios.

(1) We verify and generalize the results in Ref. [43] using Hamiltonian theory rather than Lagrangian theory in [43].

(2) We define the Majorana fermion as a real Grassmannian field, acting as an irreducible representation of eightfold real Clifford algebra to include the symplectic Majorana fermion in spacetime dimension $d+1=5,6,7 \bmod 8$ and the conventional Majorana fermion.

(3) We determine the invariant group for Majorana and Dirac fermions, including internal symmetries and CRT symmetries. The results are listed in Tables VIII, IX, XXVII, and XXVIII.

(4) We analyze the mass manifold generated by multiple mass terms. The symmetries can act nontrivially on the manifold, and the symmetry reduction under the domain wall is independent of the choice of mass term. The main results are collected in Secs. II C, II D, III C, and III D.

(5) CRT-internal symmetries are sufficient to rule out all bilinear mass terms. We systematically assign these symmetries in all dimensions for both Majorana and Dirac fermions and prove that we can use these symmetries to induce a gapless regime. Notably, the gapless regime is caused by symmetry constraint instead of anomaly, so we can still use four- or more-fermion interaction to symmetrically deform to a symmetric gapped phase as symmetric mass generation (SMG) [58].

Let us emphasize also the *condensed matter relevance* of our work.

(1) A primary motivation of this work is to understand the constraints imposed by the full C-R-T symmetry structure on fermionic Hamiltonians at the first-quantized level. In particular, we show that the combined C-R-T–internal symmetry constraint can forbid all possible fermion bilinear mass terms in certain spacetime dimensions. The absence of symmetry-allowed bilinear mass terms is a key ingredient in the study of fermionic symmetric mass generation (SMG) [58] and interacting fermionic Symmetry-Protected Topological (SPT) phases, where fermions may be gapped only through interactions rather than single-particle mass terms. Our results, therefore, provide a systematic framework for analyzing symmetry constraints relevant to the classification of interacting fermionic phases in condensed-matter systems. A central observation of this work is that the generators of the C-R-T symmetry group cannot, in general, be uniquely defined without taking into account accompanying internal symmetries. For example, the reflection generator acting on Dirac fermions can be modified by an internal U(1) phase factor, implying that a complete and well-defined symmetry group containing reflection should also incorporate the corresponding internal symmetry. Since this interplay between spacetime and internal symmetries has not been systematically analyzed in previous studies, we investigate in this work the group structure of the full C-R-T–internal symmetry and its consequences for fermionic Hamiltonians.

(2) In addition, quasicrystals are solids that exhibit long-range order but lack translational periodicity. They were first discovered experimentally [59] and theoretically described as quasiperiodic structures by Levine and Steinhardt [60]. A standard theoretical framework represents that quasicrystals are aperiodic materials, as projections of higher-dimensional periodic lattices onto physical space via the cut-and-project construction (see, e.g., Ref. [61]). In this approach, the symmetry structure of quasicrystals can naturally be formulated in terms of higher-dimensional crystallographic groups. Fermionic quasiparticles in such systems may therefore be analyzed using higher-dimensional symmetry frameworks, providing a potential condensed-matter setting in which our C-R-T-internal symmetry structures, studied in this work, shall play a key role.

In this work, we commence with an examination of the real Majorana fermion, as detailed in Sec. II. We define the Majorana fermion field as a real Grassmannian field, acting as an irreducible representation of the eightfold real Clifford algebra. When the spatial dimension $d=1,5 \bmod 8$, massless Majorana fermion splits into two isomorphic Cartan subalgebras, and the Majorana field in each subalgebra is defined as a Majorana-Weyl fermion. We define the corresponding symmetries for both massless Majorana and Majorana-Weyl fermions, encompassing Lorentz symmetry, RT symmetry, and internal symmetries that are intrinsic to the Clifford algebra $\mathcal{C}\ell(d,0)$. The results of these symmetries are compiled in Tables VIII and IX, where we observe an eightfold periodicity in the symmetry group. To further investigate the richer structures, we extend the mass terms from the representation space of $\mathcal{C}\ell(d,0)$ to that of $\mathcal{C}\ell(d,1)$, as outlined in Table XII. This extension allows distinct mass terms to form mass manifolds, on which the aforementioned symmetries can act nontrivially, as demonstrated in Tables XVI–XVIII. By employing domain wall reduction methods, we establish relations between symmetries in different dimensions, which are detailed in Tables XIX and XX.

Similarly, we also delve into the properties of the complex Dirac fermion, as discussed in Sec. III. We define the Dirac fermion field as a complex Grassmannian field, acting as an irreducible representation of twofold complex Clifford algebra. In odd spatial dimensions, a massless Dirac fermion splits into two isomorphic Cartan subalgebras, and the Dirac field in each subalgebra is defined as a Weyl fermion. We therefore define the corresponding symmetries for both massless Dirac and Weyl fermions, including Lorentz symmetry, CRT symmetry, and internal symmetries that are intrinsic to the Clifford algebra $\mathcal{C}\ell(d)$. Notably, the symmetries exhibit an eightfold periodicity rather than a twofold one. These results are systematically compiled in Tables XXVII and XXVIII. To further explore the richer structures, we extend the mass terms from the representation space of $\mathcal{C}\ell(d)$ to that of $\mathcal{C}\ell(d+1)$, as detailed in Tables XXIX and XXX. Distinct mass terms can form mass manifolds, on which the symmetries above can act nontrivially, as demonstrated in Tables XXXIII–XXXVI. By employing domain wall reduction methods, we establish relations between symmetries in different dimensions, which are detailed in Tables XXXVII and XXXVIII.

II. MAJORANA FERMION

In the well-known theory of the Dirac equation in $3+1d$ spacetime, we derive the Hamiltonian for a fermion by applying the square root towards the square of the relativistic energy $E^2 = \sum_i k_i^2 + m^2$:

$$H = \frac{1}{2}\int d^3x\psi^\dagger\left(\sum_{i=1}^{3}\alpha_i \mathrm{i}\,\partial_i + \beta m\right)\psi, \tag{3}$$

where α_i and β are 4×4 gamma matrices, satisfying

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1 \tag{4}$$

to ensure the relativistic energy spectrum, and ψ is a Dirac spinor with four components (or flavors). To include the conventional Majorana fermion ($d+1 = 0, 1, 2, 3, 4 \bmod 8$) and the symplectic Majorana fermion ($d+1 = 5, 6, 7 \bmod 8$), we uniformly define the Majorana fermion in the embedded $n_{\mathbb{R}}$ space as a real Grassmannian field acting as an irreducible representation of real Clifford algebra (see discussion in Sec. II A). In Sec. II B, we will introduce CRT and internal symmetries intrinsic to our minimal Majorana fermion model. The mass terms also form a nontrivial manifold in some dimensions as we will give a discussion in Sec. II C. The existence of mass terms will break some internal symmetries, and we will show that the CRT and internal symmetries together are enough to rule out all possible mass bilinear terms in Sec. II D. Finally, in Sec. II D 2, we will use the domain wall reduction method to give the relation between symmetry groups in different dimensions.

A. Field theory models

We first start with the field theory model of the real fermion χ whose complex conjugation χ^* is itself. To give a general field theory of the real Majorana fermion in a generalized dimension, we duplicate it to form a spinor χ with multiple components (or flavors), and the Hamiltonian is the "square root" of $\sum_i k_i^2 + m_i^2$ analogous to the case in $3+1d$ spacetime.

1. Hamiltonian

A free Majorana fermion theory in $(d+1)$-dimensional *spacetime* with n-dimensional *mass manifold* is described by the field theory Hamiltonian

$$H = \frac{1}{2}\int d^dx\chi^\mathsf{T} h\chi, \tag{5}$$

where h is the square root of $\sum_i k_i^2 + m_i^2$ with $d+n$ anticommuting matrices α_i and β_i:

$$h = \sum_{i=1}^{d}\alpha_i \mathrm{i}\,\partial_i + \sum_{i=1}^{n}\beta_i m_i. \tag{6}$$

In general, the matrix h must satisfy

$$h = -h^* = -h^\mathsf{T} = h^\dagger, \tag{7}$$

for the following reasons.

(1) *Hermiticity* of the Hamiltonian: $H = H^\dagger$. Since

$$H^\dagger = \frac{1}{2}\int d^dx\chi^\dagger h^\dagger \chi^{\dagger\mathsf{T}} = \frac{1}{2}\int d^dx\chi^\mathsf{T} h^\dagger\chi, \tag{8}$$

we must have $h = h^\dagger$. Here, we have used the fact that χ is a real field, s.t. $\chi^{\dagger\mathsf{T}} = \chi^* = \chi$.

(2) Compatible with the *fermion* statistics: $\chi_i\chi_j = -\chi_j\chi_i$, s.t.

$$\begin{aligned}\frac{1}{2}\int d^dx\chi^\mathsf{T} h\chi &= \frac{1}{2}\int d^dx\sum_{i,j}h_{ij}\chi_i\chi_j\\ &= -\frac{1}{2}\int d^dx\sum_{i,j}h_{ij}\chi_j\chi_i\\ &= -\frac{1}{2}\int d^dx\chi^\mathsf{T} h^\mathsf{T}\chi,\end{aligned} \tag{9}$$

therefore, we must have $h = -h^\mathsf{T}$.

Combining $h = h^\dagger$ and $h = -h^\mathsf{T}$, we conclude that h must satisfy Eq. (7).

Now given the specific form of h in Eq. (6), in which momentum operators $\mathrm{i}\,\partial_i$ and real scalar mass terms m_i satisfy

$$\mathrm{i}\,\partial_i = -(\mathrm{i}\,\partial_i)^\mathsf{T} = -(\mathrm{i}\,\partial_i)^* = (\mathrm{i}\,\partial_i)^\dagger, \quad m_i = m_i^\mathsf{T} = m_i^* = m_i^\dagger, \tag{10}$$

we must require

$$\alpha_i = \alpha_i^* = \alpha_i^\mathsf{T} = \alpha_i^\dagger, \quad \beta_i = -\beta_i^* = -\beta_i^\mathsf{T} = \beta_i^\dagger, \tag{11}$$

in order for the condition Eq. (7) to hold for h in Eq. (6).

In conclusion, the Hamiltonian H is fully specified by a real Clifford algebra $\mathcal{C}\ell(d, n)$ [62–64], which defines α_i and $\mathrm{i}\beta_i$ as its *Hermitian generators*, satisfying:

(1) *anticommutation* relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\mathrm{i}\beta_i, \mathrm{i}\beta_j\} = -2\delta_{ij}, \quad \{\alpha_i, \mathrm{i}\beta_j\} = 0; \tag{12}$$

(2) *real* conditions

$$\alpha_i = \alpha_i^* = \alpha_i^\mathsf{T} = \alpha_i^\dagger \in \mathbb{R}(\cdot), \quad \beta_i = -\beta_i^* = -\beta_i^\mathsf{T} = \beta_i^\dagger \in \mathbb{I}(\cdot). \tag{13}$$

TABLE I. Eightfold periodic table for real Clifford algebra $\mathcal{C}\ell(d,n)$. $2\mathbb{R}(1)$ is a short-hand notation for $\mathbb{R}(1)\oplus\mathbb{R}(1)$, and so on.

$d\backslash n$	0	1	2	3	4	5	6	7
0	$\mathbb{R}(1)$	$\mathbb{C}(1)$	$\mathbb{H}(1)$	$2\mathbb{H}(1)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$
1	$2\mathbb{R}(1)$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$2\mathbb{H}(8)$	$\mathbb{H}(16)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$2\mathbb{H}(16)$
5	$2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$2\mathbb{R}(64)$	$\mathbb{R}(128)$

2. Real Clifford algebra

After defining the free Majorana fermion field theory specified by real Clifford algebra $\mathcal{C}\ell(d,n)$, it's conducive for us to further analyze its algebraic structure.

Real Clifford algebras have the following recursive relations:

$$\begin{aligned}\mathcal{C}\ell(p,q)&\cong\mathcal{C}\ell(q,p-2)\otimes_{\mathbb{R}}\mathbb{R}(2),\\ \mathcal{C}\ell(p,q)&\cong\mathcal{C}\ell(q-2,p)\otimes_{\mathbb{R}}\mathbb{H}(1),\\ \mathcal{C}\ell(p,q)&\cong\mathcal{C}\ell(p-1,q-1)\otimes_{\mathbb{R}}\mathbb{R}(2),\end{aligned}\tag{14}$$

where $\mathbb{R}(n)$, $\mathbb{C}(n)$, and $\mathbb{H}(n)$ denote the real, complex, and quaternion algebra characterized by $n\times n$ matrices whose components are real numbers, complex numbers, and quaternions as their linear representation. The tensor product of division algebras over the real field is given by

$$\begin{array}{c|ccc}\otimes_{\mathbb{R}} & \mathbb{R} & \mathbb{C} & \mathbb{H}\\ \hline \mathbb{R} & \mathbb{R} & \mathbb{C} & \mathbb{H}\\ \mathbb{C} & \mathbb{C} & \mathbb{C}\oplus\mathbb{C} & \mathbb{C}(2)\\ \mathbb{H} & \mathbb{H} & \mathbb{C}(2) & \mathbb{R}(4)\end{array}.\tag{15}$$

These relations above suffice to derive the eightfold Bott periodicity for the real Clifford algebra:

$$\begin{aligned}\mathcal{C}\ell(p+8,q)&\cong\mathcal{C}\ell(q,p+6)\otimes_{\mathbb{R}}\mathbb{R}(2)\\ &\cong\mathcal{C}\ell(p+4,q)\otimes_{\mathbb{R}}\mathbb{H}(2)\\ &\cong\mathcal{C}\ell(q,p+2)\otimes_{\mathbb{R}}\mathbb{H}(4)\\ &\cong\mathcal{C}\ell(p,q)\otimes_{\mathbb{R}}\mathbb{R}(16),\\ \mathcal{C}\ell(p,q+8)&\cong\mathcal{C}\ell(q+6,p)\otimes_{\mathbb{R}}\mathbb{H}(1)\\ &\cong\mathcal{C}\ell(p,q+4)\otimes_{\mathbb{R}}\mathbb{H}(2)\\ &\cong\mathcal{C}\ell(q+2,p)\otimes_{\mathbb{R}}\mathbb{R}(8)\\ &\cong\mathcal{C}\ell(p,q)\otimes_{\mathbb{R}}\mathbb{R}(16).\end{aligned}\tag{16}$$

This eightfold periodicity allows us to list the real Clifford algebras $\mathcal{C}\ell(d,n)$ in a finite periodic table in Table I.

For further analytical derivation, it's intuitive to choose a specific representation for these matrices α_i and β_i in the real Clifford algebra. To be concrete, we may choose the following explicit representations in massless ($n=0$) and massive ($n=1$) cases that we are most interested in:

(1) $\mathcal{C}\ell(d,0)$ - massless Majorana fermions (boundary)

The Majorana fermion on the boundary (or domain-wall) is described by a massless Majorana fermion Hamiltonian, specified by matrices α_i in real Clifford algebra $\mathcal{C}\ell(d,0)$. The explicit representation for these matrices is given in Table II.

TABLE II. Explicit representation for $\mathcal{C}\ell(d,0)$ describing massless Majorana fermions on the boundary. $\dim_{\mathbb{R}}\chi$ is the smallest flavor number of Majorana fermions χ_i we need to write down a Hamiltonian in Eq. (6), which can be calculated through the Clifford algebra structure.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	χ	$\dim_{\mathbb{R}}\chi$
0								$1_{\mathbb{R}}$	1
1	σ^3							$1^+_{\mathbb{R}}\oplus 1^-_{\mathbb{R}}$	2
2	σ^1	σ^3						$2_{\mathbb{R}}$	2
3	σ^{10}	σ^{22}	σ^{30}					$2_{\mathbb{C}}$	4
4	σ^{100}	σ^{212}	σ^{220}	σ^{300}				$2_{\mathbb{H}}$	8
5	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}			$2^+_{\mathbb{H}}\oplus 2^-_{\mathbb{H}}$	16
6	σ^{1000}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}		$4_{\mathbb{H}}$	16
7	σ^{1000}	σ^{2002}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}	$8_{\mathbb{C}}$	16

(2) $\mathcal{C}\ell(d,1)$ - massive Majorana fermions (bulk)

The Majorana fermion in the bulk is described by a massive Majorana fermion Hamiltonian, specified by matrices α_i and β_1 in real Clifford algebra $\mathcal{C}\ell(d,1)$. The explicit representation for these matrices is given in Table III.

3. Majorana field

In this work, χ is defined as a *real Grassmannian* (Majorana) field, acting as an irreducible representation of the Clifford algebra $\mathcal{C}\ell(d,n)$. Furthermore, we notice that when (and only when) $d-n=1,5 \bmod 8$, the Clifford algebra $\mathcal{C}\ell(d,n)$ *splits* into two isomorphic Cartan subalgebras,

$$\mathcal{C}\ell(d,n)=\mathcal{C}\ell(d,n)^+\oplus\mathcal{C}\ell(d,n)^-.\tag{17}$$

The subalgebras $\mathcal{C}\ell(d,n)^\pm$ are split by the following projection operator defined by

$$\eta:=\mathrm{i}^n\prod_{i=1}^{d}\alpha_i\prod_{j=1}^{n}\beta_j=\pm1,\tag{18}$$

where η is the *pseudo scalar* in $\mathcal{C}\ell(d,n)$. In this case, the Majorana field χ can be further projected into the representation space of each subalgebra $\mathcal{C}\ell(d,n)^\pm$,

$$\chi^\pm:=\frac{1\pm\eta}{2}\chi,\tag{19}$$

defined as the Majorana-Weyl fermion, which only occurs at $d-n=1,5 \bmod 8$.

The component number of the Majorana fermion χ is calculated by the real representation dimension $\dim_{\mathbb{R}}\chi$, chosen to be the (minimal) dimension of a real vector space in which the representation χ can be faithfully embedded. The dimension is counted from the corresponding vector space of χ analogous to the Clifford algebra structure, following the rules:

$$\dim_{\mathbb{R}}2^k{}_{\mathbb{R}}=2^k,\quad \dim_{\mathbb{R}}2^k{}_{\mathbb{C}}=2^{k+1},\quad \dim_{\mathbb{R}}2^k{}_{\mathbb{H}}=2^{k+2},\tag{20}$$

TABLE III. Explicit representation for $\mathcal{C}\ell(d,1)$ describing massive Majorana fermions in the bulk. $\dim_{\mathbb{R}}\chi$ is the smallest flavor number of Majorana fermions χ_i we need to write down a Hamiltonian in Eq. (6), which can be calculated through the Clifford algebra structure.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	β_1	χ	$\dim_{\mathbb{R}}\chi$
0								σ^2	$1_{\mathbb{C}}$	2
1	σ^3							σ^2	$2_{\mathbb{R}}$	2
2	σ^{31}	σ^{33}						σ^{32}	$2^+_{\mathbb{R}}\oplus 2^-_{\mathbb{R}}$	4
3	σ^{10}	σ^{22}	σ^{30}					σ^{21}	$4_{\mathbb{R}}$	4
4	σ^{100}	σ^{212}	σ^{220}	σ^{300}				σ^{211}	$4_{\mathbb{C}}$	8
5	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}			σ^{1002}	$4_{\mathbb{H}}$	16
6	σ^{31000}	σ^{33100}	σ^{33212}	σ^{33220}	σ^{33232}	σ^{33300}		σ^{32000}	$4^+_{\mathbb{H}}\oplus 4^-_{\mathbb{H}}$	32
7	σ^{10000}	σ^{20020}	σ^{31000}	σ^{32120}	σ^{32200}	σ^{32320}	σ^{33000}	σ^{20212}	$8_{\mathbb{H}}$	32

where $n_{\mathbb{R}}$, $n_{\mathbb{C}}$, and $n_{\mathbb{H}}$ denotes the real, complex, and quaternion vector spaces characterized by n component vectors filled with real numbers, complex numbers, and quaternions.

The eightfold Bott periodicity of the real Clifford algebra $\mathcal{C}\ell(d,n)$ in both d and n [given in Eq. (16)] directly implies the eightfold periodicity in the corresponding vector space:

$$\chi_{\mathcal{C}\ell(d+8,n)} = \chi_{\mathcal{C}\ell(d,n)} \otimes 16_{\mathbb{R}}, \quad \chi_{\mathcal{C}\ell(d,n+8)} = \chi_{\mathcal{C}\ell(d,n)} \otimes 16_{\mathbb{R}}, \tag{21}$$

Due to the Bott periodicity, it will be sufficient to enumerate real Clifford algebras and their corresponding vector spaces for $d = 0,\dots,7$ in the massless Majorana fermion case and Majorana-Weyl fermion case:

(1) *Majorana fermion* (massless case, i.e., $n = 0$)

The Majorana fermion on the boundary (or domain wall) is represented in a vector space, where different components correspond to different Majorana flavors (copies). The structure of the vector space corresponds to the Clifford algebra $\mathcal{C}\ell(d,0)$ and is listed in Table IV.

(2) *Majorana-Weyl fermion* (massless case, i.e., $n = 0$)

The Majorana-Weyl fermion on the boundary (or domain wall) is also represented in a vector space. The structure of the vector space corresponds to the Clifford subalgebra $\mathcal{C}\ell(d,0)^+$ and is listed in Table V.

B. Symmetries

In the previous sections, we have listed the explicit representation of the massless (boundary) Majorana fermions in Table II, and we can clearly observe that the α matrices cannot generate the complete $\mathbb{R}(\dim_{\mathbb{R}}\chi_{\mathcal{C}\ell(d,0)})$ algebra in some dimensions. For example, for $d = 1, 5 \bmod 8$, α matrices span a diagonal matrix space with chiral symmetry operator $(-)^{\chi} = \sigma^3$ (or σ^{3000} for $d = 5$). These symmetries are called internal symmetries for the Clifford algebra, as we will define more strictly later. Apart from internal symmetries, we will also include the most well-known symmetries for physicists: Lorentz symmetry and CRT symmetries.

TABLE IV. Clifford algebra $\mathcal{C}\ell(d,0)$, vector space of massless Majorana fermion χ and its real representation dimension $\dim_{\mathbb{R}}\chi$.

d	$\mathcal{C}\ell\,(d,0)$	χ	$\dim_{\mathbb{R}}\chi$
0	$\mathbb{R}(1)$	$1_{\mathbb{R}}$	1
1	$\mathbb{R}(1)\oplus\mathbb{R}(1)$	$1^+_{\mathbb{R}}\oplus 1^-_{\mathbb{R}}$	2
2	$\mathbb{R}(2)$	$2_{\mathbb{R}}$	2
3	$\mathbb{C}(2)$	$2_{\mathbb{C}}$	4
4	$\mathbb{H}(2)$	$2_{\mathbb{H}}$	8
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$2^+_{\mathbb{H}}\oplus 2^-_{\mathbb{H}}$	16
6	$\mathbb{H}(4)$	$4_{\mathbb{H}}$	16
7	$\mathbb{C}(8)$	$8_{\mathbb{C}}$	16

1. Lorentz symmetry

Given the Clifford algebra $\mathcal{C}\ell(d,n)$, the Lorentz group is generated by

(1) *boost* generators:

$$\alpha_i \ (\text{for } i = 1,\dots,d), \tag{22}$$

with the property that $\alpha_i = \alpha_i^* = \alpha_i^{\mathsf{T}} = \alpha_i^{\dagger}$;

(2) *rotation* generators:

$$\Sigma_{ij} = \frac{\mathrm{i}}{2}[\alpha_i,\alpha_j] \quad (\text{for } i,j = 1,\dots,d), \tag{23}$$

with the property that $\Sigma_{ij} = -\Sigma_{ij}^* = -\Sigma_{ij}^{\mathsf{T}} = \Sigma_{ij}^{\dagger}$.

This can be checked in the action formulation, where $S = \int dx^0 L$ with the Lagrangian L in two possible forms depending on the *metric signature*.

(1) *Minkowski spacetime:* $g_{\mu\mu} = (-,+,\cdots)$

$$\begin{aligned} L &= \frac{1}{2}\int d^dx\,\chi^{\mathsf{T}}\,\mathrm{i}\,\partial_0\chi - H \\ &= \frac{1}{2}\int d^dx\,\chi^{\mathsf{T}}\left(\mathrm{i}\,\partial_0 - \sum_{i=1}^{d}\alpha_i\,\mathrm{i}\,\partial_i - \sum_{i=1}^{n}\beta_i m_i\right)\chi. \end{aligned} \tag{24}$$

TABLE V. Clifford subalgebra $\mathcal{C}\ell(d,0)^+$, vector space of massless Majorana-Weyl fermion χ^+ and its real representation dimension $\dim_{\mathbb{R}}\chi^+$.

d	$\mathcal{C}\ell\,(d,0)^+$	χ^+	$\dim_{\mathbb{R}}\chi^+$
1	$\mathbb{R}(1)^+$	$1^+_{\mathbb{R}}$	1
5	$\mathbb{H}(2)^+$	$2^+_{\mathbb{H}}$	8

Lorentz symmetry $\Lambda(\zeta,\theta)$ (parametrized by the rabidity ζ_i and the rotation angle θ_{ij}) is implemented as

$$\Lambda(\zeta,\theta)\begin{pmatrix}\partial_0\\ \partial_1\\ \partial_2\\ \vdots\end{pmatrix}\Lambda(\zeta,\theta)^{-1} = \exp\begin{pmatrix}0 & \zeta_1 & \zeta_2 & \cdots\\ \zeta_1 & 0 & -\theta_{12} & \cdots\\ \zeta_2 & \theta_{12} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\\ \partial_2\\ \vdots\end{pmatrix},$$

$$\Lambda(\zeta,\theta)\chi\Lambda(\zeta,\theta)^{-1} = \exp\left(\frac{1}{2}\left(\sum_i \zeta_i\alpha_i + \mathrm{i}\sum_{i<j}\theta_{ij}\Sigma_{ij}\right)\right)\chi. \tag{25}$$

such that

∂_μ transforms as the *vector* representation of SO$(d,1)$,

χ transforms as the *spinor* representation of Spin$(d,1)$.

The detailed verification of the invariance of the Lagrangian in Eq. (24) under Lorentz boost and rotation in Eq. (25) is discussed in Appendix D.

(2) *Euclidean spacetime*: $g_{\mu\mu} = (+,+,\cdots)$

$$L = \frac{1}{2}\int d^dx\chi^\mathsf{T}\partial_0\chi + H = \frac{1}{2}\int d^dx\chi^\mathsf{T}\left(\partial_0 + \sum_{i=1}^{d}\alpha_i\mathrm{i}\partial_i + \sum_{i=1}^{n}\beta_i m_i\right)\chi. \tag{26}$$

Lorentz symmetry $\Lambda(\zeta,\theta)$ (parametrized by rabidity ζ_i and the rotation angle θ_{ij}) is implemented as

$$\Lambda(\zeta,\theta)\begin{pmatrix}\partial_0\\ \partial_1\\ \partial_2\\ \vdots\end{pmatrix}\Lambda(\zeta,\theta)^{-1} = \exp\begin{pmatrix}0 & -\zeta_1 & -\zeta_2 & \cdots\\ \zeta_1 & 0 & -\theta_{12} & \cdots\\ \zeta_2 & \theta_{12} & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\\ \partial_2\\ \vdots\end{pmatrix},$$

$$\Lambda(\zeta,\theta)\chi\Lambda(\zeta,\theta)^{-1} = \exp\left(\frac{1}{2}\left(-\mathrm{i}\sum_i \zeta_i\alpha_i + \mathrm{i}\sum_{i<j}\theta_{ij}\Sigma_{ij}\right)\right)\chi, \tag{27}$$

such that

∂_μ transforms as the *vector* representation of SO$(d+1)$,

χ transforms as the *spinor* representation of Spin$(d+1)$.

Again, the detailed verification of the invariance of the Lagrangian in Eq. (26) under Lorentz boost and rotation in Eq. (27) is discussed in Appendix E.

2. Internal symmetry

The internal symmetry of the Majorana fermion corresponds to the *invariant group* $G(\mathcal{C}\ell(d,n))$ of the Clifford algebra $\mathcal{C}\ell(d,n)$, defined by the *short exact sequence*:

$$1 \to G(\mathcal{C}\ell(d,n)) \to \mathrm{O}(\dim_\mathbb{R}\chi) \to \mathrm{Aut}(\mathcal{C}\ell(d,n)) \to 1, \tag{28}$$

where

(1) $\mathrm{O}(\dim_\mathbb{R}\chi)$ - the maximal orthogonal group of χ preserving its anticommutation relation $\{\chi,\chi^\mathsf{T}\} = \mathbf{1}$.

(2) $\mathrm{Aut}(\mathcal{C}\ell(d,n))$ - the automorphism group of $\mathcal{C}\ell(d,n)$. Each automorphism is induced by a *group conjugation* for $g \in \mathrm{O}(\dim_\mathbb{R}\chi)$ and $h \in \mathcal{C}\ell(d,n)$,

$$h \to g^{-1}hg. \tag{29}$$

The short exact sequence Eq. (28) indicates that $G(\mathcal{C}\ell(d,n))$ is the normal subgroup of $\mathrm{O}(\dim_\mathbb{R}\chi)$ that leaves $\mathcal{C}\ell(d,n)$ invariant. The internal symmetries of the massless Majorana fermion and Majorana-Weyl fermion are demonstrated in the following.

(1) *Majorana fermion* (massless case, i.e., $n=0$)

The internal symmetry of a massless Majorana fermion is listed in Table VI, which includes

(a) fermion parity $\mathbb{Z}_2^F$ which flips the sign of the Majorana fermion.

(b) fermion chiral symmetry $\mathbb{Z}_2^\chi$ which add an additional minus sign to one of the Majorana-Weyl subspace in spacetime dimension $d+1 = 2, 6 \bmod 8$.

(c) continuous symmetry generated by corresponding Lie algebra. The continuous symmetry only exists for spacetime dimension $d+1 = 0, 4, 5, 6, 7 \bmod 8$.

(2) Majorana-Weyl fermion (massless case, i.e., $n=0$)

The internal symmetry of massless Majorana-Weyl fermion is listed in Table VII, which includes

(a) fermion parity $\mathbb{Z}_2^F$, which flips the sign of the Majorana fermion.

(b) continuous symmetry generated by the corresponding Lie algebra. The continuous symmetry only exists for spacetime dimension $d+1 = 6 \bmod 8$.

3. CRT symmetry

For Majorana fermions, the charge conjugation $\mathcal{C}$ is undefined since Majorana fermions are free of charges. Therefore, in the following discussion, we will only focus on the reflection $\mathcal{R}_i$ and the time reversal $\mathcal{T}$ symmetry.

To define reflection $\mathcal{R}_i$, it's convenient for us first to define parity $\mathcal{P}$ and then use it to define reflections $\mathcal{R}_i$ in different directions.

Parity $\mathcal{P}$: a *unitary* symmetry, acting as

$$\mathcal{P}\partial_i\mathcal{P}^{-1} = -\partial_i \quad (\text{for } i = 1,\ldots,d),$$
$$\mathcal{P}\chi\mathcal{P}^{-1} = M_\mathcal{P}\chi. \tag{30}$$

$M_\mathcal{P} \in \mathrm{O}(\dim_\mathbb{R}\chi)$ must be in the *maximal orthogonal group* of χ.

The massless Hamiltonian in Eq. (6) changes under the parity transformation $\mathcal{P}$ as

$$\frac{1}{2}\int d^dx\chi^\mathsf{T}h\chi \to -\frac{1}{2}\int d^dx(M_\mathcal{P}\chi)^\mathsf{T}h(M_\mathcal{P}\chi) = -\frac{1}{2}\int d^dx\chi^\mathsf{T}\left(M_\mathcal{P}^\mathsf{T}hM_\mathcal{P}\right)\chi. \tag{31}$$

To keep the Hamiltonian invariant, h should transform under $M_\mathcal{P}$ as

$$M_\mathcal{P}^\mathsf{T}hM_\mathcal{P} = -h, \tag{32}$$

TABLE VI. The generators of the internal symmetries including fermion parity $\mathbb{Z}_2^F$, fermion chiral symmetry $\mathbb{Z}_2^\chi$, and continuous symmetry generated by Lie algebra for massless Majorana fermion.

d	$\mathcal{C}\ell\,(d,0)$	$G(\mathcal{C}\ell\,(d,0))$	$\mathbb{Z}_2^F$	$\mathbb{Z}_2^\chi$	Lie algebra
0	$\mathbb{R}(1)$	$\mathbb{Z}_2$	-1		
1	$\mathbb{R}(1)\oplus\mathbb{R}(1)$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$-\sigma^0$	σ^3	
2	$\mathbb{R}(2)$	$\mathbb{Z}_2$	$-\sigma^0$		
3	$\mathbb{C}(2)$	U(1)	$-\sigma^{00}$		σ^{02}
4	$\mathbb{H}(2)$	Sp(1)	$-\sigma^{000}$		$(\sigma^{002},\sigma^{021},\sigma^{023})$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	Sp(1) $\times$ Sp(1)	$-\sigma^{0000}$	σ^{3000}	$(\sigma^{0002},\sigma^{0021},\sigma^{0023})$
6	$\mathbb{H}(4)$	Sp(1)	$-\sigma^{0000}$		$(\sigma^{0002},\sigma^{0021},\sigma^{0023})$
7	$\mathbb{C}(8)$	U(1)	$-\sigma^{0000}$		σ^{0002}

which indicates that $M_\mathcal{P}$ should act on the Clifford algebra as

$$M_\mathcal{P}^\mathsf{T}\alpha_i M_\mathcal{P}=-\alpha_i\quad(\text{for } i=1,\dots,d).\tag{33}$$

Reflection $\mathcal{R}_i$: a *unitary* symmetry, acting as

$$\mathcal{R}_i\partial_j\mathcal{R}_i^{-1}=\begin{cases}-\partial_i & j=i,\\ \partial_j & j\neq i,\end{cases}$$
$$\mathcal{R}_i\chi\mathcal{R}_i^{-1}=\alpha_i M_\mathcal{P}\chi.\tag{34}$$

Given Eq. (33), one can easily prove that $\alpha_i M_\mathcal{P}$ is also an orthogonal operator and acts on the Clifford algebra as expected:

$$\begin{aligned}(\alpha_i M_\mathcal{P})^\mathsf{T}\alpha_j(\alpha_i M_\mathcal{P})&=M_\mathcal{P}^\mathsf{T}\alpha_i^\mathsf{T}\alpha_j\alpha_i M_\mathcal{P}\\&=M_\mathcal{P}^\mathsf{T}\left(\begin{cases}\alpha_i & j=i\\ -\alpha_j & j\neq i\end{cases}\right)M_\mathcal{P}\\&=\begin{cases}-\alpha_i & j=i\\ \alpha_j & j\neq i\end{cases}.\end{aligned}\tag{35}$$

Under this construction, we always have $\forall i:\mathcal{R}_i^2=(-)^F\mathcal{P}^2$, where F denotes the fermion number and is even for boson (∂_μ).

Similarly, time-reversal $\mathcal{T}$ can be defined as follows.

Time reversal $\mathcal{T}$: an *antiunitary* symmetry, acting as

$$\mathcal{T}\mathrm{i}\mathcal{T}^{-1}=-\mathrm{i},$$
$$\mathcal{T}\chi\mathcal{T}^{-1}=\mathcal{K}M_\mathcal{T}\chi.\tag{36}$$

$\mathcal{K}$ is the complex conjugation operator. $M_\mathcal{T}\in\mathrm{O}(\dim_\mathbb{R}\chi)$ must be in the *maximal orthogonal group* of χ.

The massless Hamiltonian in Eq. (6) transforms under the time-reversion $\mathcal{T}$ as

$$\begin{aligned}\frac{1}{2}\int d^dx\chi^\mathsf{T}h\chi&\to\frac{1}{2}\int d^dx(M_\mathcal{T}\chi)^\mathsf{T}h^*(M_\mathcal{T}\chi)\\&=-\frac{1}{2}\int d^dx\chi^\mathsf{T}\big(M_\mathcal{T}^\mathsf{T}hM_\mathcal{T}\big)\chi.\end{aligned}\tag{37}$$

TABLE VII. The generators of the internal symmetries including fermion parity $\mathbb{Z}_2^F$ and continuous symmetry generated by Lie algebra for massless Majorana-Weyl fermion.

d	$\mathcal{C}\ell(d,0)^+$	$G(\mathcal{C}\ell(d,0)^+)$	$\mathbb{Z}_2^F$	Lie algebra
1	$\mathbb{R}(1)^+$	$\mathbb{Z}_2$	-1	
5	$\mathbb{H}(2)^+$	Sp(1)	$-\sigma^{000}$	$(\sigma^{002},\sigma^{021},\sigma^{023})$

To keep the Hamiltonian invariant, h should transform under $M_\mathcal{P}$ as

$$M_\mathcal{T}^\mathsf{T}hM_\mathcal{T}=-h,\tag{38}$$

which indicates that $M_\mathcal{T}$ should act on the Clifford algebra as

$$M_\mathcal{T}^\mathsf{T}\alpha_i M_\mathcal{T}=-\alpha_i\quad(\text{for } i=1,\dots,d).\tag{39}$$

To find specific representations for matrices $M_\mathcal{P}$, $M_\mathcal{T}$, we notice that the choices are ambiguous up to internal symmetry transformations $g_\mathcal{P},g_\mathcal{T}\in G(\mathcal{C}\ell(d,0))$,

$$M_\mathcal{P}\to g_\mathcal{P}M_\mathcal{P},\quad M_\mathcal{T}\to g_\mathcal{T}M_\mathcal{T}.\tag{40}$$

To give further constraints on the explicit representation, it's intuitive to assume canonical CRT conditions [29]. For Majorana fermions, $\mathcal{C}$ is trivial, the canonical conditions are given by

$$(\mathcal{R}_i\mathcal{T})^2=1,\quad\mathcal{T}(\mathcal{R}_i\mathcal{T})=(-)^F(\mathcal{R}_i\mathcal{T})\mathcal{T}\quad(\text{for } i=1,\dots,d).\tag{41}$$

To realize these conditions, one convenient *choice* is

$$M_\mathcal{P}=M_\mathcal{T}.\tag{42}$$

Under this choice, we always have $\mathcal{P}^2=\mathcal{T}^2$. In conclusion, for Majorana fermions, we can consistently assume the following:

$$\begin{aligned}\mathcal{P}^2=\mathcal{T}^2=(-)^F\mathcal{R}_i^2,\quad(\mathcal{R}_i\mathcal{T})^2=1,\\ \mathcal{T}(\mathcal{R}_i\mathcal{T})=(-)^F(\mathcal{R}_i\mathcal{T})\mathcal{T}\quad(\text{for } i=1,\dots,d).\end{aligned}\tag{43}$$

The CRT and internal symmetries are summarized in Tables VIII and IX, where we have chosen a specific direction for reflection $\mathcal{R}_1$. Other reflections can be generated through rotation in the Lorentz symmetry group, which we have not included in $G_{\text{CRTinternal}}$ for brevity.

(1) Majorana fermion (massless case, i.e., $n=0$)

The internal symmetries $\mathbb{Z}_2^F$, $\mathbb{Z}_2^\chi$, and continuous symmetry generated by the Lie algebra, along with RT symmetries, generate the invariant group $G_{\text{CRTinternal}}$ for Majorana fermions, which is independent of explicit representation basis. (See Table VIII.)

(2) Majorana-Weyl fermion (massless case, i.e., $n=0$).

The internal symmetries $\mathbb{Z}_2^F$ and continuous symmetry generated by the Lie algebra, along with combined $\mathcal{PT}$ symmetry generate the invariant group $G_{\text{CRTinternal}}$ for Majorana-Weyl fermions, which is independent of explicit representation basis. (See Table IX.)

TABLE VIII. The invariant group of massless Majorana fermions in different dimensions, including fermion parity $\mathbb{Z}_2^F$, fermion chiral symmetry $\mathbb{Z}_2^\chi$, continuous symmetry generated by Lie algebra ($\mathcal{J}_i$), reflection $\mathbb{Z}_2^{\mathcal{R}_1}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	$\mathcal{C}\ell(d,0)$	$G_{\text{CRTinternal}}$	$\mathbb{Z}_2^F$	$\mathbb{Z}_2^\chi$	Lie Algebra	$\mathbb{Z}_2^{\mathcal{P}}$	$\mathbb{Z}_2^{\mathcal{R}_1}$	$\mathbb{Z}_2^{\mathcal{T}}$
0	$\mathbb{R}(1)$	$\mathbb{Z}_2^F \times \mathbb{Z}_2^{\mathcal{T}}$	-1					$\mathcal{K}$
1	$\mathbb{R}(1)\oplus\mathbb{R}(1)$	$\mathbb{D}_8^{\mathcal{T},\chi} \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}\chi}$	$-\sigma^0$	σ^3		$\mathrm{i}\sigma^2$	σ^1	$\mathcal{K}\mathrm{i}\sigma^2$
2	$\mathbb{R}(2)$	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1}$	$-\sigma^0$			$\mathrm{i}\sigma^2$	σ^3	$\mathcal{K}\mathrm{i}\sigma^2$
3	$\mathbb{C}(2)$	$\mathrm{Pin}_-^{\mathcal{T}}(2) \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{J}\mathcal{R}_1\mathcal{T}}$	$-\sigma^{00}$		σ^{02}	$\mathrm{i}\sigma^{23}$	σ^{33}	$\mathcal{K}\mathrm{i}\sigma^{23}$
4	$\mathbb{H}(2)$	$\mathrm{Spin}(3) \times_{\mathbb{Z}_2^F} \mathbb{D}_8^{\mathcal{J}_1\mathcal{R}_1,\mathcal{J}_1\mathcal{T}}$	$-\sigma^{000}$		$(\sigma^{002},\sigma^{021},\sigma^{023})$	$\mathrm{i}\sigma^{230}$	σ^{330}	$\mathcal{K}\mathrm{i}\sigma^{230}$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathrm{Spin}(4) \rtimes_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{T}} \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}\chi}$	$-\sigma^{0000}$	σ^{3000}	$(\sigma^{0002},\sigma^{0021},\sigma^{0023})$	$\mathrm{i}\sigma^{2000}$	σ^{1100}	$\mathcal{K}\mathrm{i}\sigma^{2000}$
6	$\mathbb{H}(4)$	$\mathrm{Spin}(3) \times_{\mathbb{Z}_2^F} \mathbb{D}_8^{\mathcal{T},\mathcal{R}_1}$	$-\sigma^{0000}$		$(\sigma^{0002},\sigma^{0021},\sigma^{0023})$	$\mathrm{i}\sigma^{2000}$	σ^{3000}	$\mathcal{K}\mathrm{i}\sigma^{2000}$
7	$\mathbb{C}(8)$	$\mathrm{Pin}_+^{\mathcal{T}}(2) \times \mathbb{Z}_4^{\mathcal{J}\mathcal{R}_1\mathcal{T}}$	$-\sigma^{0000}$		σ^{0002}	σ^{2023}	$\mathrm{i}\sigma^{3023}$	$\mathcal{K}\sigma^{2023}$
8	$\mathbb{R}(16)$	$\mathbb{D}_8^{\mathcal{R}_1,\mathcal{T}}$	$-\sigma^{0000}$			σ^{2023}	$\mathrm{i}\sigma^{3023}$	$\mathcal{K}\sigma^{2023}$

4. Clifford algebra extension

Though the invariant group including internal symmetries is independent of the choice of our explicit representation, the choice of $M_{\mathcal{P}}$ is still ambiguous. In fact, $\mathcal{P}^2$ is still not fixed in some dimensions.

Notably, the definition of $M_{\mathcal{P}}$ requires the anticommutation condition with all α_i matrices. Therefore the choice of $M_{\mathcal{P}}$ is the same as extending an extra matrix to the Clifford algebra.

Given a *massless* Majorana fermion theory specified by $\mathcal{C}\ell(d,0)$, the Cliffor algebra extension concerns the ability to add extra anticommuting terms to the theory without enlarging the representation dimension of χ. The extension is given by two possible sequences:

$$\begin{aligned}&\mathcal{C}\ell(d,0) \to \mathcal{C}\ell(d+1,0)^{(+)} \to \cdots \to \mathcal{C}\ell(d+n,0)^{(+)},\\&\mathcal{C}\ell(d,0) \to \mathcal{C}\ell(d,1)^{(+)} \to \cdots \to \mathcal{C}\ell(d,n)^{(+)},\end{aligned} \tag{44}$$

where the last sequence is also called mass extension as we will discuss in the next section.

As we have shown in Table X, there are two possible extensions in each direction:

Regular extension: if

$$\begin{aligned}&\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n)} = \dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n+1)}, \quad \text{or}\\&\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n)} = \dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d+1,n)},\end{aligned} \tag{45}$$

an anticommuting term can be added directly.

Chiral extension: if $\mathcal{C}\ell(d,n+1) \cong \mathcal{C}\ell(d,n+1)^+ \oplus \mathcal{C}\ell(d,n+1)^-$ (or $\mathcal{C}\ell(d+1,n) \cong \mathcal{C}\ell(d+1,n)^+ \oplus \mathcal{C}\ell(d+1,n)^-$) splits and

$$\begin{aligned}&\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n)} = \dim_{\mathbb{R}} \chi^{\pm}_{\mathcal{C}\ell(d,n+1)^{\pm}} = \tfrac{1}{2}\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n+1)},\\ \text{or }&\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d,n)} = \dim_{\mathbb{R}} \chi^{\pm}_{\mathcal{C}\ell(d+1,n)^{\pm}} = \tfrac{1}{2}\dim_{\mathbb{R}} \chi_{\mathcal{C}\ell(d+1,n)},\end{aligned} \tag{46}$$

an anticommuting term can be added by promoting the Majorana fermion to a Majorana-Weyl fermion in one of the chiral subalgebras (say $\mathcal{C}\ell(d,n+1)^+$ or $\mathcal{C}\ell(d+1,n)^+$). No further term can be added for a Majorana-Weyl fermion, so the chiral extension is always the *end* of an extension sequence.

Extension $\mathcal{C}\ell(d,0)$ to $\mathcal{C}\ell(d+1,0)$ corresponds to the choice $M_{\mathcal{P}} = \alpha_{d+1}$, which means $\mathcal{P}^2 = 1$, while extension $\mathcal{C}\ell(d,0)$ to $\mathcal{C}\ell(d,1)$ corresponds to the choice $M_{\mathcal{P}} = \mathrm{i}\beta_1$, which means $\mathcal{P}^2 = (-)^F$. In this sense, one can simply check Table X to see the choices of $M_{\mathcal{P}}$ we can make. If only the first extension is allowed, then $\mathcal{P}^2 = 1$ cannot be modified to $(-)^F$ by the internal symmetry. Similarly, if only the second extension (mass extension) is allowed, then $\mathcal{P}^2 = (-)^F$ cannot be modified to 1 by the internal symmetry. The process to find $M_{\mathcal{P}}^+$ and $M_{\mathcal{P}}^-$ defined to satisfy $\mathcal{P}^2 = 1$ and $\mathcal{P}^2 = (-)^F$ can also be shown by searching for explicit Clifford algebra extension matrices for Majorana fermions, as demonstrated in Table XI.

TABLE IX. The invariant group of massless Majorana-Weyl fermions in different dimensions, including fermion parity $\mathbb{Z}_2^F$ and continuous symmetry generated by Lie algebra ($\mathcal{J}_i$), and combined $\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$ symmetry.

d	$\mathcal{C}\ell(d,0)^+$	$G_{\text{CRTinternal}}$	$\mathbb{Z}_2^F$	Lie algebra	$\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$
1	$\mathbb{R}(1)^+$	$\mathbb{Z}_2^F \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	-1		$\mathcal{K}$
5	$\mathbb{H}(2)^+$	$\mathrm{Spin}(3) \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	$-\sigma^{000}$	$(\sigma^{002},\sigma^{021},\sigma^{023})$	$\mathcal{K}\sigma^{100}$

C. Mass

After carefully examining the Clifford algebra theory for massless Majorana fermions, we will step forward to the massive theory by extending mass terms. In this section, we will discuss mass extensions and mass domain wall reductions.

1. Mass extension

Given a *massless* Majorana fermion theory specified by $\mathcal{C}\ell(d,0)$, the mass extension concerns the ability to add mass terms to the theory without enlarging the representation dimension of χ.

$$\mathcal{C}\ell(d,0) \to \mathcal{C}\ell(d,1)^{(+)} \to \cdots \to \mathcal{C}\ell(d,n)^{(+)}. \tag{47}$$

The mass extension for Majorana fermions is demonstrated in Table X as horizontal arrows. If a chiral mass extension is made by promoting the Majorana fermion to a Majorana-Weyl fermion in one of the chiral subalgebras, no further mass can be added for the Majorana-Weyl fermion, so the chiral mass

TABLE X. Clifford algebra extension. The green arrow means regular extension. The red arrow means chiral extension. The extension sequence ends if we meet a red arrow and acquire a chiral term.

$d\backslash n$	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}(1)$ ${\color{red}\downarrow}$	$\mathbb{C}(1)$ ${\color{green}\downarrow}$	$\mathbb{H}(1)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(1)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(2)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(4)\ {\color{green}\longrightarrow}$	$\mathbb{R}(8)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(8)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(16)$ ${\color{red}\downarrow}$
1	$2\mathbb{R}(1)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(2)$ ${\color{red}\downarrow}$	$\mathbb{C}(2)$ ${\color{green}\downarrow}$	$\mathbb{H}(2)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(2)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(4)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(8)\ {\color{green}\longrightarrow}$	$\mathbb{R}(16)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(16)$ ${\color{green}\downarrow}$
2	$\mathbb{R}(2)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(2)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(4)$ ${\color{red}\downarrow}$	$\mathbb{C}(4)$ ${\color{green}\downarrow}$	$\mathbb{H}(4)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(4)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(8)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(16)\ {\color{green}\longrightarrow}$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)\ {\color{green}\longrightarrow}$	$\mathbb{R}(4)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(4)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(8)$ ${\color{red}\downarrow}$	$\mathbb{C}(8)$ ${\color{green}\downarrow}$	$\mathbb{H}(8)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(8)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(16)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(4)\ {\color{green}\longrightarrow}$	$\mathbb{R}(8)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(8)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(16)$ ${\color{red}\downarrow}$	$\mathbb{C}(16)$ ${\color{green}\downarrow}$	$\mathbb{H}(16)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(16)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(32)$ ${\color{red}\downarrow}$
5	$2\mathbb{H}(2)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(4)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(8)\ {\color{green}\longrightarrow}$	$\mathbb{R}(16)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(16)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(32)$ ${\color{red}\downarrow}$	$\mathbb{C}(32)$ ${\color{green}\downarrow}$	$\mathbb{H}(32)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(32)$ ${\color{green}\downarrow}$
6	$\mathbb{H}(4)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(4)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(8)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(16)\ {\color{green}\longrightarrow}$	$\mathbb{R}(32)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(32)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(64)$ ${\color{red}\downarrow}$	$\mathbb{C}(64)$ ${\color{green}\downarrow}$	$\mathbb{H}(64)$ ${\color{green}\downarrow}$
7	$\mathbb{C}(8)$ ${\color{green}\downarrow}$	$\mathbb{H}(8)\ {\color{red}\longrightarrow}$ ${\color{green}\downarrow}$	$2\mathbb{H}(8)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{H}(16)\ {\color{green}\longrightarrow}$ ${\color{red}\downarrow}$	$\mathbb{C}(32)\ {\color{green}\longrightarrow}$	$\mathbb{R}(64)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(64)\ {\color{green}\longrightarrow}$ ${\color{green}\downarrow}$	$\mathbb{R}(128)$ ${\color{red}\downarrow}$	$\mathbb{C}(128)$ ${\color{green}\downarrow}$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)\ {\color{red}\longrightarrow}$	$2\mathbb{H}(16)\ {\color{green}\longrightarrow}$	$\mathbb{H}(32)\ {\color{green}\longrightarrow}$	$\mathbb{C}(64)\ {\color{green}\longrightarrow}$	$\mathbb{R}(128)\ {\color{red}\longrightarrow}$	$2\mathbb{R}(128)\ {\color{green}\longrightarrow}$	$\mathbb{R}(256)$

extension is always the *end* of a mass extension sequence. The mass extension process can also be shown by finding explicit mass extensions for Majorana fermions, as demonstrated in Table XII.

2. Mass manifold

Consider the following mass extension sequence,

$$\mathcal{C}\ell(d,0) \to \mathcal{C}\ell(d,1)^{(+)} \to \cdots . \to \mathcal{C}\ell(d,n)^{(+)}. \tag{48}$$

The *number* of mass terms corresponds to the length n of the sequence Eq. (48).

These masses span an $(n-1)$-dimensional mass manifold, formulated as a Grassmannian manifold:

$$M_d = \frac{G(\mathcal{C}\ell(d,0))}{G(\mathcal{C}\ell(d,1)^{(+)})}. \tag{49}$$

For Majorana fermions, the mass manifolds are listed in Table XIII. There are several cases for the manifold M_d.

$M_d = 0$: not admit any mass term.

$M_d = 1$: only one mass m, and $m = \pm 1$ belongs to a different topological order.

$M_d = \mathbb{Z}_2$: only one mass m, and $m = \pm 1$ are topologically trivial but may belong to different SPT phases depending on symmetry assignments.

$M_d = S^{n-1}$: admits a mass vector $\mathbf{m}$ of n components, transforming under the internal and CRT symmetries as an $\mathrm{O}(n)$ vector.

3. Domain wall reduction

Domain wall reduction reduces a massive (bulk) fermion to the mass domain wall as a massless (boundary) fermion in one lower dimension. This simultaneously removes a momentum term and a mass term, corresponding to

$$\mathcal{C}\ell(d,1) \to \mathcal{C}\ell(d-1,0). \tag{50}$$

Note that $\mathcal{C}\ell(d,1) \cong \mathcal{C}\ell(d-1,0) \otimes_{\mathbb{R}} \mathbb{R}(2)$ holds for all d, the domain wall reduction is always feasible.

Domain wall projection. Starting with the Hamiltonian

$$H = \frac{1}{2}\int d^d x \chi^\mathsf{T}\left(\sum_{i=1}^{d} \alpha_i \mathrm{i}\partial_i + \beta m(x_1)\right)\chi, \tag{51}$$

we introduce a mass domain wall $m(x_1) \sim \pm\tanh x_1$ and project the bulk fermion to the domain wall.

TABLE XI. Explicit representation of the matrix $M_\mathcal{P}$. The explicit Clifford algebra extensions correspond to the results in Table X.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	$M^+_\mathcal{P}$	$M^+_\mathcal{P}$	$M^+_\mathcal{P}$	$M^+_\mathcal{P}$	$M^-_\mathcal{P}$	$M^-_\mathcal{P}$	$M^-_\mathcal{P}$	$M^-_\mathcal{P}$
1	σ^3								σ^1				$\mathrm{i}\sigma^2$			
2	σ^1	σ^3											$\mathrm{i}\sigma^2$			
3	σ^{10}	σ^{22}	σ^{30}										$\mathrm{i}\sigma^{21}$	$\mathrm{i}\sigma^{23}$		
4	σ^{100}	σ^{212}	σ^{220}	σ^{300}					σ^{232}				$\mathrm{i}\sigma^{211}$	$\mathrm{i}\sigma^{213}$	$\mathrm{i}\sigma^{230}$	
5	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}				σ^{1000}	σ^{2002}	σ^{2021}	σ^{2023}	$\mathrm{i}\sigma^{1002}$	$\mathrm{i}\sigma^{1021}$	$\mathrm{i}\sigma^{1023}$	$\mathrm{i}\sigma^{2000}$
6	σ^{1000}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}			σ^{2002}	σ^{2021}	σ^{2023}		$\mathrm{i}\sigma^{2000}$			
7	σ^{1000}	σ^{2002}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}		σ^{2021}	σ^{2023}						
8	σ^{1000}	σ^{2002}	σ^{2021}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}	σ^{2023}							

TABLE XII. Explicit representation of the extended mass terms. The explicit mass extensions correspond to the results in Table X.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	β_1	β_2	β_3	β_4
0											
1	σ^3							σ^2			
2	σ^1	σ^3						σ^2			
3	σ^{10}	σ^{22}	σ^{30}					σ^{21}	σ^{23}		
4	σ^{100}	σ^{212}	σ^{220}	σ^{300}				σ^{211}	σ^{213}	σ^{230}	
5	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}			σ^{1002}	σ^{1021}	σ^{1023}	σ^{2000}
6	σ^{1000}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}		σ^{2000}			
7	σ^{1000}	σ^{2002}	σ^{3100}	σ^{3212}	σ^{3220}	σ^{3232}	σ^{3300}				

Given $m(x_1) \sim \pm\tanh x_1 \sim \pm x_1$, the zero mode equation is given by

$$\frac{\delta H}{\delta \chi} = \left(\sum_{i=1}^{d} \alpha_i \,\mathrm{i}\,\partial_i \pm \beta x_1\right)\chi = 0. \tag{52}$$

Assuming χ is uniform in all directions other than x_1, i.e., $\partial_i \chi = 0$ for $i = 2, \dots$, Eq. (52) reduces to

$$(\alpha_1 \,\mathrm{i}\,\partial_1 \pm \beta x_1)\chi = 0, \Rightarrow (\pm \mathrm{i}\ \beta\ \alpha_1 \partial_1 + x_1)\chi = 0. \tag{53}$$

Define $D_\pm = \frac{1}{\sqrt{2}}(x_1 \pm \mathrm{i}\ \beta\ \alpha_1 \partial_1)$, Eq. (53) can be written as

$$D_\pm \chi = 0. \tag{54}$$

$D_\pm$ satisfies the following commutation relation:

$$\begin{aligned}[D_\pm, D_\pm^\dagger] &= \frac{1}{2}[x_1 \pm \mathrm{i}\ \beta\ \alpha\partial_1, x_1 \mp \mathrm{i}\ \beta\ \alpha_1\partial_1] \\ &= \frac{\pm \mathrm{i}\ \beta\ \alpha_1}{2}([\partial_1, x_1] - [x_1, \partial_1]) \\ &= \pm \mathrm{i}\ \beta\ \alpha_1.\end{aligned} \tag{55}$$

The operator $D_\pm$ admits a zero mode if and only if $\pm \mathrm{i}\beta\alpha_1 = 1$, such that $D_\pm$ behaves as a boson annihilation operator and χ is the vacuum state to be annihilated. So the domain wall projection operator is given by

$$P_{\mathrm{DW}} = \frac{\mathbf{1} \pm \mathrm{i}\beta\alpha_1}{2}. \tag{56}$$

which always reduces the fermion spinor dimension by *half*:

$$2_{\mathbb{R}}^{k} \xrightarrow{P} 2_{\mathbb{R}}^{k-1}, \tag{57}$$

which means that a mass domain wall (defect) in each direction always traps half of the fermions. Similarly, a d dimensional monopole defect $m_i \sim x_i$ ($i = 1, \dots, d$) traps $1/2^d$ of the fermions.

The resulting domain wall fermion is then described by

$$H = \frac{1}{2}\int d^{d-1}x\chi^{\mathsf{T}} P_{\mathrm{DW}}\left(\sum_{i=2}^{d} \alpha_i \,\mathrm{i}\,\partial_i\right) P_{\mathrm{DW}}\chi. \tag{58}$$

The domain wall reduction of the real Clifford algebra and domain wall reduction of Majorana/Majorana-Weyl fermions are listed in Tables XIV and XV. To understand the table, let's focus on a specific path: the domain wall reduction from 3d bulk Majorana fermion $4_{\mathbb{R}}$ to 2d boundary $2_{\mathbb{R}}$ indicates the gapless states on the surface of 3d topological insulators. We can further extend a mass term to the system and reduce it to the 1d boundary $1_{\mathbb{R}}^{+}$, and the 1d state spontaneously acquires chirality. This 1d state is the famous integer quantum Hall effect.

D. Mass term and CRT-internal symmetry

In this section, we will examine the interplay between the symmetries and the mass terms. We will focus on the action of symmetries on the mass manifold, and how to obtain the symmetries on the domain wall using the reduction method.

1. CRT-internal symmetry acting on mass manifold

Recall that h defined in Eq. (6) is

$$h = \sum_{i=1}^{d} \alpha_i \,\mathrm{i}\,\partial_i + \sum_{i=1}^{n} \beta_i m_i = h_0 + m, \tag{59}$$

where h_0 stands for the massless part and m is the mass matrix. Since we have already proven that the h_0 part is invariant under CRT-internal symmetry in Sec. II B, we will now focus

TABLE XIII. Mass manifold M_d spanned by multiple mass terms. We can at most extend n independent mass terms.

d	n	$\mathcal{C}\ell(d,0)$	$\mathcal{C}\ell(d,1)^{(+)}$	$\mathcal{C}\ell(d,n)^{+}$	$G(\mathcal{C}\ell(d,0))$	$G(\mathcal{C}\ell(d,1)^{(+)})$	M_d
0		$\mathbb{R}(1)$			$\mathbb{Z}_2$		0
1	1	$\mathbb{R}(1)\oplus\mathbb{R}(1)$	$\mathbb{R}(2)$		$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	1	$\mathbb{R}(2)$	$\mathbb{R}(2)^{+}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	1
3	2	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)^{+}$	U(1)	$\mathbb{Z}_2$	$\frac{\mathrm{U}(1)}{\mathbb{Z}_2}\cong S^1$
4	3	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)^{+}$	Sp(1)	U(1)	$\frac{\mathrm{Sp}(1)}{\mathrm{U}(1)}\cong S^2$
5	4	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{R}(16)^{+}$	Sp(1) × Sp(1)	Sp(1)	$\frac{\mathrm{Sp}(1)\times\mathrm{Sp}(1)}{\mathrm{Sp}(1)}\cong S^3$
6	1	$\mathbb{H}(4)$	$\mathbb{H}(4)^{+}$		Sp(1)	Sp(1)	1
7		$\mathbb{C}(8)$			U(1)		0

TABLE XIV. Domain wall reduction of the real Clifford algebra. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ indicates that the Clifford algebra is real or complex or quaternionic type, the number indicates the dimension of the Clifford algebra. The black arrow means domain wall reduction. The green arrow means regular mass extension. The red arrow means chiral mass extension.

d	$\mathcal{C}\ell(d,0)$		$\mathcal{C}\ell(d,1)$
0	$\mathbb{R}(1)$		$\mathbb{C}(1)$
1	$\mathbb{R}(1)\oplus\mathbb{R}(1)$	→	$\mathbb{R}(2)$
2	$\mathbb{R}(2)$	→	$\mathbb{R}(2)\oplus\mathbb{R}(2)$
3	$\mathbb{C}(2)$	→	$\mathbb{R}(4)$
4	$\mathbb{H}(2)$	→	$\mathbb{C}(4)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	→	$\mathbb{H}(4)$
6	$\mathbb{H}(4)$	→	$\mathbb{H}(4)\oplus\mathbb{H}(4)$
7	$\mathbb{C}(8)$		$\mathbb{H}(8)$
8	$\mathbb{R}(16)$		$\mathbb{C}(16)$

on the bilinear mass $\frac{1}{2}\int d^dx\chi^{\mathsf{T}}m\chi$ and how CRT-internal symmetry acts on the mass manifold.

The bilinear mass term changes under the $\mathcal{R}_i$, $\mathcal{T}$, and internal U symmetries as follows:

$$\begin{aligned}
\frac{1}{2}\int d^dx\chi^{\mathsf{T}}m\chi &\xrightarrow{\mathcal{R}_i} \frac{1}{2}\int d^dx(\alpha_iM_{\mathcal{P}}\chi)^{\mathsf{T}}m(\alpha_iM_{\mathcal{P}}\chi)\\
&= \frac{1}{2}\int d^dx\chi^{T}\big(M_{\mathcal{P}}^{\mathsf{T}}\alpha_i\, m\,\alpha_iM_{\mathcal{P}}\big)\chi,\\
\frac{1}{2}\int d^dx\chi^{\mathsf{T}}m\chi &\xrightarrow{\mathcal{T}} \frac{1}{2}\int d^dx(M_{\mathcal{T}}\chi)^{\mathsf{T}}m^*(M_{\mathcal{T}}\chi)\\
&= -\frac{1}{2}\int d^dx\chi^{\mathsf{T}}\big(M_{\mathcal{T}}^{\mathsf{T}}\, m\, M_{\mathcal{T}}\big)\chi,\\
\frac{1}{2}\int d^dx\chi^{\mathsf{T}}m\chi &\xrightarrow{U} \frac{1}{2}\int d^dx(M_U\chi)^{\mathsf{T}}m(M_U\chi)\\
&= \frac{1}{2}\int d^dx\chi^{\mathsf{T}}\big(M_U^{\mathsf{T}}\, m\, M_U\big)\chi. \qquad (60)
\end{aligned}$$

If the bilinear mass term is invariant under the $\mathcal{R}_i$, $\mathcal{T}$, and U symmetries, then

$$\begin{aligned}
M_{\mathcal{P}}^{\mathsf{T}}\alpha_i\, m\,\alpha_iM_{\mathcal{P}} &= m,\\
M_{\mathcal{T}}^{\mathsf{T}}\, m\, M_{\mathcal{T}} &= -m, \qquad (61)\\
M_U^{\mathsf{T}}\, m\, M_U &= m.
\end{aligned}$$

Therefore, the matrices $M_{\mathcal{P}}$, $M_{\mathcal{T}}$, and M_U should satisfy the following relations (for $i = 1,\dots,d$):

$$\begin{aligned}
M_{\mathcal{P}}^{\mathsf{T}}\beta_iM_{\mathcal{P}} &= -\beta_i,\\
M_{\mathcal{T}}^{\mathsf{T}}\beta_iM_{\mathcal{T}} &= -\beta_i, \qquad (62)\\
M_U^{\mathsf{T}}\beta_iM_U &= \beta_i.
\end{aligned}$$

TABLE XV. Domain wall reduction of Majorana/Majorana-Weyl fermions. The number indicates the dimension of the representation of the fermion, the lower index indicates that the representation is real or complex or quaternionic, and the upper index indicates the chirality of the fermion. The black arrow means domain wall reduction. The green arrow means regular mass extension. The red arrow means chiral mass extension.

	Majorana-Weyl			Majorana		
d	Boundary	Bulk		Boundary		Bulk
0				$1_{\mathbb{R}}$		$1_{\mathbb{C}}$
1	$1_{\mathbb{R}}^+$			$1_{\mathbb{R}}^+\oplus 1_{\mathbb{R}}^-$	→	$2_{\mathbb{R}}$
2		$2_{\mathbb{R}}^+$	←	$2_{\mathbb{R}}$		$2_{\mathbb{R}}^+\oplus 2_{\mathbb{R}}^-$
3				$2_{\mathbb{C}}$	→	$4_{\mathbb{R}}$
4				$2_{\mathbb{H}}$	→	$4_{\mathbb{C}}$
5	$2_{\mathbb{H}}^+$			$2_{\mathbb{H}}^+\oplus 2_{\mathbb{H}}^-$	→	$4_{\mathbb{H}}$
6		$4_{\mathbb{H}}^+$	←	$4_{\mathbb{H}}$		$4_{\mathbb{H}}^+\oplus 4_{\mathbb{H}}^-$
7				$8_{\mathbb{C}}$		$8_{\mathbb{H}}$
8				$16_{\mathbb{R}}$		$16_{\mathbb{C}}$

Any violation of these relations is regarded as the corresponding symmetry breaking.

To be more concrete, in $d = 3,4,5 \bmod 8$, we have a nontrivial mass manifold spanned by multiple mass terms. The given CRT and internal symmetry operators can act on the whole manifold.

In $d = 3$ case, we have two mass matrices σ^{21} and σ^{23}, they span a general S^1 mass manifold with terms characterized by mass angle θ:

$$m(\theta) = \frac{1}{2}\int d^dx\chi^{\mathsf{T}}(\cos\theta\sigma^{21} + \sin\theta\sigma^{23})\chi. \qquad (63)$$

The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}/2}$, and its generator $\mathcal{J}$ on the S^1 mass manifold is listed in Table XVI. $\mathcal{P}$ acts as a "reflection" on the manifold about $m_1 = 0$, $\mathcal{T}$ and $\mathcal{R}_i$ act as "reflection" on the manifold about $m_2 = 0$, and $e^{\mathrm{i}\phi\mathcal{J}/2}$ acts as a "rotation" of ϕ angle on the manifold.

TABLE XVI. The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}/2}$, and its generator $\mathcal{J}$ on the S^1 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes.

	$\mathcal{J}$	$e^{\mathrm{i}\phi\mathcal{J}/2}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi+\theta$	$\theta+\phi$	$-\theta$	$\pi-\theta$	$\pi-\theta$
m_1	×	×	×	✓	✓
m_2	×	×	✓	×	×

TABLE XVII. The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}_i/2}$, and its generator $\mathcal{J}_i$ on the S^2 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes. $\theta'_{ij}(\phi_k)$ means the θ' angle after a ϕ_k rotation in the $m_i - m_j$ plane. Same for $\varphi'_{ij}(\phi_k)$.

	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$e^{\mathrm{i}\phi_1\mathcal{J}_1/2}$	$e^{\mathrm{i}\phi_2\mathcal{J}_2/2}$	$e^{\mathrm{i}\phi_3\mathcal{J}_3/2}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi-\theta$	$\pi-\theta$	θ	$\theta'_{12}(\phi_1)$	$\theta'_{13}(\phi_2)$	$\theta'_{23}(\phi_3)=\theta$	θ	$\pi-\theta$	$\pi-\theta$
φ'	$\pi-\varphi$	$-\varphi$	$\pi+\varphi$	$\varphi'_{12}(\phi_1)$	$\varphi'_{13}(\phi_2)$	$\varphi'_{23}(\phi_3)=\varphi+\phi_3$	φ	$\pi+\varphi$	$\pi+\varphi$
m_1	×	×	✓	×	×	✓	×	✓	✓
m_2	×	✓	×	×	✓	×	×	✓	✓
m_3	✓	×	×	✓	×	×	✓	×	×

In $d=4$ case, we have three mass matrices σ^{211}, σ^{213}, and σ^{230}, they span a general S^2 mass manifold with terms characterized by mass angles θ and φ:

$$m(\theta,\varphi)=\frac{1}{2}\int d^dx\chi^{\mathsf{T}}(\cos\theta\sigma^{211}+\sin\theta\cos\varphi\sigma^{213}+\sin\theta\sin\varphi\sigma^{230})\chi. \tag{64}$$

The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}_i/2}$, and its generator $\mathcal{J}_i$ on the S^2 mass manifold is listed in Table XVII.

In $d=5$ case, we have four mass terms σ^{1002}, σ^{1021}, σ^{1023}, and σ^{2000}, they span a general S^3 mass manifold with terms characterized by mass angles θ, φ, and ψ:

$$m(\theta,\varphi,\psi)=\frac{1}{2}\int d^dx\chi^{\mathsf{T}}(\cos\theta\sigma^{1002}+\sin\theta\cos\varphi\sigma^{1021}+\sin\theta\sin\varphi\cos\psi\sigma^{1023}+\sin\theta\sin\varphi\sin\psi\sigma^{2000})\chi. \tag{65}$$

The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}_i/2}$, its generator $\mathcal{J}_i$, and chiral symmetry operator $(-)^\chi$ on the S^3 mass manifold is listed in Table XVIII.

2. *CRT-internal symmetry reduction under domain wall*

By domain wall reduction, we can reduce a *bulk* Majorana fermion to a *boundary* Majorana (or Majorana-Weyl) fermion in a lower dimension. Surprisingly, the CRT-internal symmetry group in different dimensions (see Tables VIII and IX), though exhibit distinct fractionalization properties, is related by domain wall reduction. Once a mass term exist for $d+1$ spacetime dimensional Majorana theory, we can reproduce the CRT-internal symmetry group in $(d-1)+1$ dimension by projecting corresponding symmetry operators to the domain wall.

This method relies on the mass extension and works in spatial dimension $d=1,2,\dots,6$ mod 8. Once a mass term can be extended, we can randomly add a mass and reduce the fermion to the mass domain wall in either direction (say $m\sim\pm x_d$). Note that a well-defined (i.e., not mixed with broken internal symmetries) reflection $\mathcal{R}_d$ is always preserved under the mass domain wall on the dth direction, since the reflection simultaneously swaps the ground state in the P_+ and P_- projection space, and flips the mass profile $m\sim\pm x_d\to\mp x_d$. Under domain wall reduction, $\mathcal{R}_d$ always becomes an internal symmetry on the domain wall. The reduction of CRT and internal symmetries follows the *rules* below.

(1) If the reflection $\mathcal{R}_i$, time-reversion $\mathcal{T}$, or internal symmetry U is *preserved under mass extension*, then these symmetries are directly projected to the $(d-1)$-dimensional CRT-internal symmetry by projection operator $P_{DW}=\frac{1\pm\mathrm{i}\beta\alpha_1}{2}$:

$$\begin{aligned} d\text{-dimension} &\xrightarrow{DW} (d-1)\text{-dimension}\\ \mathcal{R}_d &\xrightarrow{P_{DW}} X\\ \mathcal{R}_i &\xrightarrow{P_{DW}} \mathcal{R}_i \quad (\forall i=1,\dots,d-1)\\ \mathcal{T} &\xrightarrow{P_{DW}} \mathcal{T}\\ U &\xrightarrow{P_{DW}} U, \end{aligned} \tag{66}$$

where X is an internal symmetry in the $(d-1)$-dimensional theory.

TABLE XVIII. The action of parity $\mathcal{P}$, time-reversion $\mathcal{T}$, reflection $\mathcal{R}_i$, continuous internal symmetry $e^{\mathrm{i}\phi\mathcal{J}_i/2}$, its generator $\mathcal{J}_i$, and chiral symmetry operator $(-)^\chi$ on the S^3 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes. $\theta'_{ij}(\phi_k)$ means the θ' angle after a ϕ_k rotation in the $m_i - m_j$ plane. Same for $\varphi'_{ij}(\phi_k)$, $\psi'_{ij}(\phi_k)$.

	$(-)^\chi$	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$e^{\mathrm{i}\phi_1\mathcal{J}_1/2}$	$e^{\mathrm{i}\phi_2\mathcal{J}_2/2}$	$e^{\mathrm{i}\phi_3\mathcal{J}_3/2}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi-\theta$	θ	$\pi-\theta$	$\pi-\theta$	$\theta'_{23}(\phi_1)=\theta$	$\theta'_{13}(\phi_2)$	$\theta'_{12}(\phi_3)$	$\pi-\theta$	θ	θ
φ'	$\pi-\varphi$	$\pi-\varphi$	φ	$\pi-\varphi$	$\varphi'_{23}(\phi_1)$	$\varphi'_{13}(\phi_2)$	$\varphi'_{12}(\phi_3)$	$\pi-\varphi$	φ	φ
ψ'	$\pi+\psi$	$\pi-\psi$	$\pi-\psi$	ψ	$\psi'_{23}(\phi_1)$	$\psi'_{13}(\phi_2)$	$\psi'_{12}(\phi_3)=\psi$	$\pi-\psi$	$-\psi$	$-\psi$
m_1	×	✓	×	×	✓	×	×	×	✓	✓
m_2	×	×	✓	×	×	✓	×	×	✓	✓
m_3	×	×	×	✓	×	×	✓	×	✓	✓
m_4	×	✓	✓	✓	✓	✓	✓	✓	×	×

TABLE XIX. The CRT-internal symmetry group for Majorana fermion in $d+1$ spacetime dimension $G_{M,d}$ can be reduced to the CRT-internal symmetry group for Majorana or Majorana-Weyl fermion in $(d-1)+1$ spacetime dimension $G_{M/MW,d-1}$ on the mass domain wall. The domain wall mass term m_i can be chosen in the mass manifold. The CRT-invariant group includes fermion parity $\mathbb{Z}_2^F$, fermion chiral symmetry $\mathbb{Z}_2^\chi$, continuous symmetry generated by Lie algebra $(\mathcal{J}_i)$, reflection $\mathbb{Z}_2^{\mathcal{R}_1}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	$G_{M,d}$	$G_{M/MW,d-1}$	m_1	m_2	m_3	m_4
1	$\mathbb{D}_8^{\mathcal{T},\chi}\times\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}\chi}$	$\mathbb{Z}_2^F\times\mathbb{Z}_2^{\mathcal{T}}$	$\chi^\mathsf{T}\sigma^2\chi$			
2	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1}$	$\mathbb{Z}_2^F\times\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	$\chi^\mathsf{T}\sigma^2\chi$			
3	$\mathrm{Pin}_-^{\mathcal{T}}(2)\times_{\mathbb{Z}_2^F}\mathbb{Z}_4^{\mathcal{J}\mathcal{R}_1\mathcal{T}}$	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1}$	$\chi^\mathsf{T}\sigma^{21}\chi$	$\chi^\mathsf{T}\sigma^{23}\chi$		
4	$\mathrm{Spin}(3)\times_{\mathbb{Z}_2^F}\mathbb{D}_8^{\mathcal{J}_1\mathcal{R}_1,\mathcal{J}_1\mathcal{T}}$	$\mathrm{Pin}_-^{\mathcal{T}}(2)\times_{\mathbb{Z}_2^F}\mathbb{Z}_4^{\mathcal{J}\mathcal{R}_1\mathcal{T}}$	$\chi^\mathsf{T}\sigma^{211}\chi$	$\chi^\mathsf{T}\sigma^{213}\chi$	$\chi^\mathsf{T}\sigma^{230}\chi$	
5	$\mathrm{Spin}(4)\rtimes_{\mathbb{Z}_2^F}\mathbb{Z}_4^{\mathcal{T}}\times\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}\chi}$	$\mathrm{Spin}(3)\times_{\mathbb{Z}_2^F}\mathbb{D}_8^{\mathcal{J}_1\mathcal{R}_1,\mathcal{J}_1\mathcal{T}}$	$\chi^\mathsf{T}\sigma^{1002}\chi$	$\chi^\mathsf{T}\sigma^{1021}\chi$	$\chi^\mathsf{T}\sigma^{1023}\chi$	$\chi^\mathsf{T}\sigma^{2000}\chi$
6	$\mathrm{Spin}(3)\times_{\mathbb{Z}_2^F}\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1}$	$\mathrm{Spin}(3)\times\mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	$\chi^\mathsf{T}\sigma^{2000}\chi$			
7	$\mathrm{Pin}_+^{\mathcal{T}}(2)\times\mathbb{Z}_4^{\mathcal{J}\mathcal{R}_1\mathcal{T}}$					
8	$\mathbb{D}_8^{\mathcal{R}_1,\mathcal{T}}$					

(2) If the reflection $\mathcal{R}_i$, time-reversion $\mathcal{T}$, or internal symmetry U is *broken under mass extension*, then these symmetries should be combined with the space-orientation-reversing symmetry $\mathcal{R}_d\mathcal{T}$ to obtain a new symmetry on the domain wall [29,43,65,66]:

$$
\begin{aligned}
&d\text{-dimension} \xrightarrow{DW} \quad (d-1)\text{-dimension}\\
&\quad\mathcal{R}_d \xrightarrow{P_{DW}} X\\
&\quad\mathcal{R}_i \xrightarrow{\cdot\mathcal{R}_d\mathcal{T}} X\mathcal{R}_i\mathcal{T}=\mathcal{R}_i'\mathcal{T}' \quad (\forall i=1,\dots,d-1)\\
&\quad\mathcal{T} \xrightarrow{\cdot\mathcal{R}_d\mathcal{T}} X\\
&\quad U \xrightarrow{\cdot\mathcal{R}_d\mathcal{T}} XU\mathcal{T}=\mathcal{T}'.
\end{aligned}
\tag{67}
$$

To be concrete, the CRT-internal symmetry groups in spacetime $d+1$ and $(d-1)+1$ dimension, and distinct mass terms are listed in Table XIX. On each domain wall mass, the explicit results of the domain-wall projection for symmetry operators are listed in Table XX.

III. DIRAC FERMION

Following the similar process in the discussion of Majorana fermion χ, we will move on to the complex Dirac fermion ψ case, where other important symmetries, complex conjugation symmetry $\mathbb{Z}_2^{\mathcal{C}}$ and U(1) symmetry, are well-defined and discussed. In Sec. III A, we will define the Dirac fermion as a complex Grassmannian field, acting as an irreducible representation of complex Clifford algebra, and we will construct the field theory model for Dirac fermion. In Sec. III B, we will introduce CRT and internal symmetries intrinsic to our minimal Dirac fermion model. The mass terms also form a nontrivial S^1 manifold in odd spatial dimensions as we will give a discussion in Sec. III C. The existence of mass terms will break some internal symmetries, and we will show that the CRT and internal symmetries together are enough to rule out all possible mass bilinear terms in

TABLE XX. The projected symmetry operators $P_{DW}(\cdot)$ with the domain wall mass m_{DW} from spatial dimension d to $d-1$. Symmetries include fermion parity $\mathbb{Z}_2^F$, fermion chiral symmetry $\mathbb{Z}_2^\chi$, continuous symmetry generated by Lie algebra $(\mathcal{J}_i)$, reflection $\mathbb{Z}_2^{\mathcal{R}_i}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	m_{DW}	$P_{DW}((-)^F)$	$P_{DW}((-)^\chi)$	$P_{DW}(\mathcal{J}_i)$	$P_{DW}(\mathcal{R}_{i<d})$	$P_{DW}(\mathcal{R}_d)$	$P_{DW}(\mathcal{T})$
1	m_1	$(-)^F$	0			1	$\mathcal{T}$
2	m_1	$(-)^F$			$\mathcal{R}_1\mathcal{T}$	1	1
3	m_1	$(-)^F$		0	$\mathcal{R}_i$	1	$\mathcal{T}$
	m_2	$(-)^F$		0	$\mathcal{R}_i$	1	$\mathcal{T}$
4	m_1	$(-)^F$		$(0,0,\mathcal{J})$	$\mathcal{R}_i$	$\mathcal{J}$	$\mathcal{T}$
	m_2	$(-)^F$		$(0,\mathcal{J},0)$	$\mathcal{R}_i$	$\mathcal{J}$	$\mathcal{T}$
	m_3	$(-)^F$		$(\mathcal{J},0,0)$	$\mathcal{R}_i$	1	$\mathcal{T}$
5	m_1	$(-)^F$	0	$(\mathcal{J}_3,\mathcal{J}_2,\mathcal{J}_1)$	$\mathcal{R}_i$	$\mathcal{J}_3$	$\mathcal{T}$
	m_2	$(-)^F$	0	$(\mathcal{J}_3,\mathcal{J}_2,\mathcal{J}_1)$	$\mathcal{R}_i$	$\mathcal{J}_2$	$\mathcal{T}$
	m_3	$(-)^F$	0	$(\mathcal{J}_3,\mathcal{J}_1,\mathcal{J}_2)$	$\mathcal{R}_i$	$\mathcal{J}_2$	$\mathcal{T}$
	m_4	$(-)^F$	0	$(\mathcal{J}_3,\mathcal{J}_2,\mathcal{J}_1)$	$\mathcal{R}_i$	1	$\mathcal{T}$
6	m_1	$(-)^F$		$(\mathcal{J}_3,\mathcal{J}_2,\mathcal{J}_1)$	$\mathcal{R}_i\mathcal{T}$	1	1

TABLE XXI. Explicit representation for $\mathcal{C}\ell(d)$ describing massless Dirac fermions on the boundary. $\dim_{\mathbb{C}} \psi$ is the smallest flavor number of Dirac fermions ψ_i we need to write down a Hamiltonian in Eq. (69), which can be calculated through the Clifford algebra structure.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	$\mathcal{C}\ell(d)$	ψ	$\dim_{\mathbb{C}}\psi$
0								$\mathbb{C}(1)$	$1_{\mathbb{C}}$	1
1	σ^3							$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$1^+_{\mathbb{C}}\oplus 1^-_{\mathbb{C}}$	2
2	σ^1	σ^2						$\mathbb{C}(2)$	$2_{\mathbb{C}}$	2
3	σ^{01}	σ^{02}	σ^{33}					$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$2^+_{\mathbb{C}}\oplus 2^-_{\mathbb{C}}$	4
4	σ^{01}	σ^{02}	σ^{13}	σ^{23}				$\mathbb{C}(4)$	$4_{\mathbb{C}}$	4
5	σ^{001}	σ^{002}	σ^{013}	σ^{023}	σ^{333}			$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$4^+_{\mathbb{C}}\oplus 4^-_{\mathbb{C}}$	8
6	σ^{001}	σ^{002}	σ^{013}	σ^{023}	σ^{133}	σ^{233}		$\mathbb{C}(8)$	$8_{\mathbb{C}}$	8
7	σ^{0001}	σ^{0002}	σ^{0013}	σ^{0023}	σ^{0133}	σ^{0233}	σ^{3333}	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$8^+_{\mathbb{C}}\oplus 8^-_{\mathbb{C}}$	16

Sec. III D. Finally, in Sec. III D 2, we will use the domain wall reduction method to give the relation between symmetry groups in different dimensions.

A. Field theory models

By combining real Majorana fermions (i.e., $\psi = (\chi_1 + \mathrm{i}\,\chi_2)/2$ and $\psi^\dagger = (\chi_1 - \mathrm{i}\,\chi_2)/2$), we arrive at the field theory model of the complex Dirac fermion ψ. To give a general field theory of the complex Dirac fermion in a generalized dimension, we duplicate it to form a spinor ψ with multiple components (or flavors), and the Hamiltonian is the "square root" of $\sum_i k_i^2 + m_i^2$ analogous to the case in $3+1d$ spacetime.

1. Hamiltonian

A free Dirac fermion theory in $(d+1)$-dimensional *spacetime* with n-dimensional *mass manifold* is described by the field theory Hamiltonian

$$H = \frac{1}{2}\int d^d x \psi^\dagger h \psi, \tag{68}$$

where h is the square root of $\sum_i k_i^2 + m_i^2$ with $d+n$ anticommuting matrices α_i and β_i:

$$h = \sum_{i=1}^{d} \alpha_i \,\mathrm{i}\,\partial_i + \sum_{i=1}^{n} \beta_i m_i. \tag{69}$$

In general, the matrix h must satisfy

$$h = h^\dagger, \tag{70}$$

for the following reason.

(1) *Hermiticity* of the Hamiltonian: $H = H^\dagger$. Since

$$H^\dagger = \frac{1}{2}\int d^d x \psi^\dagger h^\dagger \psi, \tag{71}$$

we must have $h = h^\dagger$.

Another significant property for Dirac fermion is the *fermion* statistics: $\psi_i\psi_j = -\psi_j\psi_i$. This gives a useful relation

$$\frac{1}{2}\int d^d x \psi^\dagger h \psi = -\frac{1}{2}\int d^d x \psi^\mathsf{T} h^\mathsf{T} \psi^*. \tag{72}$$

Now given the specific form of h in Eq. (69), in which momentum operators $\mathrm{i}\,\partial_i$ and real scalar mass terms m_i satisfy

$$\mathrm{i}\,\partial_i = -(\mathrm{i}\,\partial_i)^\mathsf{T} = -(\mathrm{i}\,\partial_i)^* = (\mathrm{i}\,\partial_i)^\dagger, \quad m_i = m_i^\mathsf{T} = m_i^* = m_i^\dagger, \tag{73}$$

we must require

$$\alpha_i = \alpha_i^\dagger, \quad \beta_i = \beta_i^\dagger. \tag{74}$$

in order for the condition Eq. (73) to hold for h in Eq. (69).

In conclusion, the Hamiltonian H is fully specified by a complex Clifford algebra $\mathcal{C}\ell(d+n)$, which defines α_i and β_i as its *Hermitian generators*, satisfying:

(1) *Anticommutation* relations

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \{\beta_i, \beta_j\} = 2\delta_{ij}, \quad \{\alpha_i, \beta_j\} = 0, \tag{75}$$

2. Complex Clifford algebra

After defining the free Dirac fermion field theory specified by complex Clifford algebra $\mathcal{C}\ell(d+n)$, it is conducive for us to further analyze its algebraic structure and give an explicit representation.

Complex Clifford algebras have the following recursive relation:

$$\mathcal{C}\ell(d+n) \cong \mathcal{C}\ell(d+n-2) \otimes_{\mathbb{C}} \mathbb{C}(2), \tag{76}$$

which gives twofold Bott periodicity for complex Clifford algebra $\mathcal{C}\ell(d+n)$.

For further analytical derivation, it's intuitive to choose a specific representation for these matrices α_i and β_i in the complex Clifford algebra. To be concrete, we may choose the following explicit representations in massless ($n=0$) and massive ($n=1$) cases that we're most interested in.[1]

(1) $\mathcal{C}\ell(d)$ - massless Dirac fermions (boundary).

The Dirac fermion on the boundary (or domain wall) is described by a massless Dirac fermion Hamiltonian, specified by matrices α_i in complex Clifford algebra $\mathcal{C}\ell(d)$. The explicit representation for these matrices is given in Table XXI.

(2) $\mathcal{C}\ell(d+1)$ - massive Dirac fermions (bulk).

The Dirac fermion in the bulk is described by a massive Dirac fermion Hamiltonian, specified by matrices α_i and β_1 in complex Clifford algebra $\mathcal{C}\ell(d+1)$. The explicit representation for these matrices is given in Table XXII.

3. Dirac field

ψ is defined as a *complex Grassmannian* (Dirac) field, acting as an irreducible representation of the Clifford algebra

[1] Intriguingly, we will find that the CRT-internal symmetry exhibits eightfold periodicity, which motivates us to write the explicit representation for $d = 0, \ldots, 7$.

TABLE XXII. Explicit representation for $\mathcal{C}\ell(d+1)$ describing massive Dirac fermions in the bulk. $\dim_{\mathbb{C}}\psi$ is the smallest flavor number of Dirac fermions ψ_i we need to write down a Hamiltonian in Eq. (69), which can be calculated through the Clifford algebra structure.

d	α_1	α_2	α_3	α_4	α_5	α_6	α_7	β_1	$\mathcal{C}\ell(d+1)$	ψ	$\dim_{\mathbb{C}}\psi$
0								σ^3	$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$1_{\mathbb{C}}^+\oplus 1_{\mathbb{C}}^-$	2
1	σ^1							σ^2	$\mathbb{C}(2)$	$2_{\mathbb{C}}$	2
2	σ^{01}	σ^{02}						σ^{33}	$\mathbb{C}(2)\oplus\mathbb{C}(2)$	$2_{\mathbb{C}}^+\oplus 2_{\mathbb{C}}^-$	4
3	σ^{01}	σ^{02}	σ^{13}					σ^{23}	$\mathbb{C}(4)$	$4_{\mathbb{C}}$	4
4	σ^{001}	σ^{002}	σ^{013}	σ^{023}				σ^{333}	$\mathbb{C}(4)\oplus\mathbb{C}(4)$	$4_{\mathbb{C}}^+\oplus 4_{\mathbb{C}}^-$	8
5	σ^{001}	σ^{002}	σ^{013}	σ^{023}	σ^{133}			σ^{233}	$\mathbb{C}(8)$	$8_{\mathbb{C}}$	8
6	σ^{0001}	σ^{0002}	σ^{0013}	σ^{0023}	σ^{0133}	σ^{0233}		σ^{3333}	$\mathbb{C}(8)\oplus\mathbb{C}(8)$	$8_{\mathbb{C}}^+\oplus 8_{\mathbb{C}}^-$	16
7	σ^{0001}	σ^{0002}	σ^{0013}	σ^{0023}	σ^{0133}	σ^{0233}	σ^{1333}	σ^{2333}	$\mathbb{C}(16)$	$16_{\mathbb{C}}$	16

$\mathcal{C}\ell(d+n)$. Furthermore, we notice that when (and only when) $d+n=1 \bmod 2$, the Clifford algebra $\mathcal{C}\ell(d+n)$ *splits* into two isomorphic Cartan subalgebras,

$$\mathcal{C}\ell(d+n)=\mathcal{C}\ell(d+n)^+\oplus\mathcal{C}\ell(d+n)^-. \tag{77}$$

The subalgebras $\mathcal{C}\ell(d+n)^\pm$ are split by the following projection operator defined by

$$\eta := \prod_{i=1}^{d}\alpha_i\prod_{j=1}^{n}\beta_j=\pm 1, \tag{78}$$

where η is the *pseudoscalar* in $\mathcal{C}\ell(d+n)$. In this case, the Dirac field ψ can be further projected into the representation space of each subalgebra $\mathcal{C}\ell(d+n)^\pm$,

$$\psi^\pm := \frac{1\pm\eta}{2}\psi, \tag{79}$$

defined as the *Weyl* fermion, which only occurs at $d+n=1 \bmod 2$.

The component number of the Dirac fermion ψ is calculated by the complex representation dimension $\dim_{\mathbb{C}}\psi$, chosen to be the (minimal) dimension of a complex vector space in which the representation ψ can be faithfully embedded. The dimension is counted from the corresponding vector space of ψ analogous to the Clifford algebra structure, following the rules:

$$\dim_{\mathbb{C}} 2^k{}_{\mathbb{C}} = 2^k, \tag{80}$$

where $n_{\mathbb{C}}$ denotes the complex vector space characterized by n component vector filled with complex numbers.

The twofold Bott periodicity of the complex Clifford algebra $\mathcal{C}\ell(d+n)$ (given in Eq. (76)) directly implies the twofold periodicity in the corresponding vector space:

$$\psi_{\mathcal{C}\ell(d+n+2)}=\psi_{\mathcal{C}\ell(d+n)}\otimes 2_{\mathbb{C}}. \tag{81}$$

Due to the Bott periodicity, it will be sufficient to enumerate complex Clifford algebras and their corresponding vector spaces for $d=0,1$ in the massless Dirac fermion case and Weyl fermion case.

(1) Dirac fermion (massless case, i.e., $n=0$).

The Dirac fermion on the boundary (or domain-wall) is represented in a vector space, where different component corresponds to different Dirac flavor (copy). The structure of the vector space corresponds to the Clifford algebra $\mathcal{C}\ell(d)$, and is listed in Table XXIII.

(2) Weyl fermion (massless case, i.e., $n=0$).

The Weyl fermion on the boundary (or domain-wall) is also represented in a vector space. The structure of the vector space corresponds to the Clifford subalgebra $\mathcal{C}\ell(d)^+$, and is listed in Table XXIV.

B. Symmetries

In the previous sections, we have listed the explicit representation of the massless (boundary) Dirac fermions in Table XXI, and we can clearly observe that the α matrices cannot generate the complete $\mathbb{C}(\dim_{\mathbb{C}}\psi_{\mathcal{C}\ell(d)})$ algebra in some dimensions. For example, for $d=1 \bmod 2$, α matrices span a diagonal matrix space with chiral symmetry operator $(-)^\chi=\sigma^3$. These symmetries are called internal symmetries for the Clifford algebra, as we will define more strictly later. Apart from internal symmetries, we will also include the most well-known symmetries for physicists: Lorentz symmetry and CRT symmetries.

The Lorentz symmetry of the Dirac fermion aligns perfectly with that of the Majorana fermion discussed in Sec. II B 1, so we will start our discussion from the internal symmetries below.

1. Internal symmetry

The internal symmetry of the Dirac fermion corresponds to the *invariant group* $G(\mathcal{C}\ell(d+n))$ of the Clifford algebra $\mathcal{C}\ell(d+n)$, defined by the *short exact sequence*:

$$1\to G(\mathcal{C}\ell(d+n))\to \mathrm{U}(\dim_{\mathbb{C}}\psi)\to \mathrm{Aut}(\mathcal{C}\ell(d+n))\to 1, \tag{82}$$

where

(1) $\mathrm{U}(\dim_{\mathbb{C}}\psi)$ - the maximal unitary group of ψ preserving its anticommutation relation $\{\psi,\psi^\dagger\}=\mathbf{1}$.

(2) $\mathrm{Aut}(\mathcal{C}\ell(d+n))$ - the automorphism group of $\mathcal{C}\ell(d+n)$. Each automorphism is induced by a *group conjugation* for

TABLE XXIII. Clifford algebra $\mathcal{C}\ell(d)$, vector space of massless Dirac fermion ψ and its complex representation dimension $\dim_{\mathbb{C}}\psi$.

d	$\mathcal{C}\ell$ (d)	ψ	$\dim_{\mathbb{C}}\psi$
0	$\mathbb{C}(1)$	$1_{\mathbb{C}}$	1
1	$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$1_{\mathbb{C}}^+\oplus 1_{\mathbb{C}}^-$	2

TABLE XXIV. Clifford subalgebra $\mathcal{C}\ell(d)^+$, vector space of massless Weyl fermion ψ^+ and its complex representation dimension $\dim_{\mathbb{C}}\psi^+$.

d	$\mathcal{C}\ell\,(d)^+$	ψ^+	$\dim_{\mathbb{C}}\psi^+$
1	$\mathbb{C}(1)^+$	$1^+_{\mathbb{C}}$	1

$g \in \mathrm{U}(\dim_{\mathbb{C}}\psi)$ and $h \in \mathcal{C}\ell(d+n)$,

$$h \to g^{-1}hg. \tag{83}$$

The short exact sequence Eq. (82) indicates that $G(\mathcal{C}\ell(d+n))$ is the normal subgroup of $\mathrm{U}(\dim_{\mathbb{C}}\psi)$ that leaves $\mathcal{C}\ell(d+n)$ invariant.

(1) Dirac fermion (massless case, i.e., $n=0$).

The internal symmetry of a massless Dirac fermion is listed in Table XXV, which includes vector $\mathrm{U}(1)^F$ and axial $\mathrm{U}(1)^\chi$ symmetries.

(2) Weyl fermion (massless case, i.e., $n=0$).

The internal symmetry of massless Weyl fermion is listed in Table XXVI, which includes vector $\mathrm{U}(1)^F$ symmetry.

2. CRT symmetry

For Dirac fermions, the charge conjugation $\mathcal{C}$, the reflection $\mathcal{R}_i$, and the time reversal $\mathcal{T}$ symmetry can be discussed.

We first define charge conjugation $\mathcal{C}$ [28,67,68], which transforms a Dirac fermion ψ to its complex conjugate ψ^*. If any operator U acts as $U\psi U^{-1} = M_U\psi$, then the action becomes $U\psi^*U^{-1} = M_U^*\psi^*$ in the complex conjugate space.[2]

Charge conjugation $\mathcal{C}$: a *unitary* symmetry, acting as

$$\begin{aligned}\mathcal{C}\partial_i\mathcal{C}^{-1} &= \partial_i \quad (\text{for } i=1,\dots,d),\\ \mathcal{C}\psi\mathcal{C}^{-1} &= \mathcal{K}_{\mathcal{C}}M_{\mathcal{C}}\psi.\end{aligned} \tag{84}$$

$\mathcal{K}_{\mathcal{C}}$ is a complex conjugation operator on Dirac fermion, sending ψ to its complex conjugate ψ^* and vice versa. $M_{\mathcal{C}} \in \mathrm{U}(\dim_{\mathbb{C}}\psi)$ must be in the *maximal unitary group* of ψ.

The massless Hamiltonian in Eq. (69) transforms under the charge conjugation $\mathcal{C}$ as

$$\begin{aligned}\frac{1}{2}\int d^dx\psi^\dagger h\psi \to \frac{1}{2}\int d^dx(M_{\mathcal{C}}\psi^*)^\dagger h(M_{\mathcal{C}}\psi^*)\\ = \frac{1}{2}\int d^dx\psi^\mathsf{T}(M_{\mathcal{C}}^\dagger hM_{\mathcal{C}})\psi^*\\ = -\frac{1}{2}\int d^dx\psi^\dagger(M_{\mathcal{C}}^\dagger hM_{\mathcal{C}})^\mathsf{T}\psi,\end{aligned} \tag{85}$$

where the last step uses the property in Eq. (72).

[2] See more strict definition as bundle map in Ref. [43].

TABLE XXV. The generators of the internal symmetries including vector $\mathrm{U}(1)^F$ and axial $\mathrm{U}(1)^\chi$ for a massless Dirac fermion. $\mathcal{Q}$ is the charge operator defined as $\mathcal{Q}\psi = \psi$ and $\mathcal{Q}\psi^* = -\psi^*$.

d	$\mathcal{C}\ell$ (d)	G($\mathcal{C}\ell$ (d))	$\mathrm{U}(1)^F$	$\mathrm{U}(1)^\chi$
0	$\mathbb{C}(1)$	U(1)	$\mathcal{Q}$	
1	$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$\mathrm{U}(1)\times\mathrm{U}(1)$	$\mathcal{Q}\sigma^0$	$\mathcal{Q}\sigma^3$

TABLE XXVI. The generator of the internal vector $\mathrm{U}(1)^F$ symmetry for massless Weyl fermion. $\mathcal{Q}$ is the charge operator defined as $\mathcal{Q}\psi = \psi$ and $\mathcal{Q}\psi^* = -\psi^*$.

d	$\mathcal{C}\ell(d)^+$	$G(\mathcal{C}\ell(d)^+)$	$\mathrm{U}(1)^F$
1	$\mathbb{C}(1)^+$	U(1)	$\mathcal{Q}$

To keep the Hamiltonian invariant, h should transform under $M_{\mathcal{P}}$ as

$$M_{\mathcal{C}}^\dagger hM_{\mathcal{C}} = -h^\mathsf{T}, \tag{86}$$

which indicates that $M_{\mathcal{C}}$ should act on the Clifford algebra as

$$M_{\mathcal{C}}^\dagger\alpha_iM_{\mathcal{C}} = \alpha_i^\mathsf{T} \quad (\text{for } i=1,\dots,d). \tag{87}$$

To define reflection $\mathcal{R}_i$, it's convenient for us first to define parity $\mathcal{P}$ and then use it to define reflections $\mathcal{R}_i$ in different directions:

Parity $\mathcal{P}$: a *unitary* symmetry, acting as

$$\begin{aligned}\mathcal{P}\partial_i\mathcal{P}^{-1} &= -\partial_i \quad (\text{for } i=1,\dots,d),\\ \mathcal{P}\psi\mathcal{P}^{-1} &= M_{\mathcal{P}}\psi.\end{aligned} \tag{88}$$

$M_{\mathcal{P}} \in \mathrm{U}(\dim_{\mathbb{C}}\psi)$ must be in the *maximal unitary group* of ψ.

The massless Hamiltonian in Eq. (69) changes under the parity transformation $\mathcal{P}$ as

$$\begin{aligned}\frac{1}{2}\int d^dx\psi^\dagger h\psi \to -\frac{1}{2}\int d^dx(M_{\mathcal{P}}\psi)^\dagger h(M_{\mathcal{P}}\psi)\\ = -\frac{1}{2}\int d^dx\psi^\dagger(M_{\mathcal{P}}^\dagger hM_{\mathcal{P}})\psi.\end{aligned} \tag{89}$$

To keep the Hamiltonian invariant, h should transform under $M_{\mathcal{P}}$ as

$$M_{\mathcal{P}}^\dagger hM_{\mathcal{P}} = -h, \tag{90}$$

which indicates that $M_{\mathcal{P}}$ should act on the Clifford algebra as

$$M_{\mathcal{P}}^\dagger\alpha_iM_{\mathcal{P}} = -\alpha_i \quad (\text{for } i=1,\dots,d). \tag{91}$$

Reflection $\mathcal{R}_i$: a *unitary* symmetry, acting as

$$\begin{aligned}\mathcal{R}_i\partial_j\mathcal{R}_i^{-1} &= \begin{cases}-\partial_i & j=i,\\ \partial_j & j\neq i,\end{cases}\\ \mathcal{R}_i\psi\mathcal{R}_i^{-1} &= \alpha_iM_{\mathcal{P}}\psi.\end{aligned} \tag{92}$$

Given Eq. (91), one can easily prove that $\alpha_iM_{\mathcal{P}}$ is also a unitary operator and acts on the Clifford algebra as expected:

$$(\alpha_iM_{\mathcal{P}})^\dagger\alpha_j(\alpha_iM_{\mathcal{P}}) = \begin{cases}-\alpha_i & j=i,\\ \alpha_j & j\neq i.\end{cases} \tag{93}$$

Under this construction, we always have $\forall i: \mathcal{R}_i^2 = (-)^F\mathcal{P}^2$, where F denotes the fermion number and is even for boson (∂_μ).

Similarly, time-reversal $\mathcal{T}$ can be defined as follows.

Time reversal $\mathcal{T}$: an *antiunitary* symmetry, acting as

$$\begin{aligned}\mathcal{T}\mathrm{i}\mathcal{T}^{-1} &= -\mathrm{i},\\ \mathcal{T}\psi\mathcal{T}^{-1} &= \mathcal{K}M_{\mathcal{T}}\psi.\end{aligned} \tag{94}$$

TABLE XXVII. The invariant group of massless Dirac fermions in different dimensions, including vector $\mathrm{U}(1)^F$ symmetry, axial $\mathrm{U}(1)^\chi$ symmetry, charge conjugation $\mathbb{Z}_2^{\mathcal{C}}$, reflection $\mathbb{Z}_2^{\mathcal{R}_1}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	$\mathcal{C}\ell\,(d)$	G_{CRT}	$G_{\mathrm{CRTinternal}}$	$\mathrm{U}(1)^F$	$\mathrm{U}(1)^\chi$	$\mathbb{Z}_2^{\mathcal{C}}$	$\mathbb{Z}_2^{\mathcal{P}}$	$\mathbb{Z}_2^{\mathcal{R}_1}$	$\mathbb{Z}_2^{\mathcal{T}}$	$\mathbb{Z}_2^{\mathcal{C}\mathcal{R}_1\mathcal{T}}$
0	$\mathbb{C}(1)$	$\mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes G_{\mathrm{CRT}}$	$\mathcal{Q}$		$\mathcal{K}_{\mathcal{C}}$			$\mathcal{K}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}$
1	$\mathbb{C}(1) \oplus \mathbb{C}(1)$	$\mathbb{D}_8^{\mathcal{C}\mathcal{T},\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^0$	$\mathcal{Q}\sigma^3$	$\mathcal{K}_{\mathcal{C}}\sigma^3$	$\mathrm{i}\sigma^2$	σ^1	$\mathcal{K}\sigma^1$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^3$
2	$\mathbb{C}(2)$	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1} \times \mathbb{Z}_2^{\mathcal{C}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^0$		$\mathcal{K}_{\mathcal{C}}\sigma^1$	$\mathrm{i}\sigma^3$	σ^2	$\mathcal{K}\sigma^2$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^1$
3	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1} \times \mathbb{Z}_2^{\mathcal{C}}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{00}$	$\mathcal{Q}\sigma^{30}$	$\mathcal{K}_{\mathcal{C}}\sigma^{11}$	$\mathrm{i}\sigma^{13}$	σ^{12}	$\mathcal{K}\sigma^{02}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{01}$
4	$\mathbb{C}(4)$	$\mathbb{D}_8^{\mathcal{T},\mathcal{C}\mathcal{R}_1} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{00}$		$\mathcal{K}_{\mathcal{C}}\sigma^{21}$	σ^{33}	$\mathrm{i}\sigma^{32}$	$\mathcal{K}\sigma^{12}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{01}$
5	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{D}_8^{\mathcal{R}_1,\mathcal{C}\mathcal{R}_1\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{000}$	$\mathcal{Q}\sigma^{300}$	$\mathcal{K}_{\mathcal{C}}\sigma^{021}$	σ^{133}	$\mathrm{i}\sigma^{132}$	$\mathcal{K}\sigma^{112}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{001}$
6	$\mathbb{C}(8)$	$\mathbb{D}_8^{\mathcal{C}\mathcal{R}_1,\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{000}$		$\mathcal{K}_{\mathcal{C}}\sigma^{121}$	$\mathrm{i}\sigma^{333}$	σ^{332}	$\mathcal{K}\sigma^{212}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{001}$
7	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{D}_8^{\mathcal{C}\mathcal{R}_1,\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{0000}$	$\mathcal{Q}\sigma^{3000}$	$\mathcal{K}_{\mathcal{C}}\sigma^{1121}$	$\mathrm{i}\sigma^{1333}$	σ^{1332}	$\mathcal{K}\sigma^{0212}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{0001}$
8	$\mathbb{C}(16)$	$\mathbb{D}_8^{\mathcal{C}\mathcal{R}_1,\mathcal{C}\mathcal{T}} \times \mathbb{Z}_2^{\mathcal{C}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{0000}$		$\mathcal{K}_{\mathcal{C}}\sigma^{2121}$	σ^{3333}	$\mathrm{i}\sigma^{3332}$	$\mathcal{K}\sigma^{1212}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{0001}$

$\mathcal{K}$ is the complex conjugation operator. $M_{\mathcal{T}} \in \mathrm{U}(\dim_{\mathbb{C}} \psi)$ must be in the *maximal unitary group* of ψ.

The massless Hamiltonian in Eq. (69) transforms under the time-reversion $\mathcal{T}$ as

$$\frac{1}{2}\int d^d x \psi^\dagger h \psi \to \frac{1}{2}\int d^d x (M_{\mathcal{T}}\psi)^\dagger h^* (M_{\mathcal{T}}\psi) = \frac{1}{2}\int d^d x \psi^\dagger (M_{\mathcal{T}}^\dagger h^* M_{\mathcal{T}})\psi. \tag{95}$$

To keep the Hamiltonian invariant, h should transform under $M_{\mathcal{T}}$ as

$$M_{\mathcal{T}}^\dagger h M_{\mathcal{T}} = h^*, \tag{96}$$

which indicates that $M_{\mathcal{C}}$ should act on the Clifford algebra as

$$M_{\mathcal{T}}^\dagger \alpha_i M_{\mathcal{T}} = -\alpha_i^* \quad (\text{for } i = 1, \ldots, d). \tag{97}$$

To find explicit representations for matrices $M_{\mathcal{C}}$, $M_{\mathcal{P}}$, and $M_{\mathcal{T}}$, we notice that the choices are ambiguous up to internal symmetry transformations $g_{\mathcal{C}}, g_{\mathcal{P}}, g_{\mathcal{T}} \in G(\mathcal{C}\ell(d))$,

$$M_{\mathcal{C}} \to g_{\mathcal{C}} M_{\mathcal{C}}, \quad M_{\mathcal{P}} \to g_{\mathcal{P}} M_{\mathcal{P}}, \quad M_{\mathcal{T}} \to g_{\mathcal{T}} M_{\mathcal{T}}. \tag{98}$$

To give further constraints on the representation, it's intuitive to assume canonical CRT conditions [29] given by

$$(\mathcal{C}\mathcal{R}_i\mathcal{T})^2 = 1, \quad \mathcal{C}(\mathcal{C}\mathcal{R}_i\mathcal{T}) = (\mathcal{C}\mathcal{R}_i\mathcal{T})\mathcal{C}, \quad \mathcal{T}(\mathcal{C}\mathcal{R}_i\mathcal{T}) = (-)^F(\mathcal{C}\mathcal{R}_i\mathcal{T})\mathcal{T} \quad (\text{for } i = 1, \ldots, d). \tag{99}$$

To realize these conditions, one convenient *choice* is

$$M_{\mathcal{P}} = M_{\mathcal{C}} M_{\mathcal{T}}. \tag{100}$$

Under this choice, we always have $\mathcal{P}^2 = (\mathcal{C}\mathcal{T})^2$. In conclusion, for Dirac fermions, we can consistently assume the following:

$$\mathcal{P}^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F\mathcal{R}_i^2, \quad (\mathcal{C}\mathcal{R}_i\mathcal{T})^2 = 1, \quad \mathcal{C}(\mathcal{C}\mathcal{R}_i\mathcal{T}) = (\mathcal{C}\mathcal{R}_i\mathcal{T})\mathcal{C}, \quad \mathcal{T}(\mathcal{C}\mathcal{R}_i\mathcal{T}) = (-)^F(\mathcal{C}\mathcal{R}_i\mathcal{T})\mathcal{T} \quad (\text{for } i = 1, \ldots, d). \tag{101}$$

Intuitively, one may assume that the invariant group is twofold periodic parallel to the Bott periodicity of the complex Clifford algebra. Intriguingly the CRT-internal symmetry group actually exhibits eightfold periodicity as in the Majorana case due to the canonical constraints. The corresponding CRT and internal symmetries for Dirac and Weyl fermions are summarized in Tables XXVII and XXVIII, where we have chosen a specific direction for reflection $\mathcal{R}_1$. Other reflections can be generated through rotation in the Lorentz symmetry group, which we have not included in $G_{\mathrm{CRTinternal}}$ for brevity.

(1) Dirac fermion (massless case, i.e., $n = 0$).

The internal symmetries $\mathrm{U}(1)^F$ and $\mathrm{U}(1)^\chi$, along with CRT symmetries generate the invariant group $G_{\mathrm{CRTinternal}}$ for Dirac fermions, which is independent of explicit representation basis. (See Table XXVII.) The invariant group $G_{\mathrm{CRTinternal}}$ is complicated, and the detailed presentation is collected in Appendix F.

(2) Weyl fermion (massless case, i.e., $n = 0$).

The internal symmetry $\mathrm{U}(1)^F$, along with CRT symmetries generates the invariant group $G_{\mathrm{CRTinternal}}$ for Weyl fermions, which is independent of explicit representation basis. (See Ta-

TABLE XXVIII. The invariant group of massless Weyl fermions in different dimensions, including vector $\mathrm{U}(1)^F$ symmetry, charge conjugation $\mathbb{Z}_2^{\mathcal{C}}$, time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$, and combined $\mathbb{Z}_2^{\mathcal{C}\mathcal{R}_1\mathcal{T}}$ symmetry.

d	$\mathcal{C}\ell(d)^+$	G_{CRT}	$G_{\mathrm{CRTinternal}}$	$\mathrm{U}(1)^F$	$\mathbb{Z}_2^{\mathcal{C}}$	$\mathbb{Z}_2^{\mathcal{T}}$	$\mathbb{Z}_2^{\mathcal{C}\mathcal{R}_1\mathcal{T}}$
1	$\mathbb{C}(1)^+$	$\mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{C}\mathcal{R}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes G_{\mathrm{CRT}}$	$\mathcal{Q}$	$\mathcal{K}_{\mathcal{C}}$		$\mathcal{K}_{\mathcal{C}}\mathcal{K}$
3	$\mathbb{C}(2)^+$	$\mathbb{Z}_4^{\mathcal{T}} \times \mathbb{Z}_2^{\mathcal{C}\mathcal{R}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^0$		$\mathcal{K}\sigma^2$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^1$
5	$\mathbb{C}(4)^+$	$\mathbb{D}_8^{\mathcal{C},\mathcal{C}\mathcal{R}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{00}$	$\mathcal{K}_{\mathcal{C}}\sigma^{21}$		$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{01}$
7	$\mathbb{C}(8)^+$	$\mathbb{D}_8^{\mathcal{C}\mathcal{R}_1,\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{\mathrm{CRT}}$	$\mathcal{Q}\sigma^{000}$		$\mathcal{K}\sigma^{212}$	$\mathcal{K}_{\mathcal{C}}\mathcal{K}\sigma^{001}$

TABLE XXIX. Mass extension. The green arrow means regular mass extension. The red arrow means chiral mass extension.

d	$\mathcal{C}\ell(d)$		$\mathcal{C}\ell(d+1)$		$\mathcal{C}\ell(d+2)$
0	$\mathbb{C}(1)$	$\longrightarrow$	$\mathbb{C}(1)\oplus\mathbb{C}(1)$		$\mathbb{C}(2)$
1	$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$\longrightarrow$	$\mathbb{C}(2)$	$\longrightarrow$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$

ble XXVIII.) The invariant group $G_{\text{CRTinternal}}$ is complicated, and the detailed presentation is demonstrated in Appendix G.

Under the canonical CRT condition, the choice of $M_{\mathcal{C}}, M_{\mathcal{P}}$, $M_{\mathcal{T}}$ is still ambiguous up to internal symmetry transformations. It's proved in Ref. [43] that $\mathcal{P}^2$ can be either 1 or $(-)^F$ in each dimension, and $\mathcal{C}^2$, $\mathcal{T}^2$ is fixed by canonical conditions.

C. Mass

After carefully examining the Clifford algebra theory for massless Dirac fermion, we will step forward to the massive theory by extending mass terms. In this section, we will discuss mass extensions and mass domain wall reductions.

1. Mass extension and mass manifold

Given a *massless* Dirac fermion theory specified by $\mathcal{C}\ell(d)$, the mass extension concerns the ability to add mass terms to the theory without enlarging the representation dimension of ψ.

$$\mathcal{C}\ell(d) \to \mathcal{C}\ell(d+1)^{(+)} \to \cdots \to \mathcal{C}\ell(d+n)^{(+)}. \quad (102)$$

The mass extension for the Dirac fermion is demonstrated in Table XXIX. There are two possible extensions:

(1) Regular mass extension: if

$$\dim_{\mathbb{C}} \psi_{\mathcal{C}\ell(d+n)} = \dim_{\mathbb{C}} \psi_{\mathcal{C}\ell(d+n+1)}, \quad (103)$$

a mass term can be added directly.

(2) Chiral mass extension: if $\mathcal{C}\ell(d+n+1) \cong \mathcal{C}\ell(d+n+1)^{+} \oplus \mathcal{C}\ell(d+n+1)^{-}$ splits and

$$\dim_{\mathbb{C}} \psi_{\mathcal{C}\ell(d+n)} = \dim_{\mathbb{C}} \psi^{\pm}_{\mathcal{C}\ell(d+n+1)^{\pm}} = \tfrac{1}{2}\dim_{\mathbb{C}} \psi_{\mathcal{C}\ell(d+n+1)}. \quad (104)$$

a mass term can be added by promoting the Dirac fermion to a Weyl fermion in one of the chiral subalgebras (say $\mathcal{C}\ell(d+n+1)^{+}$). No further mass can be added for a Weyl fermion, so the chiral mass extension is always the *end* of a mass extension sequence.

The mass extension process can also shown by finding explicit mass extensions for Dirac fermions, as demonstrated in Table XXX. These masses span an $(n-1)$-dimensional mass manifold, formulated as a Grassmannian manifold:

$$M_d = \frac{G(\mathcal{C}\ell(d))}{G(\mathcal{C}\ell(d+1)^{(+)})}. \quad (105)$$

TABLE XXX. Explicit mass extension and the mass manifold M_d.

d	α_1	β_1	β_2	$G(\mathcal{C}\ell(d))$	$G(\mathcal{C}\ell(d+1)^{(+)})$	M_d
0		σ^3		U(1)	U(1)	1
1	σ^3	σ^1	σ^2	U(1) × U(1)	U(1)	$\frac{\mathrm{U(1)\times U(1)}}{\mathrm{U(1)}} \cong S^1$

TABLE XXXI. Domain wall reduction of the complex Clifford algebra. The black arrow means domain wall reduction. The green arrow means regular mass extension. The red arrow means projective mass extension.

d	$\mathcal{C}\ell(d)$		$\mathcal{C}\ell(d+1)$
0	$\mathbb{C}(1)$	$\longrightarrow$	$\mathbb{C}(1)\oplus\mathbb{C}(1)$
1	$\mathbb{C}(1)\oplus\mathbb{C}(1)$	$\longrightarrow$	$\mathbb{C}(2)$
2	$\mathbb{C}(2)$	$\longrightarrow$	$\mathbb{C}(2)\oplus\mathbb{C}(2)$

For Dirac fermions, there are two cases for the manifold M_d:

$M_d = 1$: only one mass m, and $m = \pm 1$ belongs to a different topological order.

$M_d = S^1$: admits a mass vector $\mathbf{m}$ of 2 components, transforming under the internal and CRT symmetries as an O(2) vector.

2. Domain wall reduction

Domain wall reduction reduces a massive (bulk) fermion to the mass domain wall as a massless (boundary) fermion in one lower dimension. This simultaneously removes a momentum term and a mass term, corresponding to

$$\mathcal{C}\ell(d+1) \to \mathcal{C}\ell(d-1). \quad (106)$$

Note that $\mathcal{C}\ell(d+1) \cong \mathcal{C}\ell(d-1) \otimes_{\mathbb{C}} \mathbb{C}(2)$ holds for all d, the domain wall reduction is always feasible.

Similar to the Majorana fermion, the domain wall projection of the Dirac fermion is also given by the projection operator

$$P_{\text{DW}} = \frac{\mathbf{1} \pm i\beta\alpha_1}{2}. \quad (107)$$

which always reduces the fermion spinor dimension by *half*:

$$2^k_{\mathbb{C}} \xrightarrow{P} 2^{k-1}_{\mathbb{C}}. \quad (108)$$

The domain wall reduction of the complex Clifford algebra and domain wall reduction of Dirac/Weyl fermions are listed in Tables XXXI and XXXII.

TABLE XXXII. Domain wall reduction of Dirac/Weyl fermions. The number indicates the dimension of the representation of the fermion, the lower index indicates that the representation is complex, and the upper index indicates the chirality of the fermion.

	Weyl			Dirac		
d	Boundary	Bulk		Boundary		Bulk
0		$1^+_{\mathbb{C}}$	$\longleftarrow$	$1_{\mathbb{C}}$		$1^+_{\mathbb{C}} \oplus 1^-_{\mathbb{C}}$
1	$1^+_{\mathbb{C}}$			$1^+_{\mathbb{C}} \oplus 1^-_{\mathbb{C}}$	$\longrightarrow$	$2_{\mathbb{C}}$
2		$2^+_{\mathbb{C}}$	$\longleftarrow$	$2_{\mathbb{C}}$		$2^+_{\mathbb{C}} \oplus 2^-_{\mathbb{C}}$

TABLE XXXIII. The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes.

	$(-)^\chi$	$\mathcal{U}^F(\phi_F)$	$\mathcal{U}^\chi(\phi_\chi)$	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}$
θ'	$\pi+\theta$	θ	$\theta+\phi_\chi$	$-\theta$	$\pi-\theta$	θ	$-\theta$
m_1	×	✓	×	✓	×	✓	✓
m_2	×	✓	×	×	✓	✓	×

D. Mass term and CRT-internal symmetry

In this section, we will examine the interplay between the symmetries and the mass terms. We will focus on the action of symmetries on the manifold, and how to obtain the symmetries on the domain wall using the reduction method.

1. CRT-internal symmetry acting on mass manifold

Recall that h defined in Eq. (69) is

$$h=\sum_{i=1}^{d}\alpha_i\,\mathrm{i}\,\partial_i+\sum_{i=1}^{n}\beta_i m_i=h_0+m. \tag{109}$$

where h_0 stands for the massless part and m is the mass matrix. Since we have already proven that the h_0 part is invariant under CRT-internal symmetry in Sec. III B, we will now focus on the bilinear mass $\frac{1}{2}\int d^dx\psi^\dagger m\psi$ and how CRT-internal symmetry acts on the mass manifold.

The bilinear mass term changes under the $\mathcal{C}$, $\mathcal{R}_i$, $\mathcal{T}$, and internal U symmetries as follows:

$$\begin{aligned}
\frac{1}{2}\int d^dx\psi^\dagger m\psi &\xrightarrow{\mathcal{C}} \frac{1}{2}\int d^dx(M_{\mathcal{C}}\psi^*)^\dagger m(M_{\mathcal{C}}\psi^*)\\
&= \frac{1}{2}\int d^dx\psi^{\mathsf{T}}(M_{\mathcal{C}}^\dagger mM_{\mathcal{C}})\psi^*\\
&= -\frac{1}{2}\int d^dx\psi^\dagger(M_{\mathcal{C}}^\dagger mM_{\mathcal{C}})^{\mathsf{T}}\psi,\\
\frac{1}{2}\int d^dx\psi^\dagger m\psi &\xrightarrow{\mathcal{R}_i} \frac{1}{2}\int d^dx(\alpha_iM_{\mathcal{P}}\psi)^\dagger m(\alpha_iM_{\mathcal{P}}\psi)\\
&= \frac{1}{2}\int d^dx\psi^\dagger(M_{\mathcal{P}}^\dagger\alpha_i m\alpha_iM_{\mathcal{P}})\psi,\\
\frac{1}{2}\int d^dx\psi^\dagger m\psi &\xrightarrow{\mathcal{T}} \frac{1}{2}\int d^dx(M_{\mathcal{T}}\psi)^\dagger m^*(M_{\mathcal{T}}\psi)\\
&= \frac{1}{2}\int d^dx\psi^\dagger(M_{\mathcal{T}}^\dagger m^*M_{\mathcal{T}})\psi,\\
\frac{1}{2}\int d^dx\psi^\dagger m\psi &\xrightarrow{U} \frac{1}{2}\int d^dx(M_U\psi)^\dagger m(M_U\psi)\\
&= \frac{1}{2}\int d^dx\psi^\dagger(M_U^\dagger mM_U)\psi.
\end{aligned} \tag{110}$$

If the bilinear mass term is invariant under the $\mathcal{C}$, $\mathcal{R}_i$, $\mathcal{T}$, and internal U symmetries, then

$$\begin{aligned}
M_{\mathcal{C}}^\dagger mM_{\mathcal{C}} &= -m^{\mathsf{T}},\\
M_{\mathcal{P}}^\dagger\alpha_i m\alpha_iM_{\mathcal{P}} &= m,\\
M_{\mathcal{T}}^\dagger mM_{\mathcal{T}} &= m^*,\\
M_U^\dagger mM_U &= m.
\end{aligned} \tag{111}$$

Therefore the matrices $M_{\mathcal{C}}, M_{\mathcal{P}}, M_{\mathcal{T}}$, and U should satisfy the following relations (for $i=1,\ldots,d$):

$$\begin{aligned}
M_{\mathcal{C}}^\dagger\beta_iM_{\mathcal{C}} &= -\beta_i^*,\\
M_{\mathcal{P}}^\dagger\beta_iM_{\mathcal{P}} &= -\beta_i,\\
M_{\mathcal{T}}^\dagger\beta_iM_{\mathcal{T}} &= \beta_i^*,\\
M_U^\dagger\beta_iM_U &= \beta_i.
\end{aligned} \tag{112}$$

Any violation of these relations is regarded as the corresponding symmetry breaking.

To be more specific, in $d=$ odd, we have a nontrivial S^1 mass manifold spanned by two mass terms. The given CRT and internal symmetry operators can act on the whole manifold:

In $d=1$ case, we have two mass matrices σ^1 and σ^2, they span a general S^1 mass manifold with terms characterized by mass angle θ:

$$m(\theta)=\frac{1}{2}\int d^dx\psi^\dagger(\cos\theta\sigma^1+\sin\theta\sigma^2)\psi. \tag{113}$$

The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold is listed in Table XXXIII. $\mathcal{U}^F(\phi_F)$ and $\mathcal{T}$ act trivially on the manifold, $\mathcal{P}$ acts as a "reflection" on the manifold about $m_1=0$, $\mathcal{C}$ and $\mathcal{R}$ act as "reflection" on the manifold about $m_2=0$, and $\mathcal{U}^\chi(\phi_\chi)$ acts as a "rotation" of ϕ_χ angle on the manifold.

In $d=3$ case, we have two mass matrices σ^{13} and σ^{23}, they span a general S^1 mass manifold with terms characterized by mass angle θ:

$$m(\theta)=\frac{1}{2}\int d^dx\psi^\dagger(\cos\theta\sigma^{13}+\sin\theta\sigma^{23})\psi. \tag{114}$$

The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold is listed in Table XXXIV. $\mathcal{U}^F(\phi_F)$ and $\mathcal{C}$ act trivially on the manifold, $\mathcal{T}$ and $\mathcal{R}_i$ act as "reflection" on the manifold about $m_1=0$, $\mathcal{P}$ acts as a "reflection" on the manifold about $m_2=0$, and $\mathcal{U}^\chi(\phi_\chi)$ acts as a "rotation" of ϕ_χ angle on the manifold.

In $d=5$ case, we have two mass matrices σ^{133} and σ^{233}, they span a general S^1 mass manifold with terms characterized by mass angle θ:

$$m(\theta)=\frac{1}{2}\int d^dx\psi^\dagger(\cos\theta\sigma^{133}+\sin\theta\sigma^{233})\psi. \tag{115}$$

The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold is listed in Table XXXV. $\mathcal{U}^F(\phi_F)$ and $\mathcal{T}$ act trivially on the manifold, $\mathcal{C}$ and $\mathcal{R}_i$ act as "reflection" on the manifold about $m_1=0$, $\mathcal{P}$ acts as a "reflection" on the

TABLE XXXIV. The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes.

	$(-)^\chi$	$\mathcal{U}^F(\phi_F)$	$\mathcal{U}^\chi(\phi_\chi)$	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi+\theta$	θ	$\theta+\phi_\chi$	θ	$-\theta$	$\pi-\theta$	$\pi-\theta$
m_1	×	✓	×	✓	✓	×	×
m_2	×	✓	×	✓	×	✓	✓

manifold about $m_2 = 0$, and $\mathcal{U}^\chi(\phi_\chi)$ acts as a "rotation" of ϕ_χ angle on the manifold.

In $d = 7$ case, we have two mass matrices σ^{1333} and σ^{2333}, they span a general S^1 mass manifold with terms characterized by mass angle θ:

$$m(\theta) = \frac{1}{2}\int d^dx\psi^\dagger(\cos\theta\sigma^{1333} + \sin\theta\sigma^{2333})\psi. \quad (116)$$

The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold is listed in Table XXXVI. $\mathcal{U}^F(\phi_F)$ and $\mathcal{C}$ act trivially on the manifold, $\mathcal{T}$ and $\mathcal{R}_i$ act as "reflection" on the manifold about $m_1 = 0$, $\mathcal{P}$ acts as a "reflection" on the manifold about $m_2 = 0$, and $\mathcal{U}^\chi(\phi_\chi)$ acts as a "rotation" of ϕ_χ angle on the manifold.

2. CRT-internal symmetry reduction under domain wall

By domain wall reduction, we can reduce a *bulk* Dirac fermion to *boundary* Dirac (or Weyl) fermion in a lower dimension. Surprisingly, the CRT-internal symmetry group in different dimensions (see Tables XXVII and XXVIII), though exhibit distinct fractionalization properties, is related by domain wall reduction. We can always reproduce the CRT-internal symmetry group in $(d-1)+1$ dimension by projecting corresponding symmetry operators to the mass domain wall.

We can randomly add a mass and reduce the fermion to the mass domain wall in either direction (say $m \sim \pm x_d$). Note that a well-defined (i.e., not modified by broken internal symmetries) reflection $\mathcal{R}_d$ is always preserved under the mass domain wall on the d-th direction, since the reflection simultaneously swaps the ground state in the P_+ and P_- projection space, and flips the mass profile $m \sim \pm x_d \to \mp x_d$. Under domain wall reduction, $\mathcal{R}_d$ always becomes an internal symmetry on the domain wall. The reduction of CRT and internal symmetries follow the *rules* below:

TABLE XXXV. The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes.

	$(-)^\chi$	$\mathcal{U}^F(\phi_F)$	$\mathcal{U}^\chi(\phi_\chi)$	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi+\theta$	θ	$\theta+\phi_\chi$	$\pi-\theta$	$-\theta$	θ	$\pi-\theta$
m_1	×	✓	×	×	✓	✓	×
m_2	×	✓	×	✓	×	✓	✓

TABLE XXXVI. The action of $\mathcal{P}, \mathcal{T}, \mathcal{R}_i, (-)^\chi, \mathcal{U}^F(\phi_F), \mathcal{U}^\chi(\phi_\chi)$ on the S^1 mass manifold. ✓ means the mass manifold preserves the symmetry. × means the mass term breaks the symmetry and mass angle θ changes.

	$(-)^\chi$	$\mathcal{U}^F(\phi_F)$	$\mathcal{U}^\chi(\phi_\chi)$	$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{R}_i$
θ'	$\pi+\theta$	θ	$\theta+\phi_\chi$	θ	$-\theta$	$\pi-\theta$	$\pi-\theta$
m_1	×	✓	×	✓	✓	×	×
m_2	×	✓	×	✓	×	✓	✓

(1) If the charge conjugation $\mathcal{C}$, reflection $\mathcal{R}_i$, time-reversion $\mathcal{T}$, or internal symmetry U is *preserved under mass extension*, then these symmetries are directly projected to the $(d-1)$-dimensional CRT-internal symmetry by projection operator $P_{DW} = \frac{1\pm i\beta\alpha_1}{2}$:

$$\begin{aligned} d\text{-dimension} &\xrightarrow{DW} (d-1)\text{-dimension}\\ \mathcal{C} &\xrightarrow{P_{DW}} \mathcal{C}\\ \mathcal{R}_d &\xrightarrow{P_{DW}} X\\ \mathcal{R}_i &\xrightarrow{P_{DW}} \mathcal{R}_i \quad (\forall i = 1,\dots,d-1)\\ \mathcal{T} &\xrightarrow{P_{DW}} \mathcal{T}\\ U &\xrightarrow{P_{DW}} U, \end{aligned} \quad (117)$$

where X is an internal symmetry in the $(d-1)$-dimensional theory.

(2) If the charge conjugation $\mathcal{C}$, reflection $\mathcal{R}_i$, time-reversion $\mathcal{T}$, or internal symmetry U is *broken under mass extension*, then these symmetries should be combined with the space-orientation-reversing symmetry $\mathcal{C}\mathcal{R}_d\mathcal{T}$ to obtain a new symmetry on the domain wall [29,43,65,66]:

$$\begin{aligned} d\text{-dimension} &\xrightarrow{DW} (d-1)\text{-dimension}\\ \mathcal{C} &\xrightarrow{\cdot\mathcal{C}\mathcal{R}_d\mathcal{T}} X\mathcal{T} = \mathcal{T}'\\ \mathcal{R}_d &\xrightarrow{P_{DW}} X\\ \mathcal{R}_i &\xrightarrow{\cdot\mathcal{C}\mathcal{R}_d\mathcal{T}} X\mathcal{C}\mathcal{R}_i\mathcal{T} = \mathcal{C}'\mathcal{R}'_i\mathcal{T}' \quad (\forall i = 1,\dots,d-1)\\ \mathcal{T} &\xrightarrow{\cdot\mathcal{C}\mathcal{R}_d\mathcal{T}} X\mathcal{C} = \mathcal{C}'\\ U &\xrightarrow{\cdot\mathcal{C}\mathcal{R}_d\mathcal{T}} XU\mathcal{C}\mathcal{T} = \mathcal{C}'\mathcal{T}'. \end{aligned} \quad (118)$$

To be specific, the CRT-internal symmetry groups in spacetime $d+1$ and $(d-1)+1$ dimension, and distinct mass terms are listed in Table XXXVII. On each domain wall mass, the explicit result of the domain-wall projection for symmetry operators are listed in Table XXXVIII.

IV. CONCLUSION

In this work, we have systematically analyzed the CRT fractionalization in a single-particle Hamiltonian theory.

In Sec. II A, we have defined the Majorana field as a real Grassmannian field acting as an irreducible representation of the Clifford algebra $\mathcal{C}\ell(d,n)$. This definition is compatible with the conventional definition [43] as a single Dirac fermion with trivial charge conjugation and can extend to $d+1 =$

TABLE XXXVII. The CRT-internal symmetry group for Dirac fermion in $d+1$ spacetime dimension $G_{D,d}$ can be reduced to the CRT-internal symmetry group for a Dirac or Weyl fermion in $(d-1)+1$ spacetime dimension $G_{D/W,d-1}$ on the mass domain wall. The domain wall mass term m_i can be chosen in the mass manifold. The CRT-invariant group includes vector $\mathrm{U}(1)^F$ symmetry, axial $\mathrm{U}(1)^\chi$ symmetry, charge conjugation $\mathbb{Z}_2^{\mathcal{C}}$, reflection $\mathbb{Z}_2^{\mathcal{R}_1}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	$G_{D,d}^{\mathrm{CRT}}$	$G_{D,d}$	$G_{D/W,d-1}^{\mathrm{CRT}}$	$G_{D/W,d-1}$	m_1	m_2
1	$\mathbb{D}_8^{\mathcal{CT},\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{R}_1\mathcal{T}}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes G_{D,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^1\psi$	$\psi^\dagger\sigma^2\psi$
2	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1} \times \mathbb{Z}_2^{\mathcal{C}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{Z}_2^{\mathcal{C}} \times \mathbb{Z}_2^{\mathcal{CR}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes G_{W,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^3\psi$	
3	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1} \times \mathbb{Z}_2^{\mathcal{C}}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{D}_8^{\mathcal{T},\mathcal{R}_1} \times \mathbb{Z}_2^{\mathcal{C}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{13}\psi$	$\psi^\dagger\sigma^{23}\psi$
4	$\mathbb{D}_8^{\mathcal{T},\mathcal{CR}_1} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{Z}_4^{\mathcal{T}} \times \mathbb{Z}_2^{\mathcal{CR}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{W,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{33}\psi$	
5	$\mathbb{D}_8^{\mathcal{R}_1,\mathcal{CR}_1\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{D}_8^{\mathcal{T},\mathcal{CR}_1} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{133}\psi$	$\psi^\dagger\sigma^{233}\psi$
6	$\mathbb{D}_8^{\mathcal{CR}_1,\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{D}_8^{\mathcal{C},\mathcal{CR}_1\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{W,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{333}\psi$	
7	$\mathbb{D}_8^{\mathcal{CR}_1,\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$(\mathrm{U}(1)^F \times \mathrm{U}(1)^\chi) \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{D}_8^{\mathcal{CR}_1,\mathcal{T}} \times_{\mathbb{Z}_2^F} \mathbb{Z}_4^{\mathcal{C}F}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{1333}\psi$	$\psi^\dagger\sigma^{2333}\psi$
8	$\mathbb{D}_8^{\mathcal{CR}_1,\mathcal{CT}} \times \mathbb{Z}_2^{\mathcal{C}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{D,d}^{\mathrm{CRT}}$	$\mathbb{D}_8^{\mathcal{CR}_1,\mathcal{T}}$	$\mathrm{U}(1)^F \rtimes_{\mathbb{Z}_2^F} G_{W,d-1}^{\mathrm{CRT}}$	$\psi^\dagger\sigma^{3333}\psi$	

5, 6, 7 mod 8 where symplectic Majorana fermion emerges. We have also reviewed the Clifford algebra and its eightfold periodicity.

In Sec. II B, we have defined Lorentz symmetries, internal symmetries and RT symmetries for the massless Majorana (and Majorana-Weyl) fermion, and specified the invariant group with eightfold periodicity. The results are listed in Tables VIII and IX. To further analyze the ambiguity of the explicit representation basis of the CRT operators, we introduce Clifford algebra extension in Table X, the explicit choices are listed in Table XII.

In Sec. II C, we have studied the massive Majorana theory by extending mass terms. In different dimensions, we can add different numbers of mass terms, and they reflect distinct topological properties by expanding mass manifolds. The resulting mass manifold in different dimensions is demonstrated in Table XIII. By further introducing domain wall reduction, we can connect Majorana (and Majorana-Weyl) fermions in different dimensions by mass extension and domain wall reduction, as listed in Tables XIV and XV.

In Sec. II D, we have studied the interplay between mass terms and CRT-internal symmetries. Intriguingly, when a mass manifold is formed by multiple mass terms, CRT-internal symmetries can act on the manifold as either flipping or reflection, as demonstrated in Tables XVI–XVIII. The CRT-internal symmetries together suffices to rule out all possible mass terms. Furthermore, we have studied the symmetry reduction under domain wall by extending the mass and then apply domain wall reduction. The reduction follows the rules in Eqs. (66) and (67) and bridges the CRT-internal symmetries in different dimensions (see Tables XIX and XX).

In Sec. III A, we have similarly defined the Dirac field as a complex Grassmannian field acting as an irreducible representation of the Clifford algebra $\mathcal{C}\ell(d+n)$. By examining the complex Clifford algebra, we have found that the Dirac field exhibits twofold periodicity.

In Sec. III B, we have defined the Lorentz symmetries, internal symmetries and CRT symmetries for the massless Dirac (and Weyl) fermion. Surprisingly, by assuming the canonical CRT conditions, the invariant group is eightfold

TABLE XXXVIII. The projected symmetry operators $P_{DW}(\cdot)$ with the domain wall mass m_{DW} from spatial dimension d to $d-1$. Symmetries include vector $\mathrm{U}(1)^F$ symmetry generated by $(-)^F$, axial $\mathrm{U}(1)^\chi$ symmetry generated by $(-)^\chi$, charge conjugation $\mathbb{Z}_2^{\mathcal{C}}$, reflection $\mathbb{Z}_2^{\mathcal{R}_i}$, and time-reversal symmetry $\mathbb{Z}_2^{\mathcal{T}}$.

d	m_{DW}	$P_{DW}((-)^F)$	$P_{DW}((-)^\chi)$	$P_{DW}(\mathcal{C})$	$P_{DW}(\mathcal{R}_{i<d})$	$P_{DW}(\mathcal{R}_d)$	$P_{DW}(\mathcal{T})$
1	m_1	$(-)^F$	0		$\mathcal{C}$	$(-)^F$	$\mathcal{T}$
	m_2	$(-)^F$	0		$\mathcal{C}$	1	$\mathcal{T}$
2	m_1	$(-)^F$		$\mathcal{C}$	$\mathcal{CR}_1\mathcal{T}$	1	$\mathcal{C}$
3	m_1	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	1	$\mathcal{T}$
	m_2	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	$(-)^F$	$\mathcal{T}$
4	m_1	$(-)^F$		$\mathcal{T}$	$\mathcal{CR}_i\mathcal{T}$	1	$\mathcal{T}$
5	m_1	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	1	$\mathcal{T}$
	m_2	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	$(-)^F$	$\mathcal{T}$
6	m_1	$(-)^F$		$\mathcal{C}$	$\mathcal{CR}_i\mathcal{T}$	1	$\mathcal{C}$
7	m_1	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	1	$\mathcal{T}$
	m_2	$(-)^F$	0	$\mathcal{C}$	$\mathcal{R}_i$	$(-)^F$	$\mathcal{T}$
8	m_1	$(-)^F$		$\mathcal{T}$	$\mathcal{CR}_i\mathcal{T}$	1	$\mathcal{T}$

TABLE XXXIX. Summary of domain wall reduction of fermions. The target of the arrow is the domain wall of the source of the arrow. The number indicates the dimension of the representation of the fermion, the lower index indicates that the representation is real or complex or quaternionic, and the upper index indicates the chirality of the fermion. The dashed lines (- - -) connecting Majorana-Weyl and Majorana fermions or connecting Weyl and Dirac fermions mean that the left and right Majorana-Weyl or Weyl fermions can be combined into a Majorana or Dirac fermion. The solid lines (—) connecting Majorana and Weyl fermions in spacetime dimensions $d+1=4 \bmod 8$ mean that Majorana and Weyl fermions in spacetime dimensions $d+1=4 \bmod 8$ can be identified [43]. For instance, in $3+1d$, we can write Majorana fermion in Weyl basis as shown in Appendix A. However, for $d+1=0 \bmod 8$, though Majorana fermion and Weyl fermion share the Clifford algebra $\mathbb{C}(2^{\frac{d-1}{2}})$, the Majorana fermion cannot be written in Weyl fermion [43].

d	Majorana-Weyl (MW) & Symplectic MW	Majorana (M) & Symplectic M	Weyl (W)	Dirac (D)
0		$1_{\mathbb{R}}$		$1_{\mathbb{C}}$
1	$1^+_{\mathbb{R}}$	$1^+_{\mathbb{R}} \oplus 1^-_{\mathbb{R}}$	$1^+_{\mathbb{C}}$	$1^+_{\mathbb{C}} \oplus 1^-_{\mathbb{C}}$
2		$2_{\mathbb{R}}$		$2_{\mathbb{C}}$
3		$2_{\mathbb{C}}$	$2^+_{\mathbb{C}}$	$2^+_{\mathbb{C}} \oplus 2^-_{\mathbb{C}}$
4		$2_{\mathbb{H}}$		$4_{\mathbb{C}}$
5	$2^+_{\mathbb{H}}$	$2^+_{\mathbb{H}} \oplus 2^-_{\mathbb{H}}$	$4^+_{\mathbb{C}}$	$4^+_{\mathbb{C}} \oplus 4^-_{\mathbb{C}}$
6		$4_{\mathbb{H}}$		$8_{\mathbb{C}}$
7		$8_{\mathbb{C}}$	$8^+_{\mathbb{C}}$	$8^+_{\mathbb{C}} \oplus 8^-_{\mathbb{C}}$
8		$16_{\mathbb{R}}$		$16_{\mathbb{C}}$
9	$16^+_{\mathbb{R}}$	$16^+_{\mathbb{R}} \oplus 16^-_{\mathbb{R}}$	$16^+_{\mathbb{C}}$	$16^+_{\mathbb{C}} \oplus 16^-_{\mathbb{C}}$
10		$32_{\mathbb{R}}$		$32_{\mathbb{C}}$
11		$32_{\mathbb{C}}$	$32^+_{\mathbb{C}}$	$32^+_{\mathbb{C}} \oplus 32^-_{\mathbb{C}}$

periodic rather than twofold (as the periodicity of the complex Clifford algebra). The results are listed in Tables XXVII and XXVIII.

In Sec. III C, we have studied the massive Dirac theory by extending mass terms. In even spatial dimensions, we can only add one mass term, while in odd cases, two distinct mass terms generate a S^1 mass manifold. The resulting mass extension and mass manifold are listed in Tables XXIX and XXX. By further introducing domain wall reduction, we can connect Dirac (and Weyl) fermions in different dimensions by mass extension and domain wall reduction, as listed in Tables XXXI and XXXII.

In Sec. III D, we have studied the interplay between mass terms and CRT-internal symmetries. We have found that the axial U(1) symmetry can rotate the mass manifold and other symmetries can flip the manifold, as demonstrated in Tables XXXIII–XXXVI. The CRT-internal symmetries together suffices to rule out all possible mass terms. Furthermore, we have studied the symmetry reduction under domain wall by extending a mass and then apply domain wall reduction. The reduction follows the rules in Eqs. (117) and (118) and bridges the CRT-internal symmetries in different dimensions (see Tables XXXVII and XXXVIII).

The domain wall reduction of Majorana/Majorana-Weyl and Dirac/Weyl fermions is summarized in Table XXXIX. This map is conducive to the derivation of the relation between $N_f = 3$ families of 16 Weyl fermions of the Standard Model in 4d to the 48 Majorana-Weyl fermions in 2d [69]. Moreover, the domain wall reduction method has a profound impact on the classification of interacting fermions [70]. Another intriguing phenomenon is that Majorana fermion and Weyl fermion can be identified only when $d+1=4 \bmod 8$. For instance, in $3+1d$, we can write Majorana fermion in Weyl basis as shown in Appendix A. However, for $d+1=0 \bmod 8$, though Majorana fermion and Weyl fermion share the Clifford algebra $\mathbb{C}(2^{\frac{d-1}{2}})$, the Majorana fermion cannot be written in Weyl fermion [43].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: BRIEF REVIEW OF MAJORANA FERMION IN 3 + 1*d* CASE

In this Appendix, we briefly review the process of deriving conventional Majorana fermion from Dirac fermion.

1. Basis independent discussion

In $3+1d$ spacetime, the Dirac fermion is characterized by the following Lagrangian:

$$L=\frac{1}{2}\int d^3x\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi, \tag{A1}$$

and corresponding Dirac equation:

$$(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi=0, \tag{A2}$$

where γ^μ is chosen in $\mathcal{C}\ell_{1,3}(\mathbb{C})$, i.e., γ^μ are complex matrices satisfying $(\gamma^0)^2=1$ and $(\gamma^i)^2=-1$ for $i=1,2,3$.

The charge conjugation for the Dirac spinor is defined by

$$\psi^c\overset{\text{def}}{=\!=}\mathcal{C}\psi\mathcal{C}^{-1}=M_{\mathcal{C}}\psi^*. \tag{A3}$$

Define another matrix C by

$$M_{\mathcal{C}}=C^{-1}(\gamma^0)^\mathsf{T}. \tag{A4}$$

Therefore, by choosing C symmetric or antisymmetric ($C^\mathsf{T}=\pm C$), we obtain

$$\bar{\psi}=(\psi^c)^\mathsf{T}C^\mathsf{T}=\pm(\psi^c)^\mathsf{T}C. \tag{A5}$$

$M_{\mathcal{C}}$ is defined to satisfy

$$M_{\mathcal{C}}^\dagger\gamma^0M_{\mathcal{C}}=-(\gamma^0)^\mathsf{T},\quad M_{\mathcal{C}}^\dagger\gamma^iM_{\mathcal{C}}=(\gamma^i)^\mathsf{T}\quad(\text{for } i=1,2,3). \tag{A6}$$

Thus C needs to satisfy

$$C\gamma^\mu C^{-1}=-(\gamma^\mu)^\mathsf{T}. \tag{A7}$$

In order to find a Majorana solution, we assume $\psi^c=\psi$:

$$\psi^c=C^{-1}(\gamma^0)^\mathsf{T}\psi^*=\psi, \tag{A8}$$

or written in the form

$$\bar{\psi}=\pm\psi^\mathsf{T}C. \tag{A9}$$

Then the Lagrangian becomes ($\pm$ is neglected since it will not affect the equation of motion)

$$L=\frac{1}{2}\int d^3x\psi^\mathsf{T}C(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi. \tag{A10}$$

The equation of motion (EOM) is again the Dirac equation

$$(\mathrm{i}\gamma^\mu\partial_\mu-m)\psi=0, \tag{A11}$$

with additional constraint

$$\psi=C^{-1}(\gamma^0)^\mathsf{T}\psi^*. \tag{A12}$$

To relax the constraint, we need to introduce an explicit basis and minimize the degree of freedom.

2. Weyl basis

To faithfully write the Majorana equation in the representation of real Clifford algebra $\mathcal{C}\ell(3,0)\cong\mathbb{C}(2)$, it is intuitive to relate the Majorana fermion to one of the Weyl fermions. The Weyl basis is given by

$$\begin{aligned}\gamma^0&=\sigma^{10}=\begin{pmatrix}0&\sigma^0\\\sigma^0&0\end{pmatrix},\\ \gamma^1&=\mathrm{i}\sigma^{21}=\begin{pmatrix}0&\sigma^1\\-\sigma^1&0\end{pmatrix},\\ \gamma^2&=\mathrm{i}\sigma^{22}=\begin{pmatrix}0&\sigma^2\\-\sigma^2&0\end{pmatrix},\\ \gamma^3&=\mathrm{i}\sigma^{23}=\begin{pmatrix}0&\sigma^3\\-\sigma^3&0\end{pmatrix}.\end{aligned} \tag{A13}$$

To satisfy

$$\begin{aligned}M_{\mathcal{C}}^\dagger\gamma^0M_{\mathcal{C}}&=-(\gamma^0)^\mathsf{T}=-\gamma^0,\\ M_{\mathcal{C}}^\dagger\gamma^1M_{\mathcal{C}}&=(\gamma^1)^\mathsf{T}=-\gamma^1,\\ M_{\mathcal{C}}^\dagger\gamma^2M_{\mathcal{C}}&=(\gamma^2)^\mathsf{T}=\gamma^2,\\ M_{\mathcal{C}}^\dagger\gamma^3M_{\mathcal{C}}&=(\gamma^3)^\mathsf{T}=-\gamma^3,\end{aligned} \tag{A14}$$

we can choose $M_{\mathcal{C}}=-\mathrm{i}\gamma^2=\sigma^{22}$ and $C=(\gamma^0)^TM_{\mathcal{C}}^\dagger=\mathrm{i}\sigma^{32}$.

The Majorana constraint becomes

$$\psi=M_{\mathcal{C}}\psi^*=\sigma^{22}\psi^*, \tag{A15}$$

which gives

$$\psi_L=-\mathrm{i}\sigma^2\psi_R^*,\quad\psi_R=\mathrm{i}\sigma^2\psi_L^*. \tag{A16}$$

This means that we can construct the Majorana fermion through only one Weyl fermion, say ψ_L:

$$\psi=\begin{pmatrix}\psi_L\\\mathrm{i}\sigma^2\psi_L^*.\end{pmatrix} \tag{A17}$$

The Lagrangian

$$L = \frac{1}{2}\int d^3x\psi^{\mathsf{T}}C(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\psi \tag{A18}$$

can be reduced to the Lagrangian for ψ_L and ψ_L^*:

$$\begin{aligned}
L &= \frac{1}{2}\int d^3x\psi^{\mathsf{T}}C(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\psi \\
&= \frac{1}{2}\int d^3x\big(\psi_L^{\mathsf{T}}, -\mathrm{i}\sigma^2\psi_L^{\dagger}\big)\mathrm{i}\sigma^{32}\left(\mathrm{i}\sigma^{10}\partial_0 - \sum_i \sigma^{2i}\partial_i - m\right)\begin{pmatrix}\psi_L \\ \mathrm{i}\sigma^2\psi_L^*\end{pmatrix} \\
&= \frac{1}{2}\int d^3x\big(\psi_L^{\mathsf{T}}, -\mathrm{i}\sigma^2\psi_L^{\dagger}\big)(-\mathrm{i}\sigma^{22}\partial_0 + \mathrm{i}\sigma^{13}\partial_1 - \sigma^{10}\partial_2 - \mathrm{i}\sigma^{11}\partial_3 - \mathrm{i}m\sigma^{32})\begin{pmatrix}\psi_L \\ \mathrm{i}\sigma^2\psi_L^*\end{pmatrix} \\
&= \frac{1}{2}\int d^3x\psi_L^{\mathsf{T}}(-\mathrm{i}\partial_0 + \mathrm{i}\partial_1\sigma^1 - \mathrm{i}\partial_2\sigma^2 + \mathrm{i}\partial_3\sigma^3)\psi_L^* + \psi_L^{\dagger}(-\mathrm{i}\partial_0 + \mathrm{i}\partial_1\sigma^1 + \mathrm{i}\partial_2\sigma^2 + \mathrm{i}\partial_3\sigma^3)\psi_L \\
&\quad + \psi_L^{\mathsf{T}}(-\mathrm{i}m\sigma^2)\psi_L + \psi_L^{\dagger}(\mathrm{i}m\sigma^2)\psi_L^* \\
&= \frac{1}{2}\int d^3x\big[-\mathrm{i}\partial_0\big(\psi_L^{\mathsf{T}}\psi_L^*\big) + \mathrm{i}\partial_1(\psi_L^{\mathsf{T}}\sigma^1\psi_L^*) - \mathrm{i}\partial_2\big(\psi_L^{\mathsf{T}}\sigma^2\psi_L^*\big) + \mathrm{i}\partial_3\big(\psi_L^{\mathsf{T}}\sigma^3\psi_L^*\big)\big] \\
&\quad - \int d^3x\psi_L^{\dagger}(\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu})\psi_L + \frac{\mathrm{i}m}{2}\big(\psi_L^{\mathsf{T}}\sigma^2\psi_L - \psi_L^{\dagger}\sigma^2\psi_L^*\big) \\
&\sim \int d^3x\psi_L^{\dagger}(\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu})\psi_L + \frac{\mathrm{i}m}{2}\big(\psi_L^{\mathsf{T}}\sigma^2\psi_L - \psi_L^{\dagger}\sigma^2\psi_L^*\big).
\end{aligned} \tag{A19}$$

Therefore the EOM becomes

$$\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu}\psi_L - \mathrm{i}m\sigma^2\psi_L^* = 0. \tag{A20}$$

3. Embedded in $\mathbb{R}(4)$: Majorana basis

Instead of throwing away half of the component, we can also relax the constraint by finding a basis where all $i\gamma^{\mu}$ matrices are real so that the Dirac equation (and Lagrangian) is real and spontaneously allows a real solution. The Majorana basis is given by

$$\begin{aligned}
\gamma^0 &= \sigma^{12} = \begin{pmatrix}0 & \sigma^2 \\ \sigma^2 & 0\end{pmatrix}, \\
\gamma^1 &= \mathrm{i}\sigma^{03} = \begin{pmatrix}\mathrm{i}\sigma^3 & 0 \\ 0 & \mathrm{i}\sigma^3\end{pmatrix}, \\
\gamma^2 &= -\mathrm{i}\sigma^{22} = \begin{pmatrix}0 & -\sigma^2 \\ \sigma^2 & 0\end{pmatrix}, \\
\gamma^3 &= -\mathrm{i}\sigma^{01} = \begin{pmatrix}-\mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1\end{pmatrix}.
\end{aligned} \tag{A21}$$

In this case, the charge conjugation matrix $M_{\mathcal{C}}$ needs to satisfy

$$\begin{aligned}
M_{\mathcal{C}}^{\dagger}\gamma^0 M_{\mathcal{C}} &= -(\gamma^0)^{\mathsf{T}} = \gamma^0, \\
M_{\mathcal{C}}^{\dagger}\gamma^i M_{\mathcal{C}} &= (\gamma^i)^{\mathsf{T}} = \gamma^i, \quad (\text{for } i = 1, 2, 3),
\end{aligned} \tag{A22}$$

which means we can simply set $M_{\mathcal{C}} = I_{4\times 4}$ and $\mathcal{C} = (\gamma^0)^{\mathsf{T}} = -\sigma^{12}$.

Now the Majorana constraint becomes

$$\psi = M_{\mathcal{C}}\psi^* = \psi^*, \tag{A23}$$

which means ψ is real and we can throw the imaginary part of ψ away. A Majorana fermion is just a real fermion satisfying the Dirac equation on the Majorana basis. This embedding process can be done in all dimensions, and by embedding to $\mathbb{R}(n)$, we can always find a real solution of ψ, the trivial charge conjugation becomes trivial complex conjugation in this sense.

APPENDIX B: BRIEF REVIEW OF SYMPLECTIC MAJORANA FERMION IN 4 + 1*d* CASE

In this Appendix, we review the explicit process of deriving a symplectic Majorana fermion from a *pair* of two Dirac fermions. Also, we show how the $\mathbb{H}(2)$ theory can be embedded in $\mathbb{R}(8)$.

1. Basis independent discussion

For spacetime dimension $d+1 = 0, 1, 2, 3, 4 \bmod 8$, the representation algebra for Majorana fermion $\mathcal{C}\ell(d,0)$ and Dirac fermion $\mathcal{C}\ell(d)$ satisfy the relation:

$$\dim_{\mathbb{R}}\psi_{\mathcal{C}\ell(d)} = 2\dim_{\mathbb{R}}\chi_{\mathcal{C}\ell(d,0)}, \tag{B1}$$

therefore we can always set a "trivial charge conjugation" constraint to get a Majorana fermion from the corresponding Dirac fermion similar to the discussion above for $3+1d$.

However, for spacetime dimension $d+1 = 5, 6, 7 \bmod 8$, the representation algebra for Majorana fermion $\mathcal{C}\ell(d,0)$ and Dirac fermion $\mathcal{C}\ell(d)$ satisfy the relation:

$$\dim_{\mathbb{R}}\psi_{\mathcal{C}\ell(d)} = \dim_{\mathbb{R}}\chi_{\mathcal{C}\ell(d,0)}, \tag{B2}$$

which motivates us to use a *pair* of two Dirac fermions with trivial charge conjugation to construct the symplectic Majorana fermion.

a. Complexification of quaternion

For $d+1=5 \bmod 8$

$$\mathcal{C}\ell(d,0)=\mathbb{H}\big(2^{\frac{d-2}{2}}\big), \tag{B3}$$

for $d+1=6 \bmod 8$

$$\mathcal{C}\ell(d,0)=\mathbb{H}\big(2^{\frac{d-3}{2}}\big)\oplus\mathbb{H}\big(2^{\frac{d-3}{2}}\big), \tag{B4}$$

for $d+1=7 \bmod 8$

$$\mathcal{C}\ell(d,0)=\mathbb{H}\big(2^{\frac{d-2}{2}}\big), \tag{B5}$$

they're all quaternion type. In order to get a quaternion type spinor, we need to do "trivial complex conjugation" from a biquaternion (complexified quaternion) type spinor. Complexification means that a complex number substitutes the coefficients for the corresponding algebra. It can be denoted as tensor product $\otimes_{\mathbb{R}}\mathbb{C}$. For $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, the result of complexification is

$$\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C},\quad \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C},\quad \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{C}(2), \tag{B6}$$

and the last result is what we need. We can give a specific map from $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}$ to $\mathbb{C}(2)$. Note that

$$\begin{pmatrix} \mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \tag{B7}$$

and these three matrices have squares equal to -1 while anticommuting with each other. We can map them to the units of quaternion:

$$\mathrm{i}'\sim\begin{pmatrix} \mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix},\quad \mathrm{j}'\sim\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad \mathrm{k}'\sim\begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \tag{B8}$$

therefore a complexified quaternion $q=u+v\,\mathrm{i}'+w\,\mathrm{j}'+x\,\mathrm{k}'$ can be mapped to

$$q\sim\begin{pmatrix} u+\mathrm{i}v & w+\mathrm{i}x\\ -w+\mathrm{i}x & u-\mathrm{i}v\end{pmatrix}, \tag{B9}$$

where u, v, w, x are complex numbers. From a general 2×2 complex matrix

$$M=\begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}, \tag{B10}$$

we can similarly map back to a complexified quaternion:

$$M\sim\frac{c_{11}+c_{22}}{2}+\frac{c_{11}-c_{22}}{2\mathrm{i}}\,\mathrm{i}'+\frac{c_{12}-c_{21}}{2}\,\mathrm{j}'+\frac{c_{12}+c_{21}}{2\mathrm{i}}\,\mathrm{k}'. \tag{B11}$$

Equipped with this mapping, we can return to the analysis of the symplectic Majorana fermion.

b. Mapping from two Dirac fermions to one symplectic Majorana fermion

In order to go through the same "trivial charge conjugation" process, we need a pair of Dirac fermions now.

Let's first examine the $5+1d$ case, where a Dirac fermion is the representation of $\mathbb{C}(4)$ and a symplectic Majorana fermion is the representation of $\mathbb{H}(2)$.

The process goes as follows.

(1) Introduce 2 Dirac fermions ψ_1 and ψ_2, each ψ_a has four complex components.

(2) Rearrange their components to form a 4×2 matrix.

(3) Map each four components to a compexificated quaternion $\psi_i'^{\mathbb{C}}$. The 4×2 matrix becomes a two-component spinor with compexified quaternion value for each component.

(4) Introduce a trivial charge conjugation constraint.

To be specific, the mapping is given by

$$\begin{pmatrix}(\psi_1)_1\\(\psi_1)_2\\(\psi_1)_3\\(\psi_1)_4\end{pmatrix}\text{ and }\begin{pmatrix}(\psi_2)_1\\(\psi_2)_2\\(\psi_2)_3\\(\psi_2)_4\end{pmatrix}\to\left(\begin{array}{c|c}(\psi_1)_1 & (\psi_2)_1\\(\psi_1)_2 & (\psi_2)_2\\(\psi_1)_3 & (\psi_2)_3\\(\psi_1)_4 & (\psi_2)_4\end{array}\right)\to\left(\begin{array}{cc}(\psi_1)_1 & (\psi_2)_2\\(\psi_1)_2 & (\psi_2)_1\\ \hline (\psi_1)_3 & (\psi_2)_4\\(\psi_1)_4 & (\psi_2)_3\end{array}\right)\to\begin{pmatrix}\psi_1'^{\mathbb{C}}\\ \psi_2'^{\mathbb{C}}\end{pmatrix}\to\begin{pmatrix}\psi_1'\\ \psi_2'\end{pmatrix} \tag{B12}$$

where

$$\psi_1'^{\mathbb{C}}=\frac{(\psi_1)_1+(\psi_2)_1}{2}+\frac{(\psi_1)_1-(\psi_2)_1}{2\mathrm{i}}\,\mathrm{i}'+\frac{(\psi_2)_2-(\psi_1)_2}{2}\,\mathrm{j}'+\frac{(\psi_2)_2+(\psi_1)_2}{2\mathrm{i}}\,\mathrm{k}', \tag{B13}$$

$$\psi_2'^{\mathbb{C}}=\frac{(\psi_1)_3+(\psi_2)_3}{2}+\frac{(\psi_1)_3-(\psi_2)_3}{2\mathrm{i}}\,\mathrm{i}'+\frac{(\psi_2)_4-(\psi_1)_4}{2}\,\mathrm{j}'+\frac{(\psi_2)_4+(\psi_1)_4}{2\mathrm{i}}\,\mathrm{k}'. \tag{B14}$$

The trivial charge conjugation constraint can be set analogous to the $3+1d$ case as

$$\psi'=M_{\mathcal{C}}'\psi'^{*} \tag{B15}$$

or

$$\bar{\psi}'=\pm\psi'^{\mathsf{T}}C', \tag{B16}$$

where

$$M_{\mathcal{C}}'=C'^{-1}(\gamma^0)^{\mathsf{T}} \tag{B17}$$

and

$$\bar{\psi}'\overset{\text{def}}{=\!=}\psi'^{\dagger}\gamma^0, \tag{B18}$$

where the complex conjugation acts only on i but not on $\mathrm{i}', \mathrm{j}', \mathrm{k}'$.

Now the quaternion spinor ψ' has the following Lagrangian:

$$\begin{aligned}L&=\frac{1}{2}\int d^4x\bar{\psi}'(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m)\psi'\\&=\frac{1}{2}\int d^4x\psi'^{\mathsf{T}}C'(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m)\psi',\end{aligned} \tag{B19}$$

and the EOM is given by

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu}-m)\psi'=0 \tag{B20}$$

with a trivial charge conjugation constraint

$$\bar{\psi}'=\pm\psi'^{\mathsf{T}}C'. \tag{B21}$$

c. Summary

In the representation space, the combination process is given by

$$\begin{aligned}4_{\mathbb{C}}\text{ and }4_{\mathbb{C}}&\to M_{4\times2}(\mathbb{C})\to M_{2\times2}(\mathbb{C})\oplus M_{2\times2}(\mathbb{C})\\&\to 2_{\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}}\to 2_{\mathbb{H}}.\end{aligned} \tag{B22}$$

In the Clifford algebra sense, the combination process is given by

$$\mathbb{C}(4)\text{ and }\mathbb{C}(4)\to\mathbb{C}(4)\to\mathbb{C}(4)\to\mathbb{H}(2)\otimes_{\mathbb{R}}\mathbb{C}\to\mathbb{H}(2). \tag{B23}$$

2. Symplectic Majorana basis

Similar to the Majorana basis, we want to find a symplectic Majorana basis in $4+1d$ so that the matrices $\mathrm{i}\gamma^{\mu}$ do not contain the imaginary unit i, and are quaternion-valued (rather than biquaternion). Once the symplectic Majorana basis is found, the matrix M'_C can be set to identity, and the trivial charge conjugation constraint becomes $\psi'^{*}=\psi'$. Under this basis, ψ' can be quaternion-valued instead of biquaternion-valued.

Unfortunately, for $d+1=5$ mod 8, this symplectic Majorana basis does not exist. However, if $m=0$, we can still find a symplectic Majorana basis such that the matrices γ^{μ} do not contain the imaginary unit i, and the EOM for a massless symplectic Majorana fermion is imaginary.

To simplify the notation, we define

$$\begin{aligned}\sigma^0&=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \tilde{\sigma}^2&=-\mathrm{i}\sigma^2=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\end{aligned} \tag{B24}$$

and $\sigma^0,\sigma^1,\tilde{\sigma}^2,\sigma^3$ do not explicitly contain imaginary unit i.

The symplectic Majorana basis is given by

$$\begin{aligned}\gamma^0&=\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ \gamma^1&=\mathrm{i}'\sigma^3=\begin{pmatrix}\mathrm{i}'&0\\0&-\mathrm{i}'\end{pmatrix},\\ \gamma^2&=\mathrm{j}'\sigma^3=\begin{pmatrix}\mathrm{j}'&0\\0&-\mathrm{j}'\end{pmatrix},\\ \gamma^3&=\mathrm{k}'\sigma^3=\begin{pmatrix}\mathrm{k}'&0\\0&-\mathrm{k}'\end{pmatrix},\\ \gamma^4&=\tilde{\sigma}^2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.\end{aligned} \tag{B25}$$

When $d+1=6,7$, we can find gamma matrices satisfying $\mathrm{i}\gamma^{\mu}$ is quaternion-valued. We give the symplectic Majorana basis for $d+1=6$ and $d+1=7$.

For $d+1=6$, symplectic Majorana basis is given by

$$\begin{aligned}\mathrm{i}\gamma^0&=\sigma^0\otimes\tilde{\sigma}^2=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\\ \mathrm{i}\gamma^1&=\mathrm{i}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{i}'\\0&0&-\mathrm{i}'&0\\0&\mathrm{i}'&0&0\\\mathrm{i}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^2&=\mathrm{j}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{j}'\\0&0&-\mathrm{j}'&0\\0&\mathrm{j}'&0&0\\\mathrm{j}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^3&=\mathrm{k}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{k}'\\0&0&-\mathrm{k}'&0\\0&\mathrm{k}'&0&0\\\mathrm{k}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^4&=\sigma^1\otimes\sigma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^5&=\sigma^3\otimes\sigma^1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}.\end{aligned} \tag{B26}$$

For $d+1=7$, symplectic Majorana basis is given by

$$\begin{aligned}\mathrm{i}\gamma^0&=\sigma^0\otimes\tilde{\sigma}^2=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\\ \mathrm{i}\gamma^1&=\mathrm{i}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{i}'\\0&0&-\mathrm{i}'&0\\0&\mathrm{i}'&0&0\\\mathrm{i}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^2&=\mathrm{j}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{j}'\\0&0&-\mathrm{j}'&0\\0&\mathrm{j}'&0&0\\\mathrm{j}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^3&=\mathrm{k}'\tilde{\sigma}^2\otimes\sigma^1=\begin{pmatrix}0&0&0&-\mathrm{k}'\\0&0&-\mathrm{k}'&0\\0&\mathrm{k}'&0&0\\\mathrm{k}'&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^4&=\sigma^1\otimes\sigma^1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\\ \mathrm{i}\gamma^5&=\sigma^3\otimes\sigma^1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix},\end{aligned}$$

$$\mathrm{i}\gamma^6 = \sigma^0 \otimes \sigma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{B27}$$

They satisfy

$$(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1, \quad (\text{for } i = 1, 2, 3, \ldots), \tag{B28}$$

and they anticommute with each other.

3. Embedded in $\mathbb{C}(4)$

A more convenient choice is to embed the $\mathbb{H}(2)$ theory into $\mathbb{C}(4)$, which is the original Dirac theory. The constraint will rule out half of the DOF. Luckily, we can use only ψ_1 to describe the theory.

We can write the theory in the original two Dirac fermions ψ_1 and ψ_2, each with four complex components. Now the Lagrangian is given by

$$L = \frac{1}{2} \int d^4x \sum_{a=1}^{2} \bar{\psi}_a (\mathrm{i}\gamma^\mu \partial_\mu - m) \psi_a. \tag{B29}$$

The trivial charge conjugation condition is

$$\bar{\psi}' = \pm \psi'^{\mathsf{T}} C' \tag{B30}$$

written in ψ_1 and ψ_2, we need to use the map in Eq. (B13). To further simplify the relation, we assume $C' = \mathrm{i}' C''$ and C'' does not contain quaternion unit $\mathrm{i}', \mathrm{j}', \mathrm{k}'$:

$$(\bar{\psi}_1)_i = \pm ((\psi_1)^{\mathsf{T}} C)_i, \quad (\bar{\psi}_2)_i = \mp ((\psi_2)^{\mathsf{T}} C)_i, \tag{B31}$$

which gives

$$\bar{\psi}_a = \pm \sum_b \varepsilon_{ab} \psi_b^{\mathsf{T}} C. \tag{B32}$$

Therefore the Lagrangian can be reduced to

$$L = \frac{1}{2} \int d^4x \sum_{a,b} \psi_b^{\mathsf{T}} C \varepsilon_{ab} (\mathrm{i}\gamma^\mu \partial_\mu - m) \psi_a. \tag{B33}$$

The EOM is still the Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi_1 = (\mathrm{i}\gamma^\mu \partial_\mu - m)\psi_2 = 0 \tag{B34}$$

with the constraint

$$\bar{\psi}_a = \pm \sum_b \varepsilon_{ab} \psi_b^{\mathsf{T}} C \tag{B35}$$

or

$$\psi_a = \sum_b \varepsilon_{ab} C^{-1} (\gamma^0)^{\mathsf{T}} \psi_b^*. \tag{B36}$$

With some explicit basis, we can substitute Eq. (B36) into Eq. (B33) so that we can describe the theory with only ψ_1.

4. Embedded in $\mathbb{R}(8)$

A convenient choice is to embed the symplectic Majorana theory into $\mathbb{R}(8)$. Now we can write $\mathrm{i}\gamma^\mu$ as real matrices in $\mathbb{R}(8)$ space, and it suffices to set $M_{\mathcal{C}}$ to the identity. The Majorana fermion is embedded in $8_{\mathbb{R}}$, characterized by real numbers.

Now we give one explicit basis to show that this embedding is possible.

For $d + 1 = 5$:

$$\begin{aligned} &\mathrm{i}\gamma^0 = i\sigma^{211}, \quad \mathrm{i}\gamma^1 = \sigma^{100}, \quad \mathrm{i}\gamma^2 = \sigma^{212}, \\ &\mathrm{i}\gamma^3 = \sigma^{220}, \quad \mathrm{i}\gamma^4 = \sigma^{300}. \end{aligned} \tag{B37}$$

For $d + 1 = 6$:

$$\begin{aligned} &\mathrm{i}\gamma^0 = i\sigma^{1002}, \quad \mathrm{i}\gamma^1 = \sigma^{3100}, \quad \mathrm{i}\gamma^2 = \sigma^{3212}, \\ &\mathrm{i}\gamma^3 = \sigma^{3220}, \quad \mathrm{i}\gamma^4 = \sigma^{3232}, \quad \mathrm{i}\gamma^5 = \sigma^{3300}. \end{aligned} \tag{B38}$$

For $d + 1 = 7$:

$$\begin{aligned} &\mathrm{i}\gamma^0 = i\sigma^{2000}, \quad \mathrm{i}\gamma^1 = \sigma^{1000}, \quad \mathrm{i}\gamma^2 = \sigma^{3100}, \\ &\mathrm{i}\gamma^3 = \sigma^{3212}, \quad \mathrm{i}\gamma^4 = \sigma^{3220}, \quad \mathrm{i}\gamma^5 = \sigma^{3232}, \\ &\mathrm{i}\gamma^6 = \sigma^{3300}. \end{aligned} \tag{B39}$$

The Lagrangian is given by

$$L = \frac{1}{2} \int d^dx \bar{\psi} (\mathrm{i}\gamma^\mu \partial_\mu - m) \psi. \tag{B40}$$

The EOM is given by

$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0, \tag{B41}$$

without any constraint.

APPENDIX C: SYMPLECTIC MAJORANA FERMION IN 5 + 1*d* AND 6 + 1*d* CASES

The process of combining two Dirac fermions is similar. For $d + 1 = 6$.

In the representation space, the combination process is

$$\begin{aligned} 4_{\mathbb{C}} \oplus 4_{\mathbb{C}} \text{ and } 4_{\mathbb{C}} \oplus 4_{\mathbb{C}} &\to M_{4\times 2}(\mathbb{C}) \oplus M_{4\times 2}(\mathbb{C}) \to (M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C})) \oplus (M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C})) \\ &\to 2_{\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}} \oplus 2_{\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}} \to 2_{\mathbb{H}} \oplus 2_{\mathbb{H}}. \end{aligned} \tag{C1}$$

In the Clifford algebra sense, the combination process is

$$\mathbb{C}(4) \oplus \mathbb{C}(4) \text{ and } \mathbb{C}(4) \oplus \mathbb{C}(4) \to \mathbb{C}(4) \oplus \mathbb{C}(4) \to \mathbb{C}(4) \oplus \mathbb{C}(4) \to (\mathbb{H}(2) \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\mathbb{H}(2) \otimes_{\mathbb{R}} \mathbb{C}) \to \mathbb{H}(2) \oplus \mathbb{H}(2). \tag{C2}$$

For $d + 1 = 7$.

In the representation space, the combination process is

$$8_{\mathbb{C}} \text{ and } 8_{\mathbb{C}} \to M_{8\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C}) \to 4_{\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C}} \to 4_{\mathbb{H}}. \tag{C3}$$

In the Clifford algebra sense, the combination process is

$$\mathbb{C}(8) \text{ and } \mathbb{C}(8) \to \mathbb{C}(8) \to \mathbb{C}(8) \to \mathbb{H}(4) \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{H}(4). \tag{C4}$$

APPENDIX D: LORENTZ SYMMETRY IN MINKOWSKI SPACETIME

In this section, we will check that the Lorentz boost and rotation in Eq. (25) indeed keep the Lagrangian in Eq. (24) (and the corresponding action $S = \int dx^0 L$) invariant.

We first check the Lorentz boost: take the ζ_1 boost, for example, the differential operators transform as

$$\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix} \to \exp\begin{pmatrix}0 & \zeta_1\\ \zeta_1 & 0\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix} = \begin{pmatrix}\cosh\zeta_1 & \sinh\zeta_1\\ \sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix}, \tag{D1}$$

and the Clifford algebra transforms as

$$\begin{aligned}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} &\to \left(\mathrm{e}^{\frac{1}{2}\zeta_1\alpha_1}\right)^{\mathsf{T}}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\mathrm{e}^{\frac{1}{2}\zeta_1\alpha_1} = \mathrm{e}^{\zeta_1\alpha_1}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &= (\cosh\zeta_1 + \sinh\zeta_1\alpha_1)\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &= \begin{pmatrix}\cosh\zeta_1 & \sinh\zeta_1\\ \sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix},\end{aligned} \tag{D2}$$

therefore,

$$\begin{aligned}\mathrm{i}\,\partial_0 - \mathrm{i}\,\partial_1\alpha_1 &= \mathrm{i}\,(\partial_0 \quad \partial_1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &\to \mathrm{i}\,(\partial_0 \quad \partial_1)\begin{pmatrix}\cosh\zeta_1 & \sinh\zeta_1\\ \sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cosh\zeta_1 & \sinh\zeta_1\\ \sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &= \mathrm{i}\,(\partial_0 \quad \partial_1)\begin{pmatrix}\cosh\zeta_1 & \sinh\zeta_1\\ \sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}\cosh\zeta_1 & -\sinh\zeta_1\\ -\sinh\zeta_1 & \cosh\zeta_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &= \mathrm{i}\,(\partial_0 \quad \partial_1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\\ &= \mathrm{i}\,\partial_0 - \mathrm{i}\,\partial_1\alpha_1,\end{aligned} \tag{D3}$$

which is indeed invariant.

Then we check spatial rotation: take the θ_{12} rotation, for example, the differential operators transform as

$$\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} \to \exp\begin{pmatrix}0 & -\theta_{12}\\ \theta_{12} & 0\end{pmatrix}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} = \begin{pmatrix}\cos\theta_{12} & -\sin\theta_{12}\\ \sin\theta_{12} & \cos\theta_{12}\end{pmatrix}\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix}, \tag{D4}$$

and the Clifford algebra transforms as

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} &\to \left(\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_{12}\Sigma_{12}}\right)^{\mathsf{T}}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}\theta_{12}\Sigma_{12}}\\ &= \mathrm{e}^{-\mathrm{i}\theta_{12}\Sigma_{12}}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\\ &= (\cos\theta_{12} - \mathrm{i}\,\sin\theta_{12}\Sigma_{12})\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\\ &= (\cos\theta_{12} + \sin\theta_{12}\alpha_1\alpha_2)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\\ &= \begin{pmatrix}\cos\theta_{12} & -\sin\theta_{12}\\ \sin\theta_{12} & \cos\theta_{12}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix},\end{aligned} \tag{D5}$$

therefore,

$$\begin{aligned}-\mathrm{i}\,\partial_1\alpha_1 - \mathrm{i}\,\partial_2\alpha_2 &= -\mathrm{i}\,(\partial_1 \quad \partial_2)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\\ &\to -\mathrm{i}\,(\partial_1 \quad \partial_2)\begin{pmatrix}\cos\theta_{12} & \sin\theta_{12}\\ -\sin\theta_{12} & \cos\theta_{12}\end{pmatrix}\begin{pmatrix}\cos\theta_{12} & -\sin\theta_{12}\\ \sin\theta_{12} & \cos\theta_{12}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\end{aligned}$$

$$\begin{aligned} &= -\mathrm{i}(\partial_1 \quad \partial_2)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \\ &= -\mathrm{i}\,\partial_1\alpha_1 - \mathrm{i}\,\partial_2\alpha_2, \end{aligned} \tag{D6}$$

which is indeed invariant.

APPENDIX E: LORENTZ SYMMETRY IN EUCLIDEAN SPACETIME

In this section, we will check that the Lorentz boost and rotation in Eq. (27) indeed keep the Lagrangian in Eq. (26) (and the corresponding action $S = \int dx^0 L$) invariant.

We first check the Lorentz boost: take the ζ_1 boost, for example, the differential operators transform as

Check Lorentz boost: take the ζ_1 boost, for example, the differential operators transform as

$$\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix} \to \exp\begin{pmatrix}0 & -\zeta_1\\ \zeta_1 & 0\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix} = \begin{pmatrix}\cos\zeta_1 & -\sin\zeta_1\\ \sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}\partial_0\\ \partial_1\end{pmatrix}, \tag{E1}$$

and the Clifford algebra transforms as

$$\begin{aligned} \begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} &\to \left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta_1\alpha_1}\right)^{\mathsf{T}}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta_1\alpha_1} \\ &= \mathrm{e}^{-\mathrm{i}\zeta_1\alpha_1}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &= (\cos\zeta_1 - \mathrm{i}\sin\zeta_1\alpha_1)\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &= \begin{pmatrix}\cos\zeta_1 & -\mathrm{i}\sin\zeta_1\\ -\mathrm{i}\sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix}, \end{aligned} \tag{E2}$$

therefore,

$$\begin{aligned} \partial_0 + \mathrm{i}\,\partial_1\alpha_1 &= (\partial_0 \quad \partial_1)\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &\to (\partial_0 \quad \partial_1)\begin{pmatrix}\cos\zeta_1 & \sin\zeta_1\\ -\sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}\begin{pmatrix}\cos\zeta_1 & -\mathrm{i}\sin\zeta_1\\ -\mathrm{i}\sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &= (\partial_0 \quad \partial_1)\begin{pmatrix}\cos\zeta_1 & \sin\zeta_1\\ -\sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}\cos\zeta_1 & -\sin\zeta_1\\ \sin\zeta_1 & \cos\zeta_1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &= (\partial_0 \quad \partial_1)\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}\end{pmatrix}\begin{pmatrix}\mathbf{1}\\ \alpha_1\end{pmatrix} \\ &= \partial_0 + \mathrm{i}\,\partial_1\alpha_1. \end{aligned} \tag{E3}$$

which is indeed invariant.

The rotation in Euclidean spacetime is the same as in Minkowski spacetime, therefore, the verification of the invariance of the Lagrangian in Eq. (26) rotation is parallel to that in Appendix D.

APPENDIX F: PRESENTATION OF THE INVARIANT GROUP FOR DIRAC FERMION

For $d = 0$, the invariant group is given by the presentation:

$$\mathcal{C}^2 = \mathcal{T}^2 = (\mathcal{C}\mathcal{T})^2 = 1, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}. \tag{F1}$$

For $d = 1$, the invariant group is given by the presentation:

$$\begin{aligned} &\mathcal{C}^2 = \mathcal{R}_1^2 = \mathcal{T}^2 = (\mathcal{R}_1\mathcal{T})^2 = 1, \quad (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \\ &\mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad \mathcal{C}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(\theta)\mathcal{T}. \end{aligned} \tag{F2}$$

For $d = 2$, the invariant group is given by the presentation:

$$\begin{aligned} &\mathcal{C}^2 = \mathcal{R}_1^2 = (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{R}_1\mathcal{T})^2 = 1, \quad \mathcal{T}^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F, \\ &\mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}. \end{aligned} \tag{F3}$$

For $d = 3$, the invariant group is given by the presentation:

$$\begin{aligned} &\mathcal{C}^2 = \mathcal{R}_1^2 = (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{R}_1\mathcal{T})^2 = 1, \quad \mathcal{T}^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \\ &\mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad \mathcal{C}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{T}. \end{aligned} \tag{F4}$$

For $d = 4$, the invariant group is given by the presentation:

$$(\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{C}\mathcal{T})^2 = 1, \quad \mathcal{C}^2 = \mathcal{R}_1^2 = \mathcal{T}^2 = (\mathcal{R}_1\mathcal{T})^2 = (-)^F,$$
$$\mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}. \tag{F5}$$

For $d = 5$, the invariant group is given by the presentation:

$$\mathcal{C}^2 = \mathcal{R}_1^2 = \mathcal{T}^2 = (\mathcal{R}_1\mathcal{T})^2 = 1, \quad (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1,$$
$$\mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad \mathcal{C}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(\theta)\mathcal{T}. \tag{F6}$$

For $d = 6$, the invariant group is given by the presentation:

$$\mathcal{R}_1^2 = \mathcal{T}^2 = 1, \quad \mathcal{C}^2 = (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{R}_1\mathcal{T})^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F,$$
$$\mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}. \tag{F7}$$

For $d = 7$, the invariant group is given by the presentation:

$$\mathcal{R}_1^2 = \mathcal{T}^2 = 1, \quad \mathcal{C}^2 = (\mathcal{C}\mathcal{R}_1)^2 = (\mathcal{R}_1\mathcal{T})^2 = (\mathcal{C}\mathcal{T})^2 = (-)^F, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1,$$
$$\mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad \mathcal{C}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^\chi(\theta) = \mathcal{U}^\chi(-\theta)\mathcal{T}. \tag{F8}$$

For $d = 8$, the invariant group is given by the presentation:

$$\mathcal{C}^2 = \mathcal{T}^2 = (\mathcal{R}_1\mathcal{T})^2 = (\mathcal{C}\mathcal{T})^2 = 1, \quad \mathcal{R}_1^2 = (\mathcal{C}\mathcal{R}_1)^2 = (-)^F,$$
$$\mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad \mathcal{R}_1\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)\mathcal{R}_1, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}. \tag{F9}$$

APPENDIX G: PRESENTATION OF THE INVARIANT GROUP FOR WEYL FERMION

For $d = 1$, the invariant group is given by the presentation:

$$\mathcal{C}^2 = (\mathcal{R}_1\mathcal{T})^2 = (\mathcal{C}\mathcal{R}_1\mathcal{T})^2 = 1, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad (\mathcal{C}\mathcal{R}_1\mathcal{T})\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)(\mathcal{C}\mathcal{R}_1\mathcal{T}). \tag{G1}$$

For $d = 3$, the invariant group is given by the presentation:

$$(\mathcal{C}\mathcal{R}_1\mathcal{T})^2 = 1, \quad \mathcal{T}^2 = (\mathcal{C}\mathcal{R}_1)^2 = (-)^F, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad (\mathcal{C}\mathcal{R}_1\mathcal{T})\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)(\mathcal{C}\mathcal{R}_1\mathcal{T}). \tag{G2}$$

For $d = 5$, the invariant group is given by the presentation:

$$(\mathcal{R}_1\mathcal{T})^2 = (\mathcal{C}\mathcal{R}_1\mathcal{T})^2 = 1, \quad \mathcal{C}^2 = (-)^F, \quad \mathcal{C}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{C}, \quad (\mathcal{C}\mathcal{R}_1\mathcal{T})\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)(\mathcal{C}\mathcal{R}_1\mathcal{T}). \tag{G3}$$

For $d = 7$, the invariant group is given by the presentation:

$$\mathcal{T}^2 = (\mathcal{C}\mathcal{R}_1\mathcal{T})^2 = 1, \quad (\mathcal{C}\mathcal{R}_1)^2 = (-)^F, \quad \mathcal{T}\mathcal{U}^F(\theta) = \mathcal{U}^F(-\theta)\mathcal{T}, \quad (\mathcal{C}\mathcal{R}_1\mathcal{T})\mathcal{U}^F(\theta) = \mathcal{U}^F(\theta)(\mathcal{C}\mathcal{R}_1\mathcal{T}). \tag{G4}$$

APPENDIX H: SYMMETRY REDUCTION FOR MAJORANA FERMION

To prove that the symmetry of spatial dimension d can be reduced from the symmetry of spatial dimension $d + 1$ once a mass term can be found, we give the explicit symmetry reduction in $d = 1, 2, 3, 4, 5, 6$ cases ($d = 0, 7$ mod 8 has no mass terms) in the form:

$$G_{M,d} \sim G'_{M,d} \xrightarrow{m_i} G'_{M/MW,d-1} \sim G_{M/MW,d-1}, \tag{H1}$$

indicating the domain wall reduction of symmetry group G from a d dimensional Majorana fermion to a $d - 1$ dimensional Majorana/Majorana-Weyl fermion on the m_i mass domain wall. To do this, we first modify these symmetries by internal symmetries to maximally preserve the group G' under mass domain wall m_i. Following the rules in Eqs. (66) and (67), we obtain the reduced symmetries G', and modify the symmetry operators to conventional G. In the following content, we denote the generators of $\mathbb{Z}_2^F$, $\mathbb{Z}_2^\chi$,Lie algebra,$\mathbb{Z}_2^{\mathcal{R}_i}$, and $\mathbb{Z}_2^{\mathcal{T}}$ as $(-)^F$, $(-)^\chi$, $\mathcal{J}_i$, $\mathcal{R}_i$, and $\mathcal{T}$, respectively.

$$G_{M,d=1}((-)^F, (-)^\chi, \mathcal{R}, \mathcal{T}) \sim G_{M,d=1}((-)^F, \quad (-)^\chi, \mathcal{R}, \mathcal{T}' = (-)^\chi\mathcal{T})$$
$$\xrightarrow{m_1} G_{M,d=0}((-)^F, 0, 1, \mathcal{T}) \sim G_{M,d=0}((-)^F, \mathcal{T}), \tag{H2}$$

$$G_{M,d=2}((-)^F, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}) \xrightarrow{m_1} G_{MW,d=1}((-)^F, \mathcal{R}\mathcal{T}, 1, 1) \sim G_{MW,d=1}((-)^F, \mathcal{R}\mathcal{T}), \tag{H3}$$

$$G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}) \sim G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}'_3 = \mathcal{J}\mathcal{R}_3, \mathcal{T})$$
$$\xrightarrow{m_1} G_{M,d=2}((-)^F, 0, \mathcal{R}_1, \mathcal{R}_2, 1, \mathcal{T}) \sim G_{M,d=2}((-)^F, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}),$$

$$
\begin{aligned}
&G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}) \sim G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}'_1 = \mathcal{J}\mathcal{R}_1, \mathcal{R}'_2 = \mathcal{J}\mathcal{R}_2, \mathcal{R}_3, \mathcal{T}' = \mathcal{J}\mathcal{T}) \\
&\quad \xrightarrow{m_2} G_{M,d=2}((-)^F, 0, \mathcal{R}_1, \mathcal{R}_2, 1, \mathcal{T}) \sim G_{M,d=2}((-)^F, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}),
\end{aligned} \tag{H4}
$$

$$
\begin{aligned}
&G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}'_4 = \mathcal{J}_1\mathcal{R}_4, \mathcal{T}) \\
&\quad \xrightarrow{m_1} G_{M,d=3}((-)^F, 0, 0, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{J}, \mathcal{T}) \sim G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}), \\
&G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}'_4 = \mathcal{J}_1\mathcal{R}_4, \mathcal{T}) \\
&\quad \xrightarrow{m_2} G_{M,d=3}((-)^F, 0, \mathcal{J}, 0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{J}, \mathcal{T}) \sim G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}), \\
&G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{T}) \\
&\quad \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}'_1 = \mathcal{J}_2\mathcal{R}_1, \mathcal{R}'_2 = \mathcal{J}_2\mathcal{R}_2, \mathcal{R}'_3 = \mathcal{J}_2\mathcal{R}_3, \mathcal{R}_4, \mathcal{T}' = \mathcal{J}_2\mathcal{T}) \\
&\quad \xrightarrow{m_3} G_{M,d=3}((-)^F, \mathcal{J}, 0, 0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, 1, \mathcal{T}) \sim G_{M,d=3}((-)^F, \mathcal{J}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}),
\end{aligned} \tag{H5}
$$

$$
\begin{aligned}
&G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\
&\quad \sim G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}'_2 = (-)^\chi \mathcal{J}_2, \mathcal{J}'_3 = (-)^\chi \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}'_5 = (-)^\chi \mathcal{R}_5, \mathcal{T}) \\
&\quad \xrightarrow{m_1} G_{M,d=4}((-)^F, 0, \mathcal{J}_3, \mathcal{J}_2, \mathcal{J}_1, \mathcal{J}_2\mathcal{R}_1, \dots, \mathcal{J}_2\mathcal{R}_4, \mathcal{J}_3, \mathcal{J}_2\mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}), \\
&G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\
&\quad \sim G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}'_1 = (-)^\chi \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}'_3 = (-)^\chi \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}'_5 = (-)^\chi \mathcal{R}_5, \mathcal{T}) \\
&\quad \xrightarrow{m_2} G_{M,d=4}((-)^F, 0, \mathcal{J}_3, \mathcal{J}_2, \mathcal{J}_1, \mathcal{J}_3\mathcal{R}_1, \dots, \mathcal{J}_3\mathcal{R}_4, \mathcal{J}_2, \mathcal{J}_3\mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}), \\
&G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\
&\quad \sim G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}'_1 = (-)^\chi \mathcal{J}_1, \mathcal{J}'_2 = (-)^\chi \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}'_5 = (-)^\chi \mathcal{R}_5, \mathcal{T}) \\
&\quad \xrightarrow{m_3} G_{M,d=4}((-)^F, 0, \mathcal{J}_3, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\mathcal{R}_1, \dots, \mathcal{J}_3\mathcal{R}_4, \mathcal{J}_2, \mathcal{J}_3\mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}), \\
&G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\
&\quad \sim G_{M,d=5}((-)^F, (-)^\chi, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}'_1 = (-)^\chi \mathcal{R}_1, \dots, \mathcal{R}'_4 = (-)^\chi \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}' = (-)^\chi \mathcal{T}) \\
&\quad \xrightarrow{m_4} G_{M,d=4}((-)^F, 0, \mathcal{J}_3, \mathcal{J}_2, \mathcal{J}_1, \mathcal{J}_1\mathcal{R}_1, \dots, \mathcal{J}_1\mathcal{R}_4, 1, \mathcal{J}_1\mathcal{T}) \sim G_{M,d=4}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}),
\end{aligned} \tag{H6}
$$

$$
\begin{aligned}
&G_{M,d=6}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1, \dots, \mathcal{R}_5, \mathcal{R}_6, \mathcal{T}) \\
&\quad \xrightarrow{m_1} G_{MW,d=5}((-)^F, \mathcal{J}_3, \mathcal{J}_2, \mathcal{J}_1, \mathcal{R}_1\mathcal{T}, \dots, \mathcal{R}_5\mathcal{T}, 1, 1) \sim G_{MW,d=5}((-)^F, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{R}_1\mathcal{T}, \dots, \mathcal{R}_5\mathcal{T}).
\end{aligned} \tag{H7}
$$

For higher spatial dimensions, the same procedure is applied to prove the symmetry reduction.

APPENDIX I: SYMMETRY REDUCTION FOR DIRAC FERMION

To prove that the symmetry of spatial dimension d can be reduced from the symmetry of spatial dimension $d+1$, we give the explicit symmetry reduction in $d = 1, \dots, 8$ cases in the form:

$$
G_{D,d} \sim G'_{D,d} \xrightarrow{m_i} G'_{D/W,d-1} \sim G_{D/W,d-1}, \tag{I1}
$$

indicating the domain wall reduction of symmetry group G from a d dimensional Dirac fermion to a $d-1$ dimensional Dirac/Weyl fermion on the m_i mass domain wall. To do this, we first modify these symmetries by internal symmetries to maximally preserve the group G' under mass domain wall m_i. Following the rules in Eqs. (117) and (118), we obtain the reduced symmetries G', and modify the symmetry operators to conventional G. In the following content, we denote the generators of $\mathbb{Z}_2^F$, $\mathbb{Z}_2^\chi$, $\mathbb{Z}_2^\mathcal{C}$, $\mathbb{Z}_2^{\mathcal{R}_i}$, and $\mathbb{Z}_2^\mathcal{T}$ as $(-)^F$, $(-)^\chi$, $\mathcal{C}$, $\mathcal{R}_i$, and $\mathcal{T}$, respectively.

$$
\begin{aligned}
&G_{D,d=1}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}, \mathcal{T}) \sim G_{D,d=1}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}' = (-)^\chi \mathcal{R}, \mathcal{T}) \\
&\quad \xrightarrow{m_1} G_{D,d=0}((-)^F, 0, \mathcal{C}, (-)^F, \mathcal{T}) \sim G_{D,d=0}((-)^F, \mathcal{C}, \mathcal{T}), \\
&G_{D,d=1}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}, \mathcal{T}) \sim G_{D,d=1}((-)^F, (-)^\chi, \mathcal{C}' = (-)^\chi \mathcal{C}, \mathcal{R}, \mathcal{T}) \\
&\quad \xrightarrow{m_2} G_{D,d=0}((-)^F, 0, (-)^F \mathcal{C}, 1, \mathcal{T}) \sim G_{D,d=0}((-)^F, \mathcal{C}, \mathcal{T}),
\end{aligned} \tag{I2}
$$

$$\begin{aligned} &G_{D,d=2}((-)^F, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}) \\ &\quad \xrightarrow{m_1} G_{DW,d=1}((-)^F, \mathcal{C}, \mathcal{CRT}, 1, \mathcal{C}) \sim G_{DW,d=1}((-)^F, \mathcal{C}, \mathcal{CRT}), \end{aligned} \tag{I3}$$

$$\begin{aligned} &G_{D,d=3}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}) \\ &\sim G_{D,d=3}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1' = (-)^\chi \mathcal{R}_1, \mathcal{R}_2' = (-)^\chi \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}' = (-)^\chi \mathcal{T}) \\ &\xrightarrow{m_1} G_{D,d=2}((-)^F, 0, \mathcal{C}, (-)^F \mathcal{R}_1, (-)^F \mathcal{R}_2, 1, (-)^F \mathcal{T}) \sim G_{D,d=2}((-)^F, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}), \\ &G_{D,d=3}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{T}) \sim G_{D,d=3}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3' = (-)^\chi \mathcal{R}_3, \mathcal{T}) \\ &\xrightarrow{m_2} G_{D,d=2}((-)^F, 0, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, (-)^F, \mathcal{T}) \sim G_{D,d=2}((-)^F, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{T}), \end{aligned} \tag{I4}$$

$$\begin{aligned} &G_{D,d=4}((-)^F, \mathcal{C}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{T}) \\ &\xrightarrow{m_1} G_{DW,d=3}((-)^F, \mathcal{T}, \mathcal{CR}_1\mathcal{T}, \mathcal{CR}_2\mathcal{T}, \mathcal{CR}_3\mathcal{T}, 1, \mathcal{T}) \sim G_{DW,d=3}((-)^F, \mathcal{T}, \mathcal{CR}_1\mathcal{T}, \mathcal{CR}_2\mathcal{T}, \mathcal{CR}_3\mathcal{T}), \end{aligned} \tag{I5}$$

$$\begin{aligned} &G_{D,d=5}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\ &\sim G_{D,d=5}((-)^F, (-)^\chi, \mathcal{C}' = (-)^\chi \mathcal{C}, \mathcal{R}_1' = (-)^\chi \mathcal{R}_1, \dots, \mathcal{R}_4' = (-)^\chi \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\ &\xrightarrow{m_1} G_{D,d=4}((-)^F, 0, (-)^F \mathcal{C}, (-)^F \mathcal{R}_1, \dots, (-)^F \mathcal{R}_4, 1, \mathcal{T}) \sim G_{D,d=4}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}), \\ &G_{D,d=5}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\ &\sim G_{D,d=5}((-)^F, (-)^\chi, \mathcal{C}' = (-)^\chi \mathcal{C}, \mathcal{R}_1' = (-)^\chi \mathcal{R}_1, \dots, \mathcal{R}_4' = (-)^\chi \mathcal{R}_4, \mathcal{R}_5, \mathcal{T}) \\ &\xrightarrow{m_2} G_{D,d=4}((-)^F, 0, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_4, (-)^F, \mathcal{T}) \sim G_{D,d=4}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_4, \mathcal{T}), \end{aligned} \tag{I6}$$

$$\begin{aligned} &G_{D,d=6}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_5, \mathcal{R}_6, \mathcal{T}) \\ &\quad \xrightarrow{m_1} G_{DW,d=5}((-)^F, \mathcal{C}, \mathcal{CR}_1\mathcal{T}, \dots, \mathcal{CR}_5\mathcal{T}, 1, \mathcal{C}) \sim G_{DW,d=5}((-)^F, \mathcal{C}, \mathcal{CR}_1\mathcal{T}, \dots, \mathcal{CR}_5\mathcal{T}), \end{aligned} \tag{I7}$$

$$\begin{aligned} &G_{D,d=7}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, \mathcal{R}_7, \mathcal{T}) \\ &\sim G_{D,d=7}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1' = (-)^\chi \mathcal{R}_1, \dots, \mathcal{R}_6' = (-)^\chi \mathcal{R}_6, \mathcal{R}_7, \mathcal{T}' = (-)^\chi \mathcal{T}) \\ &\xrightarrow{m_1} G_{D,d=6}((-)^F, 0, \mathcal{C}, (-)^F \mathcal{R}_1, \dots, (-)^F \mathcal{R}_6, 1, (-)^F \mathcal{T}) \sim G_{D,d=6}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, \mathcal{T}), \\ &G_{D,d=7}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, \mathcal{R}_7, \mathcal{T}) \\ &\sim G_{D,d=7}((-)^F, (-)^\chi, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, \mathcal{R}_7' = (-)^\chi \mathcal{R}_7, \mathcal{T}) \\ &\xrightarrow{m_2} G_{D,d=6}((-)^F, 0, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, (-)^F, \mathcal{T}) \sim G_{D,d=6}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_6, \mathcal{T}), \end{aligned} \tag{I8}$$

$$\begin{aligned} &G_{D,d=8}((-)^F, \mathcal{C}, \mathcal{R}_1, \dots, \mathcal{R}_7, \mathcal{R}_8, \mathcal{T}) \\ &\quad \xrightarrow{m_1} G_{DW,d=7}((-)^F, \mathcal{T}, \mathcal{CR}_1\mathcal{T}, \dots, \mathcal{CR}_7\mathcal{T}, 1, \mathcal{T}) \sim G_{DW,d=7}((-)^F, \mathcal{T}, \mathcal{CR}_1\mathcal{T}, \dots, \mathcal{CR}_7\mathcal{T}), \end{aligned} \tag{I9}$$

For higher spatial dimensions, the same procedure is applied to prove the symmetry reduction.

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