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Irreducible Actions of the Group GL(∞) on L^2-spaces on 3 Infinite Rows *

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Abstract. Let G be an inductive limit group of general finite-dimensional linear groups which is acting from the right on the space X of three infinite rows equipped with a Gaussian measure μ. The involved action "respects" the measure; that is the right action is admissible. The unitary representation of the group G on the space $L^2(X,\mu)$ appears naturally. We give an irreducibility criterion in terms of the action of the group $\mathrm{GL}(3,\mathbb{R})$ from the left on X. Namely, we prove that this representation is irreducible if and only if all non trivial left actions are not admissible. This is also a manifestation of a phenomenon predicted by the *Ismagilov conjecture*, see below. To prove the irreducibility we show that the von Neumann algebra generated by the representation contains certain abelian subalgebras. This is a consequence of the orthogonality and can be seen as a kind of ergodic theorem (comparable to the Law of Large Numbers, but more subtle). More precisely, the elements of the corresponding commutative subalgebras can be approximated (in the *strong resolvent sense*) by combinations of generators of one-parameter groups. This approximation being optimal at every finite step, represents the best possible outcome under the given conditions. Its construction relies mainly on the properties of the generalized characteristic polynomial, an explicit expression for the minimum of the quadratic form on a hyperplane, and a theorem regarding the height of an infinite parallelotope.

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Key Words: Infinite-dimensional groups, irreducible unitary representation, Ismagilov conjecture, quasi-invariant measure, ergodic measure, generalized characteristic polynomial, parallelotope height.

Contents

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* Dedicated to all fearless Ukrainians defending not only their country, but the whole civilization against Putin's ruscism.

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1. Introduction

1.1. Representations of locally compact groups

1.1.1. What is representation theory about?

The main problem in the representation theory (RT) of a topological group G is to find its *unitary dual* $\widehat{G}$, i.e., *the set of all irreducible unitary representations* of the group G *up to the equivalence relation* and decompose any representation into a direct sum or direct integral of irreducible ones. Almost all constructions in RT for *locally compact groups* such as *regular, quasi-regular and induced representations* are based on the existence of the *Haar measure* on a group G. These constructions allow us to find the unitary dual $\widehat{G}$ for almost all locally compact groups G, except for, e.g., the group $\mathrm{SO}(p,q)$.

1.1.2. Compact groups and the regular representation

The existence of the Haar measure was proved by A. Haar [10]. A. Weil [45] proved a converse, namely that a group with a quasi-invariant measure that acts faithfully on the L^2-space is locally compact with respect to the strong operator topology from $U(L^2(G))$. The *right ρ (resp. the left λ) regular* representation $\rho,\lambda : G\to U(H)$ of the group G is defined on the Hilbert space $H=L^2(G,h)$ by

$$(\rho_tf)(x)=f(xt),\ (\lambda_sf)(x)=(dh(s^{-1}x)/dh(x))^{1/2}f(s^{-1}x),\ t,s,x\in G,\ f\in H. \quad (1.1)$$

Since $[\rho_t,\lambda_s]=0$ for all $t,s\in G$, both representations are reducible. When G is compact, the decomposition of the right regular representation contains *all* the irreducible representations:

$$\rho=\bigoplus_{\kappa\in\widehat{G}} c_\kappa\rho_\kappa. \quad (1.2)$$

1.1.3. The Dixmier commutation theorem

Lemma 1.1. (The Dixmier commutation theorem, [8]) *Let* $\mathfrak{A}_G^\rho=(\rho_t|t\in G)''$, *and* $\mathfrak{A}_G^\lambda=(\lambda_s|s\in G)''$ *be the* von Neumann algebras *generated by the right and the left regular representations of a locally compact group* G. *Then*

$$(\mathfrak{A}_G^\rho)'=\mathfrak{A}_G^\lambda. \quad (1.3)$$

This lemma is a *cornerstone* of our study of representations of infinite-dimensional groups. The *Ismagilov conjecture*, see below, is a far-reaching generalization of this statement.

1.1.4. Koopman's representation

In order to construct a unitary representation of a topological group G (locally compact or infinite-dimensional) we use a *G-space* (X,μ) with a "good" measure μ. To be more precise, let $\alpha: G\to \mathrm{Aut}(X)$ be a measurable action of a group G on a measurable space (X,μ) with G-quasi-invariant measure μ, i.e, $\mu^{\alpha_t}\sim\mu$ ($\sim$ means equivalent) for all $t\in G$, where $\mathrm{Aut}(X)$ is the group of all measurable automorphisms of X. We use the notation $\mu^f(\Delta)=\mu\big(f^{-1}(\Delta)\big)$ for $f:X\to X$, where Δ is a measurable set in X. To this data we associate the representation $\pi^{\alpha,\mu,X}: G\to U\big(L^2(X,\mu)\big)$ of the group G given by the formula:

$$(\pi_t^{\alpha,\mu,X}f)(x)=\big(d\mu\big(\alpha_{t^{-1}}(x)\big)/d\mu(x)\big)^{1/2}f\big(\alpha_{t^{-1}}(x)\big),\quad f\in L^2(X,\mu). \quad (1.4)$$

In the case of an invariant measure this representation is called *Koopman's representation.* We keep the same name for representation (1.4).

1.1.5. Quasi-regular representations

When the group is locally compact but not compact, the regular representation is not sufficient to find all irreducible representations. One should generalize the regular representation, for example, consider a quasi-regular or an induced representations, as for the group $\mathrm{SL}(2,\mathbb{R})$, see [32]. A quasi-regular representations of the group G is a particular case of Koopman's representation (1.4) with $X = H\setminus G$ the set of *right cosets* (or $X = G/H$, the set of *left cosets*), where H is a closed subgroup of G and μ is a G-quasi-invariant measure on X. In case $X = H\setminus G$ the group G acts on X from the right, in case $X = G/H$ it acts from the left.

1.1.6. Induced representation

To construct an *induced representation* of the group G we should fix a closed subgroup H of the group G and a unitary representation $S : H \to U(V)$ of a group H. The induced representation $\mathrm{Ind}_H^G S$ of the group G is defined on the space $L^2(X,V,\mu)$, where $X = H\setminus G$, for details, see [33].

1.2. A brief history of the representations of infinite-dimensional groups $G = \varinjlim_n G_n$

The representation theory of infinite-dimensional groups is a very broad area. We mention here only some results connected with unitary representations of inductive limits of classical groups, and an interesting connection with *random matrices.* Using his *orbit method* developed in [17], A. A. Kirillov described in [18] all unitary irreducible representations of the group $U_\infty(H)$, *completion in the strong operator topology of the group* $U(\infty) = \varinjlim_n U(n)$. The group $U_\infty(H)$ consists of all unitary operators of the form $1 + a$, where a is compact.

This approach was generalized by G. I. Ol'shanskii for the inductive limits of other classical groups $K(\infty){=}\varinjlim_n K(n)$, where K is U, O or Sp. In [38] the complete classification of the so-called "tame" representations of the group $K(\infty)$ was obtained, see also [34]. N. I. Nessonow [36] proved that a previously known list of indecomposable spherical functions of the group $\mathrm{GL}(\infty)$ that are bilaterally invariant with respect to the unitary subgroup is complete. A. I. Bufetov [6] showed that a Borel measure on the space of infinite Hermitian matrices, that is invariant under the action of the infinite unitary group under additional conditions, is finite.

The aim of the book [41] by S. Stratila and D. Voiculescu is to study the factor representations of the group $U(\infty)$, see more details in the review by Ola Bratteli (MR0458188).

In [35], K. H. Neeb describes the recent progress in the classification of bounded and semibounded representations of infinite-dimensional Lie groups. He starts with a discussion of the semiboundedness condition and how the new concept of a smoothing operator can be used to construct C^*-algebras (so called host algebras) whose representations are in one-to-one correspondence with certain semibounded representations of an infinite-dimensional Lie group G. This makes the full power of C^*-theory available in this context. Then he discusses the classification of bounded representations of several types of unitary groups on Hilbert spaces and of gauge groups. After explaining the method of holomorphic induction as a means to pass from bounded representations to semibounded ones, he describes the classification of semibounded representations for hermitian Lie groups of operators, loop groups (with infinite-dimensional targets), the Virasoro group and certain infinite-dimensional oscillator groups. The article [39] by G. I. Ol'shanskii deals with the representation theory of the automorphism groups of infinite-dimensional Riemannian symmetric spaces. The book [13] by R. S. Ismagilov is devoted to the representations of certain classes of infinite-dimensional Lie groups: current group, diffeomorphism group and some of their semidirect products.

Let $S_\infty = \cup_{n\geq 1} S_n$ be the group of finite permutations of natural numbers. All indecomposable central positive definite functions on S_∞, which are related to factor representations of II_1, were given by E. Thoma [42]. Later A. M. Vershik and S. V. Kerov obtained the same result by a different method in [43] and gave a realization of the representations of type II_1 in [44]. In [15] the generalized regular representations $\{T_z : z \in \mathbb{C}\}$ of the group $S_\infty \times S_\infty$ were studied. These representations are deformations of the biregular representation of S_∞ in $l^2(S_\infty)$. A two-parameter family of the generalized regular representations $T_{z,z'}$ of the group S_∞ was considered also in [15]. In [5] the corresponding spectral measure $P_{z,z'}$ was investigated. The correlation functions are of a determinantal form similar to those studied in *random matrix theory.*

Borodin [4] studied the asymptotics of the *Plancherel measures* M_n for the symmetric groups S_n. He showed that M_n converges to the delta measure supported on a certain subset Ω of $\mathbb{R}^2$ closely connected to *Wigner's semicircle law* for the distribution of eigenvalues of random matrices thus giving a positive answer to the conjecture of J. Baik, P. A. Deift and K. Johansson [2].

1.3. Our approach to representations of infinite-dimensional groups

We will consider infinite-dimensional non-locally compact groups. If the group is not locally compact, there is no Haar measure on it, see [45]. The main idea of [26] is to construct an *analogue* of the regular, quasi-regular and induced representations and study their irreducibility. Our approach to representation theory is completely different and is based on a *variety* of non-equivalent G-quasi-invariant *measures* μ on a G-space (X, μ). To study irreducibility we use the Ismagilov conjecture, a generalization of Dixmier's commutation theorem. This approach allows us to prove that *nonequivalent measures* correspond to *nonequivalent representations*! See Remark 1.3 below. Thus, the *nonequivalent measures* become *essential ingredients* in the description of the *dual* $\widehat{G}$ for infinite-dimensional groups G.

1.3.1. The Ismagilov conjecture

In order to construct an *analogue of the regular representation* of an infinite-dimensional group G, we can first try to find a triplet

$$(\widetilde{G}, G, \mu), \tag{1.5}$$

where $\widetilde{G}$ is some larger topological group containing G as a dense subgroup, and a measure μ on $\widetilde{G}$ which is right or left G-quasi-invariant, i.e., $\mu^{R_t} \sim \mu$ for all $t \in G$, (or $\mu^{L_s} \sim \mu$ for all $s \in G$), here $\sim$ means *equivalence*, for details see [26]. Consider the right and the left actions R_t, L_s of the group G on $\widetilde{G}$ defined below:

$$R_t x = xt^{-1}, \quad L_s x = sx, \quad t, s \in G,\ x \in \widetilde{G}.$$

Denote by μ^{R_t}, μ^{L_s} the images of the measure μ under the maps $R_t, L_s : \widetilde{G} \to \widetilde{G}$. The right and the left representations $T^{R,\mu}, T^{L,\mu} : G \to U(L^2(\widetilde{G},\mu))$ are naturally defined on the Hilbert space $L^2(\widetilde{G},\mu)$ by the following formulas, compare with (1.1):

$$(T_t^{R,\mu} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt), \tag{1.6}$$

$$(T_s^{L,\mu} f)(x) = (d\mu(s^{-1}x)/d\mu(x))^{1/2} f(s^{-1}x). \tag{1.7}$$

In [26, Chapter 5, Theorem 5.2.11] we proved that for the group $B_0^{\mathbb{N}} = \varinjlim_n B(n,\mathbb{R})$, where $B(n,\mathbb{R})$ is the group of upper-triangular real matrices with units on the diagonal, and a Gaussian product-measure μ_b on the group $B^{\mathbb{N}}$ the Dixmier commutation theorem holds when $\mu_b^{L_s} \sim \mu_b$ for all $s \in B_0^{\mathbb{N}}$ under some special conditions on the measure μ_b. Here $B_0^{\mathbb{N}}$ (resp. $B^{\mathbb{N}}$) is a group of infinite real matrices of the form $I + x$, where x is upper-triangular with a finite number of nonzero elements (resp. $x = \sum_{k<n} x_{kn} E_{kn}$ is arbitrary upper-triangular) and

$$\mu_b(x) = \bigotimes_{k<n} \mu_{(b_{kn},0)}(x_{kn}), \quad d\mu_{(b_{kn},a_{kn})}(x_{kn}) = \sqrt{\frac{b_{kn}}{\pi}} e^{-b_{kn}(x_{kn}-a_{kn})^2} dx_{kn}. \tag{1.8}$$

However, the right regular representation of an infinite-dimensional group can be irreducible if no left actions are *admissible* for the measure μ, i.e., when $\mu^{L_s} \perp \mu$ for all $s \in G\backslash\{e\}$. In this case the von Neumann algebra $\mathfrak{A}^{T^{L,\mu}}$ generated by the left regular representation $T^{L,\mu}$ is trivial:

Conjecture 1.2. (Ismagilov, 1985) The right regular representation defined by (1.6)

$$T^{R,\mu} : G \to U(L^2(\widetilde{G},\mu))$$

is irreducible if and only if

(1) $\mu^{L_s} \perp \mu$ for all $s \in G\backslash\{e\}$, (where $\perp$ stands for orthogonal measures),

(2) the measure μ is G-ergodic.

Recall that the probability measure μ on a G-space X is called *ergodic* if any function $f \in L^1(X,\mu)$ with property $f(\alpha_t(x)) = f(x) \bmod \mu$ is constant. Conditions (1) and (2) are necessary irreducibility conditions, at least for Gaussian measures. The challenge is to prove that they are sufficient too.

Remark 1.3. Conjecture 1.2 was expressed by R. S. Ismagilov in his referee report of the first author's PhD Thesis, 1985. It was verified for a lot of particular cases by the first author. In [20], see also [26, Theorem 2.1.1], he proved Conjecture 1.2 for the group $B_0^{\mathbb{N}}$ and a Gaussian product-measure μ_b on the group $B^{\mathbb{N}}$. Moreover, he proved that two irreducible representations T^{R,μ_b} and $T^{R,\mu_{b'}}$ are equivalent if and only if the corresponding measures μ_b and $\mu_{b'}$ are equivalent [26, Theorem 2.1.17]. In the general case, Conjecture 1.2 is an open problem, for details see [26].

1.3.2. Irreducibility of Koopman's representation

If a G-space X has a "natural" right action of the group G and a left action of another group G_1, such as in the case of Schur-Weil duality, and these actions commute: $[R_t, L_s] = 0$ for all $t \in G, s \in G_1$, then we can imagine that the Koopman representation (1.4) is irreducilble if the left action is not admissible, i.e., $\mu^{L_s} \perp \mu$ for all $s \in G_1 \setminus \{e\}$ and the measures μ is G-right-ergodic, such as in the Ismagilov conjecture. The main result of this article, Theorem 2.1 is a particular case of this situation. However, if we have only one action $\alpha : G \to \mathrm{Aut}(X)$, the right action $R(G)$ should be replaced by $\alpha(G) \subset \mathrm{Aut}(X)$ and the left action $L(G_1)$ by the *centralizer* of the subgroup $\alpha(G)$ in the group $\mathrm{Aut}(X)$. The following conjecture is a natural generalization of Ismagilov's conjecture.

Conjecture 1.4. The representation (1.4) is irreducible if and only if

(1) $\mu^g \perp \mu$ for all $g \in Z_{\mathrm{Aut}(X)}(\alpha(G))\backslash\{e\}$,

(2) the measure μ is G-ergodic.

Here $Z_G(H)$ is the *centralizer* of the subgroup H in the group G:

$$Z_G(H) = \{g \in G \mid \{g, a\} = e \text{ for all } a \in H\},$$

where $\{g, a\} = gag^{-1}a^{-1}$. In general, Conjecture 1.4 is false. In the case of a finite field $\mathbb{F}_p$ we need some additional conditions for the irreducibility [25]. Our aim is to determine them.

1.3.3. Representations of the groups $G = \varinjlim_n G_n$

Consider an inductive limit $G = \varinjlim_n G_n$, with all $\widehat{G_n}$ known.

Problem 1.5. Is it sufficient to determine $\widehat{G}$, i.e., whether $\widehat{G} = \varprojlim_n \widehat{G_n}$?

In the case of commutative groups $G_n = \mathbb{R}^n$ or $G_n = T^n = T \times \cdots \times T$ the answer to Problem 1.5 is positive. In the first case we have $\widehat{G} = \varprojlim_n \widehat{G_n} = \mathbb{R}^\infty$. In the second $\widehat{G} = \varprojlim_n \widehat{G_n} = \mathbb{Z}^\infty$. If we have $\rho_n \in \widehat{G_n}$ defined on the Hilbert space H_n and there is an embedding of Hilbert spaces $i_n : H_n \to H_{n+1}$, we can define the representation $\rho = \varinjlim_n \rho_n$ in the space $H = \varinjlim_n H_n = \cup_{n\in\mathbb{N}} H_n$ and this representation should be irreducible. But usually, the space $H = \varinjlim_n H_n$ *has no Hilbert structure* see e.g., [46], like the space $\mathbb{R}_0^\infty = \varinjlim_n \mathbb{R}^n$. But when the representations $T = \varinjlim_n T_n$ can be obtained as limits of Koopman's representations $T_n = \pi^{\alpha_n,\mu_n,X_n}$ on the space $H_n = L^2(X_n, \mu_n)$ and we have an additional structure:

$$X_n = X^{(1)} \times \cdots \times X^{(n)}, \quad \mu_n = \bigotimes_{k=1}^{n} \mu^{(k)}, \tag{1.9}$$

in this case the final object $\varinjlim_n H_n$ can be embedded in a Hilbert space $\bigotimes_{k=1}^\infty H^{(k)}$. The construction of *von Neumann infinite tensor product* of the Hilbert spaces $H^{(k)}$

$$\mathcal{H}_e = \bigotimes_{k=1,e}^{\infty} H^{(k)}, \tag{1.10}$$

can be found in [3], but see also [16], here $e = (f_k)_{k=1}^\infty$ is some *stabilisation* where $f_k \in H^{(k)}$. Two infinite products $\mathcal{H}_e$ and $\mathcal{H}_l$ corresponding to two different stabilisations are *equivalent* if and only if the *corresponding stabilisations are equivalent* $e \sim l$. As it was shown in [26, Chapter 8] the answer to Problem 1.5 is negative, at least for the group $B_0^{\mathbb{N}} = \varinjlim_n G_n$, where $G_n = B(n, \mathbb{R})$. The dual $\widehat{G_n}$ is described by the orbit method, but this procedure will not allow us to obtain all irreducible representations, in particular, the regular representation. By [26, Theorem 2.1.1] the representation $T^{R,\mu_b} : B_0^{\mathbb{N}} \to U\big(L^2(B^{\mathbb{N}}, \mu_b)\big)$ is irreducible if and only if $\mu_b^{L_s} \perp \mu_b$

for all $s\in B_0^{\mathbb{N}}\setminus\{e\}$, where μ_b is defined by (1.8). Denote by R_n the regular representation of the group $B(n,\mathbb{R})$ in the Hilbert space $H_n=L^2(G_n,h_n)$, where h_n is the Haar measure on G_n. But the restriction $T^{R,\mu_b}|_{G_n}$ is equivalent to the regular representation R_n of G_n all of which are reducible! We have $T^{R,\mu_b}=\varinjlim_n R_n$, for the details see [26, Chapter 2.4]. *So we can obtain the irreducible representation as inductive limits of reducible representations.* Moreover, we prove [26, Theorem 2.1.17] that two irreducible representations T^{R,μ_b} and $T^{R,\mu_{b'}}$ are equivalent if and only if the corresponding measures μ_b and $\mu_{b'}$ are equivalent. Nonequivalent measures, in fact, give us two nonequivalent infinite tensor products $\mathcal{H}_e\not\sim\mathcal{H}_l$. This means that different embeddings of the spaces $i_n: H_n\to H_{n+1}$ can give nonequivalent representations, see details in [26, Chapter 8].

1.3.4. The inductive limit of reducible Koopman representations can be irreducible

In this article we consider Koopman representations of the inductive limit

$$\mathrm{GL}_0(2\infty,\mathbb{R})=\varinjlim_{n,i^s}\mathrm{GL}(2n+1,\mathbb{R})$$

with respect to the symmetric embedding (2.2). Theorem 2.1 states that $T^{R,\mu,3}$ is irreducible if and only if $(\mu^3_{(b,a)})^{L_s}\perp\mu^3_{(b,a)}$ for all $s\in\mathrm{GL}(3,\mathbb{R})\backslash\{e\}$. But the restriction $T^{R,\mu,3}|_{G_n}$ of the representation $T^{R,\mu,3}$ to the subgroup G_n is the Koopman representation of G_n, which is reducible, since any action of the group $\mathrm{GL}(3,\mathbb{R})$ from the left on the space $X_{3,n}$ is admissible, i.e., $\left(\mu^{3,n}_{(b,a)}\right)^{L_s}\sim\mu^{3,n}_{(b,a)}$ for all $s\in\mathrm{GL}(3,\mathbb{R})$.

1.4. The general idea to prove the irreducibility

Let G be some infinite-dimensional group acting on a G-space (X,μ) equipped with some quasi-invariant measure. In the concrete examples considered in [20]–[30] the possibility to approximate a lot of functions in $L^\infty(X,\mu)$ using Lemma 6.1, follows from the fact

$$\lim_{n\to\infty}(C_n(\lambda)^{-1}a_n,a_n)=\infty,\quad\text{where}\quad C_n(\lambda)=\mathrm{diag}\big(\lambda_1,\dots,\lambda_n\big)+C_n.\tag{1.11}$$

By Theorem 5.3 proved in [30], (see also Lemma 6.4 for $m=3$), we have

$$\big(C_n(\lambda)^{-1}a_n,a_n\big)=\Delta(y_1^{(n)},y_2^{(n)},\dots,y_m^{(n)})=\frac{\det\big(I_m+\gamma(y_1^{(n)},y_2^{(n)},\dots,y_m^{(n)})\big)}{\det\big(I_{m-1}+\gamma(y_2^{(n)},\dots,y_m^{(n)})\big)}-1.\tag{1.12}$$

Finally, by Lemma 1.2, [29] (see Lemmas 6.3 and 6.5 for $m=3$) we get

$$\lim_{n\to\infty}\frac{\det\big(I_m+\gamma(y_1^{(n)},y_2^{(n)},\dots,y_m^{(n)})\big)}{\det\big(I_{m-1}+\gamma(y_2^{(n)},\dots,y_m^{(n)})\big)}=\infty.$$

The article is organized as follows. The main result and the idea of the proof are formulated in Section 2.2, the orthogonality problem in measure theory is studied in Section 3, irreducibility is considered in Section 4 and the approximation of x_{kn} or D_{kn} in Section 5. In the Appendix we explain the use of the generalized characteristic polynomial, the explicit expression for the minimum of the quadratic form restricted to a hyperplane and present a result of independent interest on the height of an infinite parallelotope. In Section 7 we indicate what we can do in the case $m>3$.

2. Representations of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$

2.1. Finite-dimensional case

Consider the space $X_{m,n} = \Big\{x = \sum_{1\le k\le m}\ \sum_{-n\le r\le n} x_{kr}E_{kr},\ x_{kr}\in\mathbb{R}\Big\}$, where E_{kn}, $k,n\in\mathbb{Z}$ are infinite matrix unities, with the measure (see (2.5))

$$\mu^{m,n}_{(b,a)}(x) = \bigotimes_{k=1}^{m}\bigotimes_{r=-n}^{n}\mu_{(b_{kr},a_{kr})}(x_{kr}).$$

Two groups act on the space $X_{m,n}$: $\mathrm{GL}(m,\mathbb{R})$ from the left, and $\mathrm{GL}(2n{+}1,\mathbb{R})$ from the right, and their actions commute. Therefore, two von Neumann algebras $\mathfrak{A}_{1,n}$ and $\mathfrak{A}_{2,n}$ in the Hilbert space $L^2(X_{m,n},\mu^{m,n}_{(b,a)})$ generated respectively by the left and the right actions of the corresponding groups have the property that $\mathfrak{A}'_{1,n}\subseteq\mathfrak{A}_{2,n}$, where $\mathfrak{A}'$ is a commutant of a von Neuman algebra $\mathfrak{A}$. We study what happens as $n\to\infty$. In the limit we obtain some unitary representation $T^{R,\mu,m}$ (see (2.6)) of the group $G := \mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_{n,i^s}\mathrm{GL}(2n+1,\mathbb{R})$ acting from the right on X_m. In the generic case, the representation $T^{R,\mu,m}$ is reducible. Indeed, if there exists a non-trivial element $s\in\mathrm{GL}(m,\mathbb{R})$ such that the left action is *admissible* for the measure $\mu^m_{(b,a)}$, i.e., $(\mu^m_{(b,a)})^{L_s}\sim\mu^m_{(b,a)}$ the operator $T^{L,\mu,m}_s$ naturally associated with the left action, is well defined and $[T^{R,\mu,m}_t, T^{L,\mu,m}_s]=0$ for all $t\in G$, $s\in\mathrm{GL}(m,\mathbb{R})$.

Here, as in the case of the regular [19, 20] and quasi-regular representations of the group $B_0^{\mathbb{N}}$, which is an inductive limit of upper-triangular real matrices, we obtain the remarkable result that the *irreducible representations* can be obtained as the *inductive limit of reducible representations*!

The action of $\mathrm{GL}(2n+1,\mathbb{R})$ on the space $X_{m,n}$ can be seen as a product of the natural action on $\mathbb{R}^{2n+1}$. To see this set

$$X^{(k)}_{m,n} = \{x = \sum_{-n\le r\le n} x_{kr}E_{kr},\ x_{kr}\in\mathbb{R}\Big\}\sim\mathbb{R}^{2n+1},\quad \mu^{m,n,k}_{(b,a)}(x) := \otimes_{r=-n}^{n}\mu_{(b_{kr},a_{kr})}(x_{kr}).$$

Then $$X_{m,n} = \otimes_{k=1}^m\mathbb{R}^{2n+1},\quad L^2(X_{m,n},\mu^{m,n}_{(b,a)}) = \otimes_{k=1}^m L^2(\mathbb{R}^{2n+1},\mu^{m,n,k}_{(b,a)}).$$

2.2. The main result

Let us denote by $\mathrm{Mat}(2\infty,\mathbb{R})$ the space of all real matrices that are infinite in both directions:

$$\mathrm{Mat}(2\infty,\mathbb{R}) = \Big\{x = \sum_{k,n\in\mathbb{Z}} x_{kn}E_{kn},\ x_{kn}\in\mathbb{R}\Big\}. \tag{2.1}$$

The group $\mathrm{GL}_0(2\infty,\mathbb{R}) = \varinjlim_{n,i^s}\mathrm{GL}(2n+1,\mathbb{R})$ is defined as the inductive limit of the general linear groups $G_n = \mathrm{GL}(2n+1,\mathbb{R})$ with respect to the *symmetric embedding* i^s:

$$G_n\ni x\mapsto i^s_{n+1}(x) = x + E_{-(n+1),-(n+1)} + E_{n+1,n+1}\in G_{n+1}. \tag{2.2}$$

For a fixed natural number m, consider a G-space X_m as the following subspace of the space $\mathrm{Mat}(2\infty,\mathbb{R})$:

$$X_m = \Big\{x\in\mathrm{Mat}(2\infty,\mathbb{R})\mid x = \sum_{k=1}^{m}\sum_{n\in\mathbb{Z}} x_{kn}E_{kn}\Big\}. \tag{2.3}$$

The right action of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ is correctly defined on the space X_m by the formula $R_t(x)=xt^{-1}$, $t\in G$, $x\in X_m$. We define a Gaussian non-centered product measure $\mu:=\mu^m:=\mu^m_{(b,a)}$ on the space X_m :

$$\mu^m_{(b,a)}(x)=\otimes_{k=1}^m\otimes_{n\in\mathbb{Z}}\mu_{(b_{kn},a_{kn})}(x_{kn}), \tag{2.4}$$

where
$$d\mu_{(b_{kn},a_{kn})}(x_{kn})=\sqrt{\frac{b_{kn}}{\pi}}e^{-b_{kn}(x_{kn}-a_{kn})^2}dx_{kn} \tag{2.5}$$

and $b=(b_{kn})_{k,n}$, $b_{kn}>0$, $a=(a_{kn})_{k,n}$, $a_{kn}\in\mathbb{R}$, $1\le k\le m$, $n\in\mathbb{Z}$. Next we define the unitary representation $T^{R,\mu,m}$ of the group $\mathrm{GL}_0(2\infty,\mathbb{R})$ on the space $L^2(X_m,\mu^m_{(b,a)})$ by the formula:

$$(T_t^{R,\mu,m}f)(x)=\big(d\mu^m_{(b,a)}(xt)/d\mu^m_{(b,a)}(x)\big)^{1/2}f(xt),\ f\in L^2(X_m,\mu^m_{(b,a)}). \tag{2.6}$$

The *centralizer* $Z_{\mathrm{Aut}(X_m)}(R(G))\subset\mathrm{Aut}(X_m)$ contains the group $L(\mathrm{GL}(m,\mathbb{R}))$, i.e., the image of the group $\mathrm{GL}(m,\mathbb{R})$ with respect to the left action

$$L:\mathrm{GL}(m,\mathbb{R})\to\mathrm{Aut}(X_m),\ L_s(x)=sx,\ s\in\mathrm{GL}(m,\mathbb{R}),\ x\in X_m.$$

Theorem 2.1. *The representation* $T^{R,\mu,m}:\mathrm{GL}_0(2\infty,\mathbb{R})\to U\big(L^2(X_m,\mu^m_{(b,a)})\big)$ *is irreducible, for* $m=3$, *if and only if*

(i) $(\mu^m_{(b,a)})^{L_s}\perp\mu^m_{(b,a)}$ *for all* $s\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$;

(ii) *the measure* $\mu^m_{(b,a)}$ *is* G-ergodic.

In [26, 27] this result was proved for $m\le 2$. Note that Theorem 2.1 is a particular case of a generalisation of the Ismagilov conjecture, see Conjecture 7.7 in [30], for the group G acting on some space X.

Remark 2.2. Any Gaussian product-measure $\mu^m_{(b,a)}$ on X_m is $\mathrm{GL}_0(2\infty,\mathbb{R})$-right-ergodic [40, §3, Corollary 1]. For non-product-measures this is not true in general.

In order to study the condition $(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$ for $t\in\mathrm{GL}(m,\mathbb{R})\setminus\{e\}$ set

$$t=(t_{rs})_{r,s=1}^m\in\mathrm{GL}(m,\mathbb{R}),\ \ B_n=\mathrm{diag}(b_{1n},b_{2n},...,b_{mn}),\ \ X_n(t)=B_n^{1/2}tB_n^{-1/2}. \tag{2.7}$$

Let $M^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}(t)$ be the *minors* of the matrix t with $i_1,i_2,...,i_r$ rows and $j_1,j_2,...,j_r$ columns, $1\le r\le m$. Let δ_{rs} be the Kronecker symbols.

Lemma 2.3 ([26], Lemma 10.2.3; [27], Lemma 2.2)**.** *For the measures* $\mu^m_{(b,a)}$, *with* m *a natural number, the following relation holds true:*

$$(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}\ \ \textit{for all}\ t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}\quad\textit{if and only if}$$

$$\prod_{n\in\mathbb{Z}}\frac{1}{2^m|\det t|}\det(I+X_n^*(t)X_n(t))+\sum_{n\in\mathbb{Z}}\sum_{r=1}^m b_{rn}\Big(\sum_{s=1}^m(t_{rs}-\delta_{rs})a_{sn}\Big)^2=\infty,$$

$$\det\Big(I+X_n^*(t)X_n(t)\Big)=1+\sum_{r=1}^m\sum_{\substack{1\le i_1<i_2<\dots<i_r\le m\\1\le j_1<j_2<\dots<j_r\le m}}\big(M^{i_1i_2\dots i_r}_{j_1j_2\dots j_r}(X_n(t))\big)^2. \tag{2.8}$$

Let us define the following measures on the spaces $\mathbb{R}^m$ and X_m:

$$\mu_m^{(B_n,0)}=\otimes_{k=1}^m\mu_{(b_{kn},0)},\quad \mu_m^{(B_n,a_n)}=\otimes_{k=1}^m\mu_{(b_{kn},a_{kn})},$$

where $a_n=(a_{1n},...,a_{mn})\in\mathbb{R}^m$ and $B_n=\mathrm{diag}(b_{1n},...,b_{mn})\in\mathrm{Mat}(m,\mathbb{R})$.

Set $\quad H_{m,n}(t)=H\Big(\Big(\mu_m^{(B_n,0)}\Big)^{L_t},\mu_m^{(B_n,0)}\Big)=\Big(\frac{1}{2^m|\det t|}\det\left(I+X_n^*(t)X_n(t)\right)\Big)^{-1/2}. \quad (2.9)$

Remark 2.4. (Idea of the proof of irreducibility.) Let us denote by $\mathfrak{A}^m$ the *von Neumann algebra* generated by the representation $T^{R,\mu,m}$, that is, we have $\mathfrak{A}^m=(T_t^{R,\mu,m}\mid t\in G)''$, where M' is the *commutant* of the von Neunmann algebra $M\subset B(H)$. For $\alpha=(\alpha_k)\in\{0,1\}^m$ define the von Neumann algebra $L^\infty_\alpha(X_m,\mu^m)$ as follows:

$$L^\infty_\alpha(X_m,\mu^m)=\Big(\exp(itB^\alpha_{kn})\mid 1\le k\le m,\ t\in\mathbb{R},\ n\in\mathbb{Z}\Big)'',$$

where $B^\alpha_{kn}=\begin{cases} x_{kn}, & \text{if}\quad \alpha_k=0\\ i^{-1}D_{kn}, & \text{if}\quad \alpha_k=1\end{cases}$ and $D_{kn}=\partial/\partial x_{kn}-b_{kn}(x_{kn}-a_{kn})$.

The proof of the irreducibility is based on four facts:

(1) We can approximate by the generators $A_{kn}=A^{R,m}_{kn}=\frac{d}{dt}T^{R,\mu,m}_{I+tE_{kn}}|_{t=0}$ the set of operators $(B^\alpha_{kn})_{k=1}^m$, $n\in\mathbb{Z}$ *for some* $\alpha\in\{0,1\}^m$ depending on the measure μ^m using the orthogonality condition $(\mu^m)^{L_s}\perp\mu^m$ for all $s\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$;

(2) it is sufficient to verify the approximation only for the *cyclic vector* $\mathbf{1}(x)\equiv 1$, since the representation $T^{R,\mu,m}$ is *cyclic*;

(3) the subalgebra $L^\infty_\alpha(X_m,\mu^m)$ is a *maximal abelian subalgebra* in $\mathfrak{A}^m$;

(4) the measure μ^m is G-ergodic.

Here the *generators* A_{kn} are given by the formulas:

$$A_{kn}=\sum_{r=1}^m x_{rk}D_{rn},\quad k,n\in\mathbb{Z},\quad\text{where}\quad D_{kn}=\partial/\partial x_{kn}-b_{kn}(x_{kn}-a_{kn}). \quad (2.10)$$

Remark 2.5. *Scheme of the proof.* We prove the irreducibility as follows

$$\big(\mu^{L_s}\perp\mu\ \text{ for all }\ s\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}\big)\Leftrightarrow\left(\begin{smallmatrix}\text{criteria}\\ \text{of}\\ \text{orthogonality}\end{smallmatrix}\right)\ \& \quad (2.11)$$

$$\left(\begin{smallmatrix}\text{Lemma 6.3}\\ \text{about}\\ \text{three vectors } f,g,h\notin l_2\end{smallmatrix}\right)\Rightarrow\left(\begin{smallmatrix}\text{some of} & \Delta^{(1)},\ \Delta_1\\ \text{the expressions} & \Delta^{(2)},\ \Delta_2\\ \text{are divergent:} & \Delta^{(3)},\ \Delta_3\end{smallmatrix}\right)\Rightarrow\text{irreducibility},$$

$$\text{where}\quad \Delta^{(i)}:=\Delta(Y_i^{(i)},Y_j^{(i)},Y_k^{(i)}),\quad \Delta_i:=\Delta(Y_i,Y_j,Y_k), \quad (2.12)$$

$\Delta(f,g,h)$ is defined by (2.15), and $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$, see for details Lemma 5.1, Lemma 5.2 and Lemma 5.4.

We use the following notation. For k vectors $f_1,f_2,\dots,f_k\in\mathbb{R}^n$ with $k\le n$ set

$$\Delta(f_1,f_2,\dots,f_k)=\frac{\det(I+\gamma(f_1,f_2,\dots,f_k))}{\det(I+\gamma(f_2,\dots,f_k))}-1,\ \text{ for } k=2,3: \quad (2.13)$$

$$\Delta(f_1,f_2)=\frac{\det(I+\gamma(f_1,f_2))}{\det(I+\gamma(f_2))}-1=\frac{I+\Gamma(f_1)+\Gamma(f_2)+\Gamma(f_1,f_2)}{I+\Gamma(f_2)}, \quad (2.14)$$

$$\Delta(f_1,f_2,f_3)=\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}. \quad (2.15)$$

Remark 2.6. The fact that the conditions $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for all $t \in \mathrm{GL}(3,\mathbb{R})\backslash\{e\}$ imply the possibility of the approximation of x_{kn} or D_{kn} by combinations of generators is based on Lemma 6.1 and Lemma 6.4 about explicit expression for $(C^{-1}(\lambda)a,a)$, see [30], where $C(\lambda)$ is defined by (6.3). Finally the last lemma is based on some completely *independent statement* about three infinite vectors $f_1, f_2, f_3 \notin l_2(\mathbb{N})$ such that $\sum_{k=1}^3 C_k f_k \notin l_2(\mathbb{N})$, see Lemma 6.3 for general m in [29]. These lemmas are the *key ingredients* of the proof of the irreducibility of the representation.

Remark 2.7. Note that in the case of the "nilpotent group" $B_0^{\mathbb{N}}$ and the infinite product of *arbitrary* Gaussian measures on $\mathbb{R}^m$ (see [1]) the proof of the *irreducibility* is also based on another completely *independent statement* namely, *the Hadamard-Fischer inequality*, see Lemma 2.8.

Lemma 2.8. (Hadamard-Fischer inequality [11, 12]) *For any positive definite matrix* $C\in\mathrm{Mat}(m,\mathbb{R})$, $m\in\mathbb{N}$, *and any two subsets* α *and* β *with* $\emptyset \subseteq \alpha,\ \beta \subseteq \{1,...,m\}$ *the following inequality holds:*

$$\begin{vmatrix} M(\alpha) & M(\alpha\bigcap\beta) \\ M(\alpha\bigcup\beta) & M(\beta) \end{vmatrix} = \begin{vmatrix} A(\hat\alpha) & A(\hat\alpha\bigcup\hat\beta) \\ A(\hat\alpha\bigcap\hat\beta) & A(\hat\beta) \end{vmatrix} \geq 0, \tag{2.16}$$

where $M(\alpha) = M^\alpha_\alpha(C)$, $A(\alpha) = A^\alpha_\alpha(C)$ *are* factors *and* cofactors *of the matrix* C *and* $\hat\alpha = \{1,...,m\}\setminus\alpha$.

2.3. Equivalent series and equivalent sequences

Definition 2.9. We say that two series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} b_n$ with positive a_n, b_n are *equivalent* if they are divergent or convergent simultaneously. We will write $\sum_{n\in\mathbb{N}} a_n \sim \sum_{n\in\mathbb{N}} b_n$. We say that two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are *equivalent* if for some $C_1, C_2 > 0$ we have $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n\in\mathbb{N}$. We will use the same notation $a_n \sim b_n$.

Lemma 2.10. *Let* $1+c_n > 0$ *for all* $n\in\mathbb{Z}$. *Then the following two series are equivalent:*

$$\Sigma_1 := \sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n}, \quad \Sigma_2 := \sum_{n\in\mathbb{Z}} c_n^2. \tag{2.17}$$

Proof. Fix some $\varepsilon \in (0,1)$ and a large enough N. We have three cases:

(a) $1+c_n \in (\varepsilon, N)$,

(b) for an infinite subset $\mathbb{Z}_1$ we have $\lim_{n\in\mathbb{Z}_1}(1+c_n) = \infty$,

(c) for an infinite subset $\mathbb{Z}_1$ we have $\lim_{n\in\mathbb{Z}_1}(1+c_n) = 0$.

When we are not in the case (a) there is an infinite subset $\mathbb{Z}_1 \subset \mathbb{Z}$ such that (b) or (c) holds true. In the case (a) we have

$$\frac{1}{N}\sum_{n\in\mathbb{Z}} c_n^2 < \sum_{n\in\mathbb{Z}} \frac{c_n^2}{1+c_n} < \frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}} c_n^2, \tag{2.18}$$

In the cases (b) and (c) both series are divergent. ∎

We will make systematic use of the following statement.

Remark 2.11. ([26]) Let $a_n, b_n > 0$ for all $n \in \mathbb{N}$. The following two series are equivalent

$$\sum_{n\in\mathbb{N}} \frac{a_n}{a_n+b_n} \sim \sum_{n\in\mathbb{N}} \frac{a_n}{b_n}. \tag{2.19}$$

3. Some orthogonality problems in measure theory

3.1. General setting

Our aim now is to find the minimal generating set of conditions for the orthogonality $(\mu^m_{(b,a)})^{L_t} \perp \mu^m_{(b,a)}$ for all $t \in \mathrm{GL}(m,\mathbb{R})\backslash\{e\}$. To be more precise, consider the following more general situation. Let $\alpha : G \to \mathrm{Aut}(X)$ be a *measurable action* of a group G on a measurable space (X,μ) with the following property: $\mu^{\alpha_t} \perp \mu$ for all $t \in G \setminus \{e\}$. Define a *generating subset* $G^{\perp}(\mu)$ in the group G as follows:

$$\text{if} \quad \mu^{\alpha_t} \perp \mu \ \text{ for all } t \in G^{\perp}(\mu), \quad \text{then} \quad \mu^{\alpha_t} \perp \mu \ \text{ for all } t \in G \setminus \{e\}. \tag{3.1}$$

Problem 3.1. Find a minimal generating subset $G^{\perp}_0(\mu)$ satisfying (3.1).

3.2. Orthogonality criteria $\mu^{L_t} \perp \mu$ for $t \in \mathrm{GL}(2,\mathbb{R}) \setminus \{e\}$

Remark 3.2. By Lemma 4.1 proved in [27] or Lemma 10.4.1 in [26] for $m = 2$ we conclude that the minimal generating set $G^{\perp}_0(\mu) = \mathrm{GL}(2,\mathbb{R})^{\perp}_0(\mu)$ (see Problem (3.1)) is reduced to the following subgroups, families and elements:

$$\exp(tE_{12}) = I + tE_{12} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ \exp(tE_{21}) = I + tE_{21} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \tag{3.2}$$

$$\exp(tE_{12})P_1 = \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad \exp(tE_{21})P_2 = \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}, \tag{3.3}$$

$$\tau_-(\phi,s) = \begin{pmatrix} \cos\phi & s^2\sin\phi \\ s^{-2}\sin\phi & -\cos\phi \end{pmatrix} = D_2(s)\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} D_2^{-1}(s)P_2. \tag{3.4}$$

The families (3.2) are one-parameter subgroups, the families (3.3) are just reflections of (3.2) and the family (3.4) depends on two parameters. All elements are of order 2 except the elements in subgroups given in (3.2). It suffices to verify the conditions (3.2) only for some $t \in \mathbb{R} \setminus \{0\}$. Actually, the family $\tau_-(\phi,s)$ coincides with $D_2(s)\mathrm{O}(2)D_2^{-1}(s)P_2$, where $D_2(s) = \mathrm{diag}(s,s^{-1})$. All points t in (3.3) and all points (ϕ,s) in (3.4) are essential, i.e., they cannot be removed.

The conditions for the orthogonality with respect to elements defined by (3.2)–(3.4) are transformed in the divergence of the following series:

$$S^L_{12}(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{2}\left(\frac{1}{2b_{2n}} + a^2_{2n}\right), \ S^L_{21}(\mu) = \sum_{n\in\mathbb{Z}} \frac{b_{2n}}{2}\left(\frac{1}{2b_{1n}} + a^2_{1n}\right), \tag{3.5}$$

$$S^{L,-}_{kn}(\mu,t) = \frac{t^2}{4}\sum_{m\in\mathbb{Z}} \frac{b_{km}}{b_{nm}} + \sum_{m\in\mathbb{Z}} \frac{b_{km}}{2}(-2a_{km} + ta_{nm})^2, \ t \in \mathbb{R}, \tag{3.6}$$

$$\Sigma^-_{12}\big(\tau_-(\phi,s)\big) = \sin^2\phi\, \Sigma_{12}(s) + \Sigma^-_{12}\big(\tau_-(\phi,s)\big), \ \phi \in [0,2\pi), \ s > 0, \tag{3.7}$$

$$\text{where} \quad \Sigma_{12}(s) := \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2, \tag{3.8}$$

$$\Sigma_{12}^{-}\big(\tau_{-}(\phi,s)\big)=\sum_{n\in\mathbb{Z}}\Big(4b_{1n}\sin^2\frac{\phi}{2}+4s^{-4}b_{2n}\cos^2\frac{\phi}{2}\Big)\Big(a_{1n}\sin\frac{\phi}{2}-s^2a_{2n}\cos\frac{\phi}{2}\Big)^2. \quad (3.9)$$

Remark 3.3. [see [27]] The following three conditions are equivalent:

(i) $\mu^{L_{\tau_{-}(\phi,s)}}\perp\mu,\qquad \phi\in[0,2\pi),\ s>0,$

(ii) $\Sigma_{12}\big(\tau_{-}(\phi,s)\big)=\sin^2\phi\,\Sigma_{12}(s)+\Sigma_{12}^{-}\big(\tau_{-}(\phi,s)\big)=\infty,\qquad \phi\in[0,2\pi),\ s>0,$

(iii) $\Sigma_{12}(s)+\Sigma_{12}(C_1,C_2)=\infty,\qquad s>0,\ (C_1,C_2)\in\mathbb{R}^2\setminus\{0\},$

where $\Sigma_{12}(s)$ is defined by (3.8) and

$$\Sigma_{12}(C_1,C_2):=\sum_{n\in\mathbb{Z}}(C_1^2b_{1n}+C_2^2b_{2n})(C_1a_{1n}+C_2a_{2n})^2. \quad (3.10)$$

3.3. Orthogonality criteria $\mu^{L_t}\perp\mu$ for $t\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}$

Recall [27] that for $m=2$ and $\det t>0$ we have

$$2^2\mid\det t\mid\big(H_{2,n}^{-2}(t)-1\big)=\Big[(1-\mid\det t\mid)^2+(t_{11}-t_{22})^2+\Big(t_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}+t_{21}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2\Big]$$
$$=\Big[\big(M_\emptyset^\emptyset(X(t))-A_\emptyset^\emptyset(X(t))\big)^2+\big(M_1^1(X(t))-A_1^1(X(t))\big)^2+\big(M_2^1(X(t))-A_2^1(X(t))\big)^2\Big],$$

where $H_{m,n}(t)$ is defined by (2.9). For $m=3$, using the relations of (2.7), we have $X(t)=B^{1/2}tB^{-1/2}$, hence

$$X(t)=\begin{pmatrix}b_{1n}&0&0\\0&b_{2n}&0\\0&0&b_{3n}\end{pmatrix}^{1/2}\begin{pmatrix}t_{11}&t_{12}&t_{13}\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{pmatrix}\begin{pmatrix}b_{1n}&0&0\\0&b_{2n}&0\\0&0&b_{3n}\end{pmatrix}^{-1/2}$$
$$=\begin{pmatrix}t_{11}&\sqrt{\frac{b_{1n}}{b_{2n}}}t_{12}&\sqrt{\frac{b_{1n}}{b_{3n}}}t_{13}\\\sqrt{\frac{b_{2n}}{b_{1n}}}t_{21}&t_{22}&\sqrt{\frac{b_{2n}}{b_{3n}}}t_{23}\\\sqrt{\frac{b_{3n}}{b_{1n}}}t_{31}&\sqrt{\frac{b_{3n}}{b_{2n}}}t_{32}&t_{33}\end{pmatrix}.$$

Therefore, using (2.8) and the fact that $X=X^*(t)X(t)$ we obtain

$$2^3\mid\det t\mid H_{3,n}^{-2}(t)=\Big(1+\mid\det t\mid^2+t_{11}^2+\frac{b_{1n}}{b_{2n}}t_{12}^2+\frac{b_{1n}}{b_{3n}}t_{13}^2+\frac{b_{2n}}{b_{1n}}t_{21}^2+t_{22}^2+\frac{b_{2n}}{b_{3n}}t_{23}^2$$
$$+\frac{b_{3n}}{b_{1n}}t_{31}^2+\frac{b_{3n}}{b_{2n}}t_{32}^2+t_{33}^2+\big(M_{12}^{12}(t)\big)^2+\frac{b_{2n}}{b_{3n}}\big(M_{13}^{12}(t)\big)^2+\frac{b_{1n}}{b_{3n}}\big(M_{23}^{12}(t)\big)^2+\frac{b_{3n}}{b_{2n}}\times$$
$$\big(M_{12}^{13}(t)\big)^2+\big(M_{13}^{13}(t)\big)^2+\frac{b_{1n}}{b_{2n}}\big(M_{23}^{13}(t)\big)^2+\frac{b_{3n}}{b_{1n}}\big(M_{12}^{23}(t)\big)^2+\frac{b_{2n}}{b_{2n}}\times$$
$$\big(M_{13}^{23}(t)\big)^2+\big(M_{23}^{23}(t)\big)^2\Big)=1+\mid\det t\mid^2+\sum_{1\le i\le j\le 3}\Big[\Big(t_j^i\sqrt{\frac{b_{in}}{b_{jn}}}\Big)^2+\Big(A_j^i\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2\Big]$$
$$=1+\mid\det t\mid^2+\sum_{1\le i\le j\le 3}\big(|M_j^i(X(t))|^2+|A_j^i(X(t))|^2\big).$$

Let $A_j^i(t)$ be the *cofactors* of a matrix $t\in\mathrm{GL}(3,\mathbb{R})$.

Using the notation $t^i_j = t_{ij}$ and the relations

$$\det t = t^k_1 A^k_1(t) + t^k_2 A^k_2(t) + t^k_3 A^k_3(t), \quad k = 1,2,3, \quad \text{we get}$$

$$2^3 \mid \det t \mid \left(H^{-2}_{3,n}(t) - 1\right) = (1- \mid \det t \mid)^2 + \sum_{1\leq i,j\leq 3} \left(M^i_j(X(t)) - A^i_j(X(t))\right)^2$$

$$= (1- \mid \det t \mid)^2 + \sum_{1\leq i\leq j\leq 3} \left(t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} - A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2. \tag{3.11}$$

Similar to [27, Lemmas 2.22] in the case $m = 2$, or [26, Lemma 10.4.30] we get the following lemma, for $m = 3$.

Lemma 3.4. *For* $t \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}$, *if* $\pm \det t > 0$, *we have respectively*

$$(\mu^3_{(b,0)})^{L_t} \perp \mu^3_{(b,0)} \quad \Leftrightarrow \tag{3.12}$$

$$\sum_{n\in\mathbb{Z}} \left[(1- \mid \det t \mid)^2 + \sum_{1\leq i\leq 3} \left(t^i_i \mp A^i_i(t)\right)^2 + \sum_{1\leq i<j\leq 3} \left(t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} \mp A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\right] = \infty.$$

By Lemma 2.3 the following lemma holds true.

Lemma 3.5. *For* $t \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}$ *we have*

$$(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)} \quad \text{if} \quad \mid \det t \mid \neq 1.$$

If $\det t = \pm 1$, *we have respectively*

$$(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)} \quad \Leftrightarrow \quad \Sigma^{\pm}(t) := \Sigma^{\pm}_1(t) + \Sigma_2(t) = \infty, \quad \text{where}$$

$$\Sigma^+_1(t) = \sum_{n\in\mathbb{Z}} \left[\sum_{k=1}^{3} (t_{kk} - A^k_k(t))^2 + \sum_{1\leq i<j\leq 3} \left(t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} - A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\right], \tag{3.13}$$

$$\Sigma^-_1(t) = \sum_{n\in\mathbb{Z}} \left[\sum_{k=1}^{3} (t_{kk} + A^k_k(t))^2 + \sum_{1\leq i<j\leq 3} \left(t^i_j \sqrt{\frac{b_{in}}{b_{jn}}} + A^i_j(t)\sqrt{\frac{b_{jn}}{b_{in}}}\right)^2\right], \tag{3.14}$$

$$\Sigma_2(t^{-1}) = \sum_{n\in\mathbb{Z}} \Big[b_{1n}\big((t_{11}-1)a_{1n} + t_{12}a_{2n} + t_{13}a_{3n}\big)^2 + \tag{3.15}$$

$$b_{2n}\big(t_{21}a_{1n} + (t_{22}-1)a_{2n} + t_{23}a_{3n}\big)^2 + b_{3n}\big(t_{31}a_{1n} + t_{32}a_{2n} + (t_{33}-1)a_{3n}\big)^2\Big].$$

Remark 3.6. By Lemma 3.5, it suffices to verify, the condition of orthogonality

$$(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)} \quad \text{for} \quad t \in \mathrm{GL}(3,\mathbb{R}) \setminus \{e\}$$

for the following two subsets of the group $\pm\mathrm{SL}(3,\mathbb{R})$:

$$G^{\pm}_3 := \{t \in \pm\mathrm{SL}(3,\mathbb{R}) \mid t_{kk} = \pm A^k_k(t),\ 1 \leq k \leq 3\}. \tag{3.16}$$

Lemma 3.7. *If $t \in G_3^{\pm}$, we have respectively*

$$(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)} \quad \Leftrightarrow \quad \Sigma^{\pm}(t) = \Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty,$$

$$\Sigma_1^{\pm}(t) = \sum_{1\le i<j\le 3}\sum_{n\in\mathbb{Z}}\Big(t_j^i\sqrt{\frac{b_{in}}{b_{jn}}} \mp A_j^i(t)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2 = \sum_{1\le i<j\le 3}\Sigma_{ij}^{\pm}(t), \tag{3.17}$$

$$\Sigma_{ij}^{\pm}(t)=\sum_{n\in\mathbb{Z}}\Big(t_j^i\sqrt{\frac{b_{in}}{b_{jn}}}\mp A_j^i(t)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2, \tag{3.18}$$

where $\Sigma_2(t)$ is defined by (3.15).

Next we will show that the set G_3^+ can be reduced to the six families of one-parameter subgroups $\exp(tE_{kr})$, $1 \le k \ne r \le 3$, see (3.20), or the three families of two-parameter subgroups, see (3.21). The set G_3^- can be reduced to the three two-parameter family (3.22), reflections of (3.21) by P_r. The remaining part is reduced to the sets $D_3(s)\mathrm{O}(3)D_3^{-1}(s)P_r$ or five parameter family of elements $\tau_r(t,s) = D_3(s)tD_3^{-1}(s)P_r$, see (3.26).

Lemma 3.8. *In case $m = 3$ the* minimal generating set $\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu)$ *is defined as follows (compare with Remark 3.2):*

$$\mathrm{GL}(3,\mathbb{R})_0^{\perp}(\mu) = \{e_r(t,s), e_r(t,s)P_r, \mid 1\le r\le 3,\ (t,s)\in\mathbb{R}^2\}$$

$$\cup\ \{\mathrm{O}_r^A(3),\ 1\le r\le 3\}, \quad \textit{where} \tag{3.19}$$

$$e_{kn}(t) := \exp(tE_{kn}) = I + tE_{kn},\ 1\le k\ne n\le 3,\ t\in\mathbb{R}, \tag{3.20}$$

$$e_1(t,s) = \begin{pmatrix} 1 & t & s\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ e_2(t,s) = \begin{pmatrix} 1 & 0 & 0\\ t & 1 & s\\ 0 & 0 & 1\end{pmatrix},\ e_3(t,s) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ t & s & 1\end{pmatrix}, \tag{3.21}$$

$$e_1(t,s)P_1=\begin{pmatrix} -1 & t & s\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ e_2(t,s)P_2=\begin{pmatrix} 1 & 0 & 0\\ t & -1 & s\\ 0 & 0 & 1\end{pmatrix},\ e_3(t,s)P_3=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ t & s & -1\end{pmatrix}, \tag{3.22}$$

$$P_1 = \begin{pmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad P_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad P_3 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \tag{3.23}$$

$$\mathrm{O}^A(3) := \{D_3(s)\mathrm{O}(3)D_3^{-1}(s) \mid D_3(s)\in A\}, \tag{3.24}$$

$$\mathrm{O}_r^A(3) := \{D_3(s)\mathrm{O}(3)D_3^{-1}(s)P_r \mid D_3(s)\in A\},\ 1\le r\le 3, \tag{3.25}$$

$$\tau_r(t,s) = D_3(s)tD_3^{-1}(s)P_r,\ t\in O(3),\ A := \{D_3(s)=\mathrm{diag}(s_1,s_2,s_3)\}. \tag{3.26}$$

The families (3.20) give us respectively the divergence of the following series:

$$S_{kr}^L(\mu) = \sum_{n\in\mathbb{Z}}\frac{b_{kn}}{2}\Big(\frac{1}{2b_{rn}} + a_{rn}^2\Big), \quad 1\le k,r\le 3,\ k\ne r. \tag{3.27}$$

The families (3.21) give us the divergence of the following series:

$$S_{1,23}^L(\mu,t,s) = \sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}} + \frac{s^2}{4}\frac{b_{1n}}{b_{3n}} + \frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2\right], \tag{3.28}$$

$$S_{2,31}^L(\mu,t,s) = \sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{2n}}{b_{1n}} + \frac{s^2}{4}\frac{b_{2n}}{b_{3n}} + \frac{b_{2n}}{2}\big(ta_{1n}-2a_{2n}+sa_{3n}\big)^2\right], \tag{3.29}$$

$$S_{3,12}^L(\mu,t,s) = \sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{3n}}{b_{1n}} + \frac{s^2}{4}\frac{b_{3n}}{b_{2n}} + \frac{b_{3n}}{2}\big(ta_{1n}+sa_{2n}-2a_{3n}\big)^2\right], \tag{3.30}$$

$$\text{in particular}\quad S_{rr}^L(\mu) := S_{r,st}^L(\mu,0,0) = \sum_{n\in\mathbb{Z}}\frac{b_{rn}}{2}a_{rn}^2, \tag{3.31}$$

where (r,s,t) is a cyclic permutation of $(1,2,3)$. The families (3.26) give us the condition (3.34), see Lemma 3.9 below.

Proof. Consider the subset $\mathrm{GL}(3,\mathbb{R})_0^\perp(\mu)$ of $\mathrm{GL}(3,\mathbb{R})$ described by (3.19). The fact that this set is *minimal generating* will follow from Lemma 4.1, more precisely, from the following implications:

$$\Big(\mu^{L_t}\perp\mu\ \text{ for all }\ t\in\mathrm{GL}(3,\mathbb{R})_0^\perp(\mu)\Big)\Rightarrow\Big(\text{irreducibility}\Big)\qquad(3.32)$$
$$\Rightarrow\Big(\mu^{L_t}\perp\mu\ \text{ for all }\ t\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}\Big).$$

The first implication follows from Lemma 4.1, and the second from the irreducibility. Indeed, suppose that $\mathrm{GL}(3,\mathbb{R})_0^\perp(\mu)$ is not a minimal generating set, then we can find an $s\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}$ such that

$$\big(\mu^3_{(b,a)}\big)^{L_s}\sim\mu^3_{(b,a)}.$$

Hence the non-trivial operator $T_s^{L,\mu,3}$ can be defined by

$$(T_s^{L,\mu,3}f)(x)=\big(d\mu^3_{(b,a)}(s^{-1}x)/d\mu^3_{(b,a)}(x)\big)^{1/2}f(s^{-1}x),\ f\in L^2(X_3,\mu^3_{(b,a)}).\qquad(3.33)$$

This operator commutes with the representations $T^{R,\mu,3}$:

$$[T_t^{R,\mu,3},T_s^{L,\mu,3}]=0\quad\text{for all}\quad t\in G,$$

contradicting the irreducibility. The relations (3.27)–(3.30) follows from (3.13)–(3.15). The relation (3.29), for example, follows from (3.14) and (3.15). The relation (3.34) is obtained from (3.13) for $\tau_r(t,s)$, $t\in\mathrm{O}(3)$, $s\in\big(\mathbb{R}^*\big)^3$ defined by (3.26). ∎

Lemma 3.9. *Set* $\tau(s,t):=D_3(s)tD_3^{-1}(s)$ *and* $\tau_r(s,t):=\tau(s,t)P_r$ *for* $t\in\pm\mathrm{O}(3)$, $D_3(s)=\mathrm{diag}(s_1,s_2,s_3)$, $s=(s_1,s_2,s_3)\in(\mathbb{R}^*)^3$ *and* $1\le r\le 3$. *Then*

$$\big(\mu^3_{(b,a)}\big)^{L_{\tau_r(s,t)}}\perp\mu^3_{(b,a)}\Leftrightarrow\Sigma_1^\pm\big(\tau_r(s,t)\big)+\Sigma_2\big(\tau_r(s,t)\big)=\infty,\qquad(3.34)$$

where $\Sigma_1^\pm(t)$ *are defined by* (3.17), *and* $\Sigma_2(t)$ *is defined by* (3.15). *In particular, if we denote* $s_{ij}=s_is_j^{-1}$ *we get*

$$\Sigma_1^+\big(\tau(t,s)\big)=\Sigma_1^+(t,s)=t_{12}^2\Sigma_{12}(s_{12}^{1/2})+t_{13}^2\Sigma_{13}(s_{13}^{1/2})+t_{23}^2\Sigma_{23}(s_{23}^{1/2}).\qquad(3.35)$$

Proof. For $T:=\tau(s,t)$ and $T(3):=\tau_3(s,t)$ we have respectively:

$$T=D_3(s)tD_3^{-1}(s)=\begin{pmatrix}t_{11}&\frac{s_1}{s_2}t_{12}&\frac{s_1}{s_3}t_{13}\\ \frac{s_2}{s_1}t_{21}&t_{22}&\frac{s_2}{s_3}t_{23}\\ \frac{s_3}{s_1}t_{31}&\frac{s_3}{s_2}t_{32}&t_{33}\end{pmatrix},\qquad(3.36)$$

$$\begin{pmatrix}t_{11}&\frac{s_1}{s_2}t_{12}&-\frac{s_1}{s_3}t_{13}\\ \frac{s_2}{s_1}t_{21}&t_{22}&-\frac{s_2}{s_3}t_{23}\\ \frac{s_3}{s_1}t_{31}&\frac{s_3}{s_2}t_{32}&-t_{33}\end{pmatrix}=D_3(s)tD_3^{-1}(s)P_3=:T(3).\qquad(3.37)$$

By Lemma 3.10 for $t\in\mathrm{O}(3)$ we have $t_{kr}=A_r^k(t)$, $1\le k,r\le 3$. Therefore, for T and $T(3)$ we have for $1\le k,r\le 3$:

$$M_r^k(T)=T_{kr}=\frac{s_k}{s_r}t_{kr},\ A_r^k(T)=\frac{s_r}{s_k}A_r^k(t)\overset{(3.40)}{=}\frac{s_r}{s_k}t_{kr},\quad M_r^k(T(3))\qquad(3.38)$$
$$=(-1)^{\delta_{3,r}}\frac{s_k}{s_r}t_{kr},\ A_r^k(T(3))=(-1)^{\delta_{3,r}}\frac{s_r}{s_k}A_r^k(t)=(-1)^{\delta_{3,r}}\frac{s_r}{s_k}t_{kr}.\qquad(3.39)$$

Finally, we get

$$\Sigma_1^+(T) = \Sigma_1^+\big(\tau(s,t)\big) = \sum_{n\in\mathbb{Z}}\Big[\sum_{1\le i<j\le 3}\Big(M_j^i(T)\sqrt{\frac{b_{in}}{b_{jn}}} - A_j^i(T)\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2\Big]$$

$$=\sum_{n\in\mathbb{Z}}\Big[t_{12}^2\Big(s_{12}\sqrt{\frac{b_{1n}}{b_{2n}}}-s_{12}^{-1}\sqrt{\frac{b_{2n}}{b_{1n}}}\Big)^2+t_{13}^2\Big(s_{13}\sqrt{\frac{b_{1n}}{b_{3n}}}-s_{13}^{-1}\sqrt{\frac{b_{3n}}{b_{1n}}}\Big)^2+$$

$$t_{23}^2\Big(s_{23}\sqrt{\frac{b_{2n}}{b_{3n}}}-s_{23}^{-1}\sqrt{\frac{b_{3n}}{b_{2n}}}\Big)^2\Big] = t_{12}^2\Sigma_{12}(s_{12}^{1/2})+t_{13}^2\Sigma_{13}(s_{13}^{1/2})+t_{23}^2\Sigma_{23}(s_{23}^{1/2}).$$

Hence, $\Sigma_1^-(T(3)) = t_{12}^2\Sigma_{12}(s_{12}^{1/2})+t_{13}^2\Sigma_{13}(s_{13}^{1/2})+t_{23}^2\Sigma_{23}(s_{23}^{1/2})$. ■

Lemma 3.10. *For an arbitrary orthogonal matrix $t\in\pm\mathrm{O}(3)$ we have*

$$t_{kn} = \pm A_n^k(t),\ 1\le k,n\le 3,\quad \textit{where}\quad t=\begin{pmatrix}t_{11}&t_{12}&t_{13}\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{pmatrix}. \tag{3.40}$$

Proof. Denote the three rows of the matrix t by, respectively, $t_1,t_2,t_3\in\mathbb{R}^3$. Since $t\in\pm\mathrm{O}(3)$ we get

$$\|t_1\|^2=\|t_2\|^2=\|t_3\|^2=1\quad\text{and}\quad t_l\perp t_r,\ l\ne r. \tag{3.41}$$

Moreover, since t_1 is orthogonal to the hyperplane V_{23} generated by the vectors t_2 and t_3 and $t\in\pm\mathrm{O}(3)$ we get respectively $t_l=\pm[t_r,t_s]$, where $[x,y]$ is the *vector product* or *cross product* of two vectors $x,y\in\mathbb{R}^3$ and the triple $\{l,r,s\}$ denotes any cyclic permutations of $\{1,2,3\}$. For $t\in\mathrm{O}(3)$ and $l=1$ we get

$$t_1=[t_2,t_3]=\begin{vmatrix}i&j&k\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{vmatrix}=i\begin{vmatrix}t_{22}&t_{23}\\t_{32}&t_{33}\end{vmatrix}-j\begin{vmatrix}t_{21}&t_{23}\\t_{31}&t_{33}\end{vmatrix}+k\begin{vmatrix}t_{21}&t_{22}\\t_{31}&t_{32}\end{vmatrix}, \tag{3.42}$$

where i,j,k is the standard orthonormal basis in $\mathbb{R}^3$, i.e.,

$$i=(1,0,0),\quad j=(0,1,0),\quad k=(0,0,1).$$

Define X *formally* as the matrix:

$$X=\begin{pmatrix}i&j&k\\x_1&x_2&x_3\\y_1&y_2&y_3\end{pmatrix},\quad\text{then}\quad t_1=(t_{11},t_{12},t_{13})=\big(A_1^1(X),A_2^1(X),A_3^1(X)\big).$$

This proves (3.40) for $k=1$. For the other rows the proof is similar. ■

Remark 3.11. For $t\in\pm\mathrm{O}(n)$ we can prove a similar statement.

4. Irreducibility, the case $m=3$

Lemma 4.1. *If $\mu^{L_t}\perp\mu$ for all $t\in\mathrm{GL}(3,\mathbb{R})\setminus\{e\}$, we can approximate by the generators A_{kn} defined by (2.10) at least one of the following eight triplets of operators:*

$$(x_{1n},x_{2n},x_{3n}),\ (x_{1n},x_{2n},D_{3n}),\ (x_{1n},D_{2n},x_{3n}),\ (D_{1n},x_{2n},x_{3n}),$$
$$(x_{1n},D_{2n},D_{3n}),(D_{1n},x_{2n},D_{3n}),\ (D_{1n},D_{2n},x_{3n}),\ (D_{1n},D_{2n},D_{3n}).$$

Idea of the proof. By Lemma 3.7, the condition of orthogonality $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{SL}(3,\mathbb{R}) \setminus \{e\}$ are:

$$\Sigma^{\pm}(t) = \Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty, \tag{4.1}$$

where $\Sigma_2(t)$ is defined by (3.15) and $\Sigma_1^{\pm}(t)$ are defined by (3.17). Let $\mathfrak{A}^3$ be the von Neumann algebra generated by the representation.

Case 1/2. We write compactly Lemma 5.1 and Lemma 5.2 as follows, see Definition 4.7 for the notation η:

$$x_{rn}x_{rt}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta^{(r)} = \infty, \quad D_{rn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta_r = \infty, \tag{4.2}$$

$$\text{where } \Delta^{(r)} := \Delta(Y_r^{(r)}, Y_s^{(r)}, Y_t^{(r)}), \quad \Delta_r := \Delta(Y_r, Y_s, Y_t), \tag{4.3}$$

and $\{r, s, t\}$ is a cyclic permutation of $\{1, 2, 3\}$. Here

$$\|Y_s^{(r)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{rk}^2}{B_{3k}^2 - (b_{1k}^2 + b_{2k}^2 + b_{3k}^2 - b_{sk}^2)}, \quad 1 \le r, s \le 3, \tag{4.4}$$

$$B_{3k} = b_{1k} + b_{2k} + b_{3k}, \quad \text{and} \quad \|Y_r\|^2 = \sum_{k\in\mathbb{Z}} \frac{a_{rk}^2}{\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}}. \tag{4.5}$$

Case 3. *Approximation of* D_{rn} *by* $x_{rk}A_{kn}$ *and* D_{3n} *by* $\sin\big(s_k(x_{3k} - a_{3k})\big)A_{kn}$, (*respectively by* $\cos\big(s_k(x_{3k} - a_{3k})\big)A_{kn}$). By Lemma 5.3 and Lemma 5.4 we have

$$\begin{aligned} D_{rn}\ \eta\ \mathfrak{A}^3 \quad &\Leftrightarrow \quad \Delta(Y_{rr}, Y_{rs}, Y_{rt}) = \infty, \\ D_{3n}\ \eta\ \mathfrak{A}^3 \quad &\Leftrightarrow \quad \Sigma_3(D, s) = \infty, \quad \text{resp.} \quad \Sigma_3^{\vee}(D, s) = \infty, \end{aligned}$$

where Y_{kr} for $1 \le k, r \le 3$ are defined by (5.13) and $\Sigma_3(D, s)$ and $\Sigma_3^{\vee}(D, s)$ are defined by (5.18). The rest of this section is devoted to the proof of Lemma 4.1.

4.1. Notations and the change of the variables

In what follows we will systematically use the following notations:

$$S_r(3) = \sum_{n\in\mathbb{Z}} \frac{b_{rn}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}}, \quad 1 \le r \le 3, \tag{4.6}$$

$$\Sigma_r := \sum_{n\in\mathbb{Z}} \frac{b_{rn}}{b_{1n} + b_{2n} + b_{3n}}, \quad 1 \le r \le 3, \tag{4.7}$$

$$\Sigma^{rs} := \sum_{k\in\mathbb{Z}} \frac{b_{rk}}{b_{sk}}, \quad 1 \le r \ne s \le 3, \quad C_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}, \tag{4.8}$$

$$y_{123} = (y_1, y_2, y_3), \quad \text{where} \quad y_r := \|Y_r\|^2, \tag{4.9}$$

$$y^{(k)} = (y_1^{(k)}, y_2^{(k)}, y_3^{(k)}) := (\|Y_1^{(k)}\|^2, \|Y_2^{(k)}\|^2, \|Y_3^{(k)}\|^2), \quad 1 \le k \le 3, \tag{4.10}$$

$$y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \end{pmatrix} = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} & y_3^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} \\ y_1^{(3)} & y_2^{(3)} & y_3^{(3)} \end{pmatrix}, \quad \text{where} \quad y_s^{(r)} := \|Y_s^{(r)}\|^2, \tag{4.11}$$

$$\Sigma_{123}(s) = (\Sigma_{12}(s_{12}), \Sigma_{23}(s_{23}), \Sigma_{13}(s_{13})), \quad s = (s_{12}, s_{23}, s_{13}). \tag{4.12}$$

We show that $S_r(3)$ is infinite for at least one $1 \le r \le 3$.

Lemma 4.2. *We have*

$$S_1(3)+S_2(3)+S_3(3)=\infty, \tag{4.13}$$

$$\|Y_r^{(r)}\|^2\sim S_r(3) \quad for\ \ all \quad 1\le r\le 3, \tag{4.14}$$

$$\|Y_r^{(s)}\|^2<\frac{1}{2}S_r(3) \quad for\ \ all \quad 1\le r\neq s\le 3, \tag{4.15}$$

$$\|Y_1^{(i_1)}\|^2+\|Y_2^{(i_2)}\|^2+\|Y_3^{(i_3)}\|^2=\infty, \quad i_1,i_2,i_3\in\{1,2,3\}. \tag{4.16}$$

Proof. Since $3(a^2+b^2+c^2)\ge 2(ab+ac+bc)$ we get

$$S_1(3)+S_2(3)+S_3(3)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}^2+b_{2n}^2+b_{3n}^2}{b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n}}\ge\sum_{k\in\mathbb{Z}}2/3=\infty.$$

Further, by (4.4)

$$\|Y_r^{(r)}\|^2=\sum_{n\in\mathbb{Z}}\frac{b_{rn}^2}{b_{rn}^2+2(b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n})}\stackrel{(2.19)}{\sim}S_r(3),$$

$$\|Y_r^{(s)}\|^2=\sum_{n\in\mathbb{Z}}\frac{b_{rn}^2}{b_{sn}^2+2(b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n})}<\frac{1}{2}S_r(3),\ \ s\neq r.$$

To prove (4.16) we observe that by (4.4)

$$\|Y_1^{(i_1)}\|^2+\|Y_2^{(i_2)}\|^2+\|Y_3^{(i_3)}\|^2=$$
$$\sum_{r=1}^3\sum_{n\in\mathbb{Z}}\frac{b_{rn}^2}{b_{i_rn}^2+2(b_{1n}b_{2n}+b_{1n}b_{3n}+b_{2n}b_{3n})}>\sum_{n\in\mathbb{Z}}\frac{\sum_{r=1}^3 b_{rn}^2}{(\sum_{r=1}^3 b_{rn})^2}=\infty. \quad \blacksquare$$

We make the following change of the variables:

$$\begin{pmatrix} b_{1n} & b_{2n} & b_{3n}\\ a_{1n} & a_{2n} & a_{3n}\end{pmatrix}\to\begin{pmatrix} b'_{1n} & b'_{2n} & b'_{3n}\\ a'_{1n} & a'_{2n} & a'_{3n}\end{pmatrix}=\begin{pmatrix} 1 & d_{2n}:=\frac{b_{2n}}{b_{1n}} & d_{3n}:=\frac{b_{3n}}{b_{1n}}\\ a_{1n}\sqrt{b_{1n}} & a_{2n}\sqrt{b_{1n}} & a_{3n}\sqrt{b_{1n}}\end{pmatrix}, \tag{4.17}$$

motivated by the following formulas:

$$d\mu_{(b,a)}(x)=\sqrt{\frac{b}{\pi}}\exp(-b(x-a)^2)dx=\sqrt{\frac{1}{\pi}}\exp(-(x'-a')^2)dx'=d\mu_{(b',a')}(x'),$$

$$d\mu_{(b_2,a_2)}(x)=\sqrt{\frac{b_2}{\pi}}\exp(-b_2(x-a_2)^2)dx=\sqrt{\frac{b_2}{b_1\pi}}\exp\Big(-\frac{b_2}{b_1}(x'-a_2')^2\Big)dx'$$
$$=d\mu_{(b_2',a_2')}(x'),\quad (b',a')=(1,a\sqrt{b}),\quad (b_2',a_2')=\Big(b_2/b_1,a_2\sqrt{b_1}\Big).$$

Remark 4.3. All the expressions, given in the list (3.13) (3.14), (3.15) and (4.1) are invariant under the transformations (4.17)

$$S_{kr}^L(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{kn}}{2}\Big(\frac{1}{2b_{rn}}+a_{rn}^2\Big),\quad Y_r=\Big(a_{rk}\Big(\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}\Big)^{-1/2}\Big)_{k\in\mathbb{Z}},$$

etc., and $S_r(3)$ (as defined by (4.6)).

4.2. Approximation scheme

Remark 4.4. In what follows if some expression $<\infty$ (resp. $=\infty$), we denote this case by 0 (respectively, by 1).

We use the following notation $S := (S_1(3), S_2(3), S_3(3))$. By Lemma 4.2 we get $\sum_{r=1}^3 S_r(3) = \infty$. Therefore, without loss of generality, it suffices to consider the following three cases:

$$(1)\ \ S = (0,0,1), \quad (2)\ \ S = (0,1,1), \quad (3)\ \ S = (1,1,1). \tag{4.18}$$

By Lemma 3.7, the condition of orthogonality $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{SL}(3,\mathbb{R})\backslash\{e\}$, i.e., $\Sigma^{\pm}(t) = \Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty$, splits into two cases:

$$\begin{array}{llll} (A) & \Sigma_1^{\pm}(t) = \infty, & & \Sigma_1^{\pm}(t) = \sum_{1\leq i<j\leq 3} \Sigma_{ij}^{\pm}(t), \\ (B) & \Sigma_1^{\pm}(t) < \infty, & \text{but} & \Sigma_2(t) = \infty, \end{array} \tag{4.19}$$

where $\Sigma_1^{\pm}(t)$, $\Sigma_{ij}^{\pm}(t)$ and $\Sigma_2(t)$ are defined by (3.17), (3.18) and (3.15).

4.3. Case $S = (0,0,1)$

Lemma 4.5. *The case $S = (0,0,1)$ is equivalent with*

$$\Sigma^{13} + \Sigma^{23} < \infty, \quad S_3(3) \sim \sum_n \frac{b_{3n}^2}{b_{1n} b_{2n}} = \infty. \tag{4.20}$$

Proof. To prove the first part of (4.20) we set $c_n = \dfrac{b_{3n}}{b_{1n} + b_{2n}}$ and note that

$$\infty > S_1(3) + S_2(3) = \sum_{n\in\mathbb{Z}} \frac{b_{1n}^2 + b_{2n}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}} \overset{(2.19)}{\sim}$$

$$\sum_{n\in\mathbb{Z}} \frac{b_{1n}^2 + b_{2n}^2}{(b_{1n} + b_{2n} + b_{3n})^2 - b_{3n}^2} \sim \sum_{n\in\mathbb{Z}} \frac{(b_{1n} + b_{2n})^2}{(b_{1n} + b_{2n} + b_{3n})^2 - b_{3n}^2} =$$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(1+c_n)^2 - c_n^2} = \sum_{n\in\mathbb{Z}} \frac{1}{1+2c_n} \overset{(2.19)}{\sim} \sum_{n\in\mathbb{Z}} \frac{1}{c_n} = \sum_{n\in\mathbb{Z}} \frac{b_{1n} + b_{2n}}{b_{3n}} = \Sigma^{13} + \Sigma^{23}.$$

To prove the second part of (4.20) we have by the first part of (4.20)

$$S_3(3) = \sum_{n\in\mathbb{Z}} \frac{b_{3n}^2}{b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n}} = \sum_{n\in\mathbb{Z}} \frac{1}{\frac{b_{1n}b_{2n}}{b_{3n}^2} + \frac{b_{1n}}{b_{3n}} + \frac{b_{2n}}{b_{3n}}} \sim \sum_{n\in\mathbb{Z}} \frac{b_{3n}^2}{b_{1n}b_{2n}}. \qquad \blacksquare$$

Lemma 4.6 ([27]). *For any $k \in \mathbb{Z}$ we have*

$$x_{1k}\mathbf{1} \in \langle x_{1k}x_{1n}\mathbf{1} \mid n \in \mathbb{Z}\rangle \Leftrightarrow S_{11}^L(\mu) = \infty.$$

Definition 4.7. A non necessarily bounded self-adjoint operator A in a Hilbert space H, is said to be *affiliated* with a von Neumann algebra M of operators in H if $e^{itA} \in M$ for all $t \in \mathbb{R}$. This is denoted by $A\ \eta\ M$, see [7].

In the case $S = (0,0,1)$ we have

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) \sim \Delta(Y_3^{(3)}) \sim \|Y_3^{(3)}\|^2 = \infty,$$

so we can approximate $x_{3n}x_{3t}$ using Lemma 5.1 and after that we can approximate x_{3n} using an analogue of Lemma 4.6. *From now on we will say that we can approximate x_{3n} using Lemma 5.1, without mentioning Lemma 4.6.*

We can not approximate x_{1n} and x_{2n} using Lemma 5.1, since we have

$$\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})+\Delta(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})<\infty.$$

We can try to approximate some of D_{rn} for $1\leq r\leq 3$ using Lemma 5.2, see Section 4.4.4 for details. We have for $1\leq k\leq 3$ (see (4.3)):

$$D_{kn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta_k=\infty, \quad \text{where} \quad \Delta_k:=\Delta(Y_k,Y_r,Y_s),$$

and $\{k,r,s\}$ is a cyclic permutation of $\{1,2,3\}$. Recall that by $\Sigma^{12}+\Sigma^{13}<\infty$ we get (see (4.5) for the expressions of $\|Y_r\|^2$, $1\leq r\leq 3$)

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{3n}^2. \tag{4.21}$$

By (4.20) we have $\Sigma^{13}+\Sigma^{23}<\infty$. We distinguish two cases:
(1) $\Sigma^{12}<\infty$, and **(2)** $\Sigma^{12}=\infty$.
In case (1), since $\Sigma^{12}+\Sigma^{13}<\infty$ we have

$$\infty=S^L_{1,23}(\mu,t,s)\overset{(3.28)}{=}\sum_{n\in\mathbb{Z}}\left[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}}+\frac{s^2}{4}\frac{b_{1n}}{b_{3n}}+\frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2\right]$$
$$\sim\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\big(-2a_{1n}+ta_{2n}+sa_{3n}\big)^2\overset{(4.21)}{\sim}\|C_1Y_1+C_2Y_2+C_3Y_3\|^2.$$

Finally, in the case (1) we can approximate all D_{rn}, $1\leq r\leq 3$ using Lemma 5.2 and Lemma 6.3, and the proof is finished. The case (2) can be divided into three cases, if necessary, we can choose an appropriate subsequence of $\left(\frac{b_{1n}}{b_{2n}}\right)_n$:

$$\lim_n\frac{b_{1n}}{b_{2n}}=\begin{cases}\text{(a)} & 0\\ \text{(b)} & b>0\\ \text{(c)} & \infty\end{cases}. \tag{4.22}$$

Case (c) is reduced to case (a) by exchanging (b_{2n},a_{2n}) with (b_{1n},a_{1n}). This exchange does not change the first condition in (4.20). In the cases (2.a) and (2.b), by (4.5) we obtain the following expressions for $\|Y_r\|^2$, $1\leq r\leq 3$:

$$\|Y_1\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{2b_{1n}a_{1n}^2}{1+\frac{b_{1n}}{b_{2n}}+\frac{b_{1n}}{b_{3n}}}\overset{\Sigma^{13}<\infty}{\sim}\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2,$$
$$\|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2,\quad \|Y_3\|^2=\sum_{n\in\mathbb{Z}}b_{1n}a_{3n}^2.$$

Since $\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2\sim S^L_{11}(\mu)=\infty$, we have in consequence four possibilities for $y_{23}:=(y_2,y_3)\in\{0,1\}^2$ as in (4.52), see Section 4.4.4:

	(1.0)	(1.1)	(1.2)	(1.3)
y_1	1	1	1	1
y_2	0	1	0	1
y_3	0	0	1	1

We just follow the instructions *given in Remark 4.17*. We note that the cases (1.0) and (1.1) can not occur since the following conditions are contradictory:

$$S_{13}^L(\mu) \overset{(3.27)}{=} \sum_{n\in\mathbb{Z}} \frac{b_{1n}}{2}\Big(\frac{1}{2b_{3n}}+a_{3n}^2\Big)=\infty,\ \ \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}} b_{1n}a_{3n}^2<\infty,\ \ \Sigma^{13}\overset{(4.20)}{<}\infty.$$

We have two cases (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.57) or (4.58) are divergent. We can approximate in these cases respectively D_{1n} and D_{3n}, see (4.54), and all D_{1n}, D_{2n}, D_{3n}, see (4.55). The proof of irreducibility is finished in both cases because we have $x_{3n},\ D_{3n}\ \eta\ \mathfrak{A}^3$ and the problem is reduced to the case $m=2$ [27], since $A_{kn}=\sum_{r=1}^3 x_{rk}D_{rn}-x_{3k}D_{3n}=\sum_{r=1}^2 x_{rk}D_{rn}$. If the opposite holds, we have two different cases (1.2.0) and (1.3.0). We try to approximate D_{3n} using Lemma 5.4. If one of the expressions $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent for some sequence $s=(s_k)_{k\in\mathbb{Z}}$, we can approximate D_{3k} and the proof is finished, since we have $x_{3n},\ D_{3n}\ \eta\ \mathfrak{A}^3$ and the problem is reduced to the case $m=2$. Let us suppose, as in Remark 4.21, that for every sequence $s=(s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_3(D,s)+\Sigma_3^\vee(D,s)<\infty.$$

Then, in particular, we have for $s^{(3)}=(s_k)_{k\in\mathbb{Z}}$ with $\frac{s_k^2}{b_{3k}}\equiv 1$

$$\begin{aligned}\infty&>\Sigma_3(D,s^{(3)})+\Sigma_3^\vee(D,s^{(3)})\sim\Sigma_3(D)+\Sigma_3^\vee(D)\\&=\sum_k\frac{\frac{1}{2b_{3k}}+a_{3k}^2}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}\overset{(2.19)}{\sim}\sum_k\frac{\frac{1}{2b_{3k}}+a_{3k}^2}{\frac{1}{2b_{1k}}+a_{1k}^2+\frac{1}{2b_{2k}}+a_{2k}^2}\\&=\sum_k\frac{\frac{b_{1k}}{b_{3k}}+2b_{1k}a_{3k}^2}{1+2b_{1k}a_{1k}^2+\frac{b_{1k}}{b_{2k}}+2b_{1k}a_{2k}^2}\overset{(4.22)}{\sim}\sum_k\frac{2b_{1k}a_{3k}^2}{1+2b_{1k}a_{1k}^2+2b_{1k}a_{2k}^2}=:\Sigma_3^+(D).\end{aligned}$$

Remark 4.8. Finally, we have $\Sigma_3^+(D)\sim\sum_k\frac{2a_{3k}^2}{1+2a_{1k}^2+2a_{2k}^2}$, since we take $b_{1n}\equiv 1$ by (4.17). In the case (1.2.0) we have $\|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2<\infty$, and therefore $\Sigma_3^+(D)\sim\sum_k\frac{2a_{3k}^2}{1+2a_{1k}^2}$, and hence $\Sigma_3^+(D)=\infty$ by Lemma 4.19. In the case (1.3.0) we have $a_3=\pm a_1\pm a_2+h$ or $a_3-h=\pm a_1\pm a_2$, see the proof of Lemma 4.20. Therefore,

$$\begin{aligned}\infty&>\Sigma_3^+(D)\sim\sum_k\frac{a_{3k}^2}{1+a_{1k}^2+a_{2k}^2}\geq\sum_k\frac{a_{3k}^2}{1+a_{1k}^2+2|a_{1k}||a_{2k}|+a_{2k}^2}\\&=\sum_k\frac{a_{3k}^2}{1+\big(|a_{1k}|+|a_{2k}|\big)^2},\quad\infty>\Sigma_3^+(D)\sim\sum_k\frac{a_{3k}^2}{1+a_{1k}^2+a_{2k}^2}\qquad(4.23)\\&\geq\sum_k\frac{a_{3k}^2}{1+a_{1k}^2+a_{2k}^2+(|a_{1k}|-|a_{2k}|)^2}\sim\sum_k\frac{a_{3k}^2}{1+2a_{1k}^2-2|a_{1k}||a_{2k}|+2a_{2k}^2}\\&\sim\sum_k\frac{a_{3k}^2}{1+a_{1k}^2-2|a_{1k}||a_{2k}|+a_{2k}^2}\sim\sum_k\frac{a_{3k}^2}{1+\big(|a_{1k}|-|a_{2k}|\big)^2}.\qquad(4.24)\end{aligned}$$

Hence, we have by (4.23) and (4.24)

$$\infty>\Sigma_3^+(D)\geq\sum_k\frac{a_{3k}^2}{1+\big(\pm a_{1k}\pm a_{2k}\big)^2}=\sum_k\frac{a_{3k}^2}{1+\big(a_{3k}-h_k\big)^2}=\infty\qquad(4.25)$$

by Lemma 4.19, this is a contradiction. Therefore, in both cases we can approximate D_{3n} and the proof is finished.

4.4. Case $S=(0,1,1)$

Lemma 4.9. *In the case* $S=(0,1,1)$ *we have*

$$\lim_n d_{2n}=\lim_n d_{3n}=\infty. \tag{4.26}$$

Proof. Setting as before $d_{rn}=b_{rn}/b_{1n}$, we obtain by (4.6) and (2.19)

$$S_1(3)=\sum_{n\in\mathbb{Z}}\frac{1}{d_{2n}+d_{3n}+d_{2n}d_{3n}}\overset{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}\frac{1}{(1+d_{2n})(1+d_{3n})}<\infty, \tag{4.27}$$

$$S_2(3)=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}}\overset{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{(1+d_{2n})(d_{2n}+d_{3n})}=\infty, \tag{4.28}$$

$$S_3(3)=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}}\overset{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{(1+d_{3n})(d_{2n}+d_{3n})}=\infty. \tag{4.29}$$

Suppose that $d_{2n}\le C$ for all $n\in\mathbb{Z}$. Then by (4.27) and (4.28) we conclude

$$S_1(3)\sim\sum_{n\in\mathbb{Z}}\frac{1}{(1+d_{2n})(1+d_{3n})}\sim\sum_{n\in\mathbb{Z}}\frac{1}{1+d_{3n}}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_{3n}}<\infty,\quad \infty=S_2(3)$$
$$\sim\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{(1+d_{2n})(d_{2n}+d_{3n})}\sim\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}+d_{3n}}\le\sum_{n\in\mathbb{Z}}\frac{C^2}{C+d_{3n}}\overset{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}\frac{1}{d_{3n}}<\infty,$$

which is a contradiction. We use the fact that for any fixed $D>0$ the function $f_D(x)=\frac{x^2}{x+D}$ is strictly increasing when $x>0$. Similarly, if we suppose that $d_{3n}\le C$ for all $n\in\mathbb{Z}$ we will obtain a contradiction too. ■

Lemma 4.10. *The case* $S=(0,1,1)$ *is equivalent with*

$$S_1(3)\sim\sum_n\frac{b_{1n}^2}{b_{2n}b_{3n}}<\infty,\quad S_2(3)\sim\sum_n\frac{1}{d_n}=\infty,\quad S_3(3)\sim\sum_n d_n=\infty. \tag{4.30}$$

Proof. Recall that $d_n=\dfrac{d_{3n}}{d_{2n}}$. Denote $D_n:=1+d_{2n}^{-1}+d_{3n}^{-1}$. By Lemma 4.9 we have

$$1\le D_n=1+d_{2n}^{-1}+d_{3n}^{-1}\le C,\quad \text{for all}\quad n\in\mathbb{Z}. \tag{4.31}$$

Therefore, we get

$$S_1(3)=\sum_{n\in\mathbb{Z}}\frac{1}{d_{2n}+d_{3n}+d_{2n}d_{3n}}=\sum_{n\in\mathbb{Z}}\frac{1}{D_nd_{2n}d_{3n}}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_{2n}d_{3n}}=\sum_n\frac{b_{1n}^2}{b_{2n}b_{3n}},$$

$$S_2(3)=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}}=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{D_nd_{2n}d_{3n}}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_n},$$

$$S_3(3)=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{2n}+d_{3n}+d_{2n}d_{3n}}=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{D_nd_{2n}d_{3n}}\sim\sum_{n\in\mathbb{Z}}d_n.$$ ■

By Lemma 4.2, (4.15) we get $\|Y_1^{(r)}\|^2<\infty,\ 1\le r\le 3$, therefore, we get

Lemma 4.11. *In the case $S=(0,1,1)$ we have*

$$\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})<\infty,\quad \Delta(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})\sim\Delta(Y_2^{(2)},Y_3^{(2)}),$$
$$\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})\sim\Delta(Y_3^{(3)},Y_2^{(3)}). \tag{4.32}$$

Proof. Set $(f_1,f_2,f_3)=(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})$. Then

$$\Delta(f_1,f_2,f_3)\overset{(2.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}$$
$$>\frac{\Gamma(f_1)+\Gamma(f_1,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\overset{(4.33)}{\geq}\frac{\Gamma(f_1)+\Gamma(f_1,f_3)}{(1+\Gamma(f_2))(1+\Gamma(f_3))}\sim\Delta(f_1,f_3),$$

since $f_2\in l_2(\mathbb{Z})$, see (2.14). Indeed, for $f,g\in l_2(\mathbb{Z})$ and $f\in l_2(\mathbb{Z}),g\notin l_2(\mathbb{Z})$ we have respectively

$$\Gamma(f,g)\leq\Gamma(f)\Gamma(g)<\infty,\ \ \Gamma(f,g)\leq\Gamma(f)\Gamma(g),\ \text{ where }\ \Gamma(f,g), \tag{4.33}$$
$$\Gamma(g)\quad\text{are defined by}\quad\Gamma(f,g):=\lim_n\Gamma(f_{(n)},g_{(n)})\quad\Gamma(g):=\lim_n\Gamma(g_{(n)}),$$

and $g_{(n)}:=(g_k)_{k=-n}^n\in\mathbb{R}^{2n+1}$. Similarly, set $(f_1,f_2,f_3)=(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)})$, then

$$\Delta(f_1,f_2,f_3)\overset{(2.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}$$
$$>\frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}\overset{(4.33)}{\geq}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)}{(1+\Gamma(f_2))(1+\Gamma(f_3))}\sim\Delta(f_1,f_2),$$

since $f_3\in l_2(\mathbb{Z})$. Finally, we derive both equivalences in (4.32). To prove $\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})<\infty$ we set $(f_1,f_2,f_3)=(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)})$, and note that

$$\Delta(f_1,f_2,f_3)\overset{(2.15)}{=}\frac{\Gamma(f_1)+\Gamma(f_1,f_2)+\Gamma(f_1,f_3)+\Gamma(f_1,f_2,f_3)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}$$
$$\leq\frac{\Gamma(f_1)\Big(1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)\Big)}{1+\Gamma(f_2)+\Gamma(f_3)+\Gamma(f_2,f_3)}=\Gamma(f_1)<\infty.$$ ■

In order to approximate x_{2n} or x_{3n}, it remains to study the case

$$\Delta(Y_2^{(2)},Y_3^{(2)})=\infty,\quad\Delta(Y_3^{(3)},Y_2^{(3)})=\infty, \tag{4.34}$$

where $\Delta(f_1,f_2)=\dfrac{\Gamma(f_1)+\Gamma(f_1,f_2)}{1+\Gamma(f_2)}$. For $2\leq r\leq 3$, denote

$$\rho_r(C_2,C_3):=\|C_2Y_2^{(r)}+C_3Y_3^{(r)}\|^2,\quad(C_2,C_3)\in\mathbb{R}^2, \tag{4.35}$$
$$\nu(C_1,C_2,C_3):=\|C_1Y_1+C_2Y_2+C_3Y_3\|^2,\ (C_1,C_2,C_3)\in\mathbb{R}^3. \tag{4.36}$$

Lemma 4.12. *In the case $S=(0,1,1)$ we have*

$$\rho_2(C_2,C_3)\sim\sum_{n\in\mathbb{Z}}\frac{\big(C_2+C_3d_n\big)^2}{1+2d_n},\ \rho_3(C_2,C_3)\sim\sum_{n\in\mathbb{Z}}\frac{\big(C_2+C_3d_n\big)^2}{d_n^2+2d_n} \tag{4.37}$$
$$=\sum_{n\in\mathbb{Z}}\frac{\big(C_2l_n+C_3\big)^2}{1+2l_n},\quad\nu(C_1,C_2,C_3)\sim\sum_{n\in\mathbb{Z}}b_{1n}\Big(\sum_{r=1}^3C_ra_{rn}\Big)^2. \tag{4.38}$$

Proof. Set as before $d_n=\frac{d_{3n}}{d_{2n}}$. By (4.4) and (4.5) we get

$$\|Y_2^{(2)}\|^2=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}^2+2(d_{2n}+d_{3n}+d_{2n}d_{3n})}=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}^2+2D_nd_{2n}d_{3n}}\sim\sum_{n\in\mathbb{Z}}\frac{1}{D_nd_n},$$

$$\|Y_3^{(2)}\|^2=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{2n}^2+2(d_{2n}+d_{3n}+d_{2n}d_{3n})}=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{2n}^2+2D_nd_{2n}d_{3n}}=\sum_{n\in\mathbb{Z}}\frac{d_n^2}{1+2D_nd_n},$$

$$\|Y_2^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{3n}^2+2(d_{2n}+d_{3n}+d_{2n}d_{3n})}=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{3n}^2+2D_nd_{2n}d_{3n}}=\sum_{n\in\mathbb{Z}}\frac{1}{d_n^2+2D_nd_n},$$

$$\|Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{3n}^2+2(d_{2n}+d_{3n}+d_{2n}d_{3n})}=\sum_{n\in\mathbb{Z}}\frac{d_{3n}^2}{d_{3n}^2+2D_nd_{2n}d_{3n}}\sim\sum_{n\in\mathbb{Z}}\frac{d_n}{D_n},\tag{4.39}$$

$$\|Y_1\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{2b_{1n}a_{1n}^2}{1+d_{2n}^{-1}+d_{3n}^{-1}}=\sum_{n\in\mathbb{Z}}\frac{2b_{1n}a_{1n}^2}{D_n},\tag{4.40}$$

$$\|Y_2\|^2=\sum_{n\in\mathbb{Z}}\frac{2b_{1n}a_{2n}^2}{D_n},\quad \|Y_3\|^2=\sum_{n\in\mathbb{Z}}\frac{b_{1n}a_{3n}^2}{D_n}.$$

Recall that $d_{rn}=\frac{b_{rn}}{b_{1n}}$. By (4.31), we obtain

$$\|Y_2^{(2)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{1}{1+2d_n}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_n},\quad \|Y_3^{(2)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{d_n^2}{1+2d_n},\tag{4.41}$$

$$\|Y_2^{(3)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_n^2+2d_n},\quad \|Y_3^{(3)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{d_n^2}{d_n^2+2d_n}\sim\sum_{n\in\mathbb{Z}}d_n,$$

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2,\ \|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{2n}^2,\ \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a_{3n}^2,$$

$$\|C_1Y_1+C_2Y_2+C_3Y_3\|^2\overset{(4.31)}{\sim}\sum_{n\in\mathbb{Z}}b_{1n}\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2.\tag{4.42}$$

By (4.41) and (4.42) the proof is finished. ■

4.4.1. Approximation of x_{2n}, x_{3n}

To approximate x_{2n}, x_{3n}, we need several lemmas. Denote $l_n=d_n^{-1}$.

Lemma 4.13. *The following five series are equivalent:*

(i–ii) $$\sum_{n\in\mathbb{Z}}\frac{\big(C_2-C_3d_n\big)^2}{1+2d_n}\sim\sum_{n\in\mathbb{Z}}c_n^2,\tag{4.43}$$

(iii–iv) $$\sum_{n\in\mathbb{Z}}\frac{\big(C_2l_n-C_3\big)^2}{1+2l_n}\sim\sum_{n\in\mathbb{Z}}e_n^2,\tag{4.44}$$

(v) $$\Sigma_{23}(s)=\sum_{n\in\mathbb{Z}}\left(s^2\sqrt{\frac{b_{2n}}{b_{3n}}}-s^{-2}\sqrt{\frac{b_{3n}}{b_{2n}}}\right)^2=\sum_{n\in\mathbb{Z}}\left(\frac{s^2}{\sqrt{d_n}}-\frac{\sqrt{d_n}}{s^2}\right)^2,\tag{4.45}$$

where

$$d_n=C_2C_3^{-1}(1+c_n),\ \ l_n=C_3C_2^{-1}(1+e_n),\ \ s^4=C_2C_3^{-1}>0,\quad l_n=d_n^{-1}.\tag{4.46}$$

Proof. To prove (4.43) and (4.44) we get by Lemma 2.10 using (4.46)

$$\sum_{n\in\mathbb{Z}}\frac{\big(C_2-C_3d_n\big)^2}{1+2d_n}=\sum_{n\in\mathbb{Z}}\frac{C_2^2c_n^2}{1+2C_2C_3^{-1}(1+c_n)}\sim\sum_{n\in\mathbb{Z}}c_n^2,$$
$$\sum_{n\in\mathbb{Z}}\frac{\big(C_2l_n-C_3\big)^2}{1+2l_n}=\sum_{n\in\mathbb{Z}}\frac{C_3^2e_n^2}{1+2C_3C_2^{-1}(1+e_n)}\sim\sum_{n\in\mathbb{Z}}e_n^2.$$

To finish the proof we make use of the following lemma ■

Lemma 4.14. *Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers with $1+c_n>0$ and $(1+c_n)(1+e_n)=1$. Then the following three series are equivalent:*

$$\sum_{n\in\mathbb{Z}}\big((1+e_n)^{1/2}-(1+e_n)^{-1/2}\big)^2,\quad \sum_{n\in\mathbb{Z}}c_n^2\quad \textit{and}\quad \sum_{n\in\mathbb{Z}}e_n^2.$$

Proof. Set $s^4=C_2C_3^{-1}$, replacing $1+c_n$ by $(1+e_n)^{-1}$ in Lemma 6.7 gives

$$\Sigma_{23}(s)=\sum_{n\in\mathbb{Z}}\big((1+c_n)^{-1/2}-(1+c_n)^{1/2}\big)^2=\sum_{n\in\mathbb{Z}}\big((1+e_n)^{1/2}-(1+e_n)^{-1/2}\big)^2.$$

Therefore, $\sum_{n\in\mathbb{Z}}\frac{c_n^2}{1+c_n}=\sum_{n\in\mathbb{Z}}\frac{e_n^2}{1+e_n}$ and hence, by Lemma 2.10, the two series are equivalent: $\sum_{n\in\mathbb{Z}}c_n^2\sim\sum_{n\in\mathbb{Z}}e_n^2$. ■

4.4.2. Two remaining possibilities

By Lemma 4.13 there are only two cases:

(1) when $\rho_2(C_2,C_3)=\rho_3(C_2,C_3)=\infty$ for all $(C_2,C_3)\in\mathbb{R}^2\setminus\{0\}$,

(2) when both $\rho_2(C_2,C_3)$ and $\rho_3(C_2,C_3)$ are finite and hence, $\Sigma_{23}(s)<\infty$.

To illustrate this we start with the following example

Example 4.15. Set $d_n=n^\alpha$ for $n\in\mathbb{N}$ with $\alpha\in\mathbb{R}$. We have

$$\lim_n d_n=\begin{cases}\infty & \text{if}\quad \alpha>0;\\ 1 & \text{if}\quad \alpha=0;\\ 0 & \text{if}\quad \alpha<0.\end{cases}\tag{4.47}$$

For the general sequence $(d_n)_{n\in\mathbb{Z}}$ we have four cases (if necessary, we can chose an appropriate subsequence):

$$\lim_n d_n=\begin{cases}\text{(a)} & \infty\\ \text{(b)} & d>0\quad\text{with}\quad \sum_n c_n^2=\infty\\ \text{(c)} & d>0\quad\text{with}\quad \sum_n c_n^2<\infty\\ \text{(d)} & d=0\end{cases}\tag{4.48}$$

where $d_n=d(1+c_n)$ and $\lim_n c_n=0$.

4.4.3. Cases (a), (b), (d)

Remark 4.16. In the case (a) we see by (4.37) that

$$\rho_2(C_2,C_3)=\rho_3(C_2,C_3)=\infty\quad\text{for all}\quad (C_2,C_3)\in\mathbb{R}^2\setminus\{0\}.$$

The case (d) is reduced to the case (a) by exchanging (b_{2n}, a_{2n}) with (b_{3n}, a_{3n}). In case (b) by Lemma 4.13 and (4.48) we conclude that

$$\rho_2(C_2, -C_3) = \rho_3(C_2, -C_3) = \infty \quad \text{for} \quad C_2C_3^{-1} > 0.$$

Hence, $\rho_2(C_2, C_3) = \rho_3(C_2, C_3) = \infty$ for all $(C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}$. Therefore, in cases (a), (b) and (d) we get $x_{2n},\ x_{3n}\, \eta\ \mathfrak{A}^3$.

To finish the proof in these cases, it is sufficient to approximate one of the operators $D_{rn},\ 1 \le r \le 3$ by the operators $(A_{kn})_{k\in\mathbb{Z}}$ using Lemmas 5.2, see Section 4.4.4. Alternatively we can try to approximate $D_{3n},\ D_{2n}$ using Lemma 5.4 and its analogue, see Section 4.4.5.

Note that by Lemma 4.9 we have $\lim_n b_{2n} = \lim_n b_{3n} = \infty$. In the cases (a) and (b) the conditions (4.30) are expressed by (4.48) as follows:

$$b = (1, b_{2n}, d_n b_{2n}),\ \sum_n \frac{1}{b_{2n}^2 d_n} < \infty,\ \sum_n \frac{1}{d_n} = \infty,\ \lim_n d_n = \infty, \tag{4.49}$$

$$b = \big(1, b_{2n}, d b_{2n}(1 + c_n)\big), \quad \sum_n \frac{1}{b_{2n}^2} < \infty, \quad \sum_n c_n^2 = \infty. \tag{4.50}$$

Indeed, to get (4.49) we observe that (4.30) are expressed as follows:

$$S_1(3) \sim \sum_n \frac{1}{b_{2n}b_{3n}} = \sum_n \frac{1}{b_{2n}^2 d_n} < \infty, \quad S_2(3) \sim \sum_n \frac{1}{d_n} = \infty.$$

The condition $S_3(3) \sim \sum_n d_n = \infty$ holds by $\lim_n d_n = \infty$.

In order to get (4.50), we express the conditions (4.30) as follows:

$$S_1(3) \sim \sum_n \frac{1}{b_{2n} d b_{2n}(1 + c_n)} \sim \sum_n \frac{1}{b_{2n}^2} < \infty,$$

$$S_2(3) \sim \sum_n \frac{1}{d_n} = \sum_n \frac{1}{1 + c_n} = \infty, \quad S_3(3) \sim \sum_n d_n = \sum_n (1 + c_n) = \infty.$$

The condition $S_2(3) = \infty$ holds by $\lim_n c_n = 0$.

4.4.4. Approximation of $D_{rn},\ 1 \le r \le 3$

By Lemma 5.2 we have for $1 \le k \le 3$ (see (4.3)):

$$D_{kn}\ \eta\ \mathfrak{A}^3 \Leftrightarrow \Delta_k = \infty, \quad \text{where} \quad \Delta_k := \Delta(Y_k, Y_r, Y_s),$$

and $\{k, r, s\}$ is a cyclic permutation of $\{1, 2, 3\}$. Recall that by (4.40)

$$\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2, \quad \|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{2n}^2, \quad \|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{3n}^2. \tag{4.51}$$

Since $\|Y_1\|^2 \sim \sum_{n\in\mathbb{Z}} b_{1n} a_{1n}^2 \sim S_{11}^L(\mu) = \infty$, we have in consequence four possibilities for $y_{23} := (y_2, y_3) \in \{0, 1\}^2$:

$$\begin{array}{ccccc} & (1.0) & (1.1) & (1.2) & (1.3) \\ y_1 & 1 & 1 & 1 & 1 \\ y_2 & 0 & 1 & 0 & 1 \\ y_3 & 0 & 0 & 1 & 1 \end{array} \tag{4.52}$$

In the case (1.0) we have $\Delta(Y_1,Y_2,Y_3)\sim\|Y_1\|^2=\infty$, so we can approximate D_{1n} using Lemma 5.3 and the proof is finished. We should consider the three following cases: (1.1), (1.2), (1.3). In the cases (1.1), (1.2) and (1.3) we have respectively (see the proof of Lemma 4.11)

$$\Delta(Y_1,Y_2,Y_3)\sim\Delta(Y_1,Y_2),\quad \Delta(Y_2,Y_3,Y_1)\sim\Delta(Y_2,Y_1), \tag{4.53}$$
$$\Delta(Y_1,Y_2,Y_3)\sim\Delta(Y_1,Y_3),\quad \Delta(Y_3,Y_1,Y_2)\sim\Delta(Y_3,Y_1), \tag{4.54}$$
$$\Delta(Y_1,Y_2,Y_3),\quad \Delta(Y_2,Y_3,Y_1),\quad \Delta(Y_3,Y_1,Y_2). \tag{4.55}$$

By (4.40) and Lemma 4.9 we have respectively in the cases (1.1)–(1.3):

$$\nu_{12}(C_1,C_2):=\|C_1Y_1+C_2Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}\Big(C_1a_{1n}+C_2a_{2n}\Big)^2, \tag{4.56}$$

$$\nu_{13}(C_1,C_3):=\|C_1Y_1+C_3Y_3\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}\Big(C_1a_{1n}+C_3a_{3n}\Big)^2, \tag{4.57}$$

$$\nu(C_1,C_2,C_3)=\sum_{n\in\mathbb{Z}}b_{1n}\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2. \tag{4.58}$$

Remark 4.17. We have three cases (1.1.1), (1.2.1) and (1.3.1) according to whether respectively the expressions in (4.56), (4.57) or (4.58) are divergent. We can approximate in these cases respectively D_{1n} and D_{2n} in (4.53), D_{1n} and D_{3n} in (4.54) all D_{1n}, D_{2n}, D_{3n} in (4.55). The proof of irreducibility is finished in these cases because we have $D_{rn},\ x_{2n},\ x_{3n}\eta\,\mathfrak{A}^3$ for some $1\le r\le 3$. If the opposite holds, we have three different cases:

$$\begin{aligned}
&(1.1.0)\quad \|C_1Y_1+C_2Y_2\|<\infty \quad\text{for some}\quad (C_1,C_2)\in\mathbb{R}^2\setminus\{0\},\\
&(1.2.0)\quad \|C_1Y_1+C_3Y_3\|<\infty \quad\text{for some}\quad (C_1,C_3)\in\mathbb{R}^2\setminus\{0\},\\
&(1.3.0)\quad \nu(C_1,C_2,C_3)<\infty \quad\text{for some}\quad (C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}.
\end{aligned}$$

Recall that by (3.28) we have

$$S^L_{1,23}(\mu,t,s)=\sum_{n\in\mathbb{Z}}\Big[\frac{t^2}{4}\frac{b_{1n}}{b_{2n}}+\frac{s^2}{4}\frac{b_{1n}}{b_{3n}}+\frac{b_{1n}}{2}\left(-2a_{1n}+ta_{2n}+sa_{3n}\right)^2\Big].$$

Remark 4.18. In the case (1.1.0) we have $\Sigma^{12}=\sum_{n\in\mathbb{Z}}\dfrac{b_{1n}}{b_{2n}}=\infty$, because of $S^L_{1,23}(\mu,t,0)=\infty$, but $\nu_{12}(C_1,C_2)<\infty$, and $\Sigma^{13}=\infty$, since

$$S^L_{13}(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\left(\frac{1}{2b_{3n}}+a^2_{3n}\right)=\infty,$$

but $\|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a^2_{3n}<\infty$; see (3.27) for the definition of $S^L_{kr}(\mu)$.

In the case (1.2.0) we conclude that $\Sigma^{13}=\infty$, since $S^L_{1,23}(\mu,0,s)=\infty$, but $\nu_{13}(C_1,C_3)<\infty$, and $\Sigma^{12}=\infty$, since

$$S^L_{12}(\mu)=\sum_{n\in\mathbb{Z}}\frac{b_{1n}}{2}\left(\frac{1}{2b_{2n}}+a^2_{2n}\right)=\infty,$$

but $\|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}b_{1n}a^2_{2n}<\infty$.

In the case (1.3.0) we have $\Sigma^{12}=\Sigma^{13}=\infty$, since $S^L_{1,23}(\mu,t,s)=\infty$, but we have

$$\nu(C_1,C_2,C_3)\sim\textstyle\sum_{n\in\mathbb{Z}}b_{1n}\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2<\infty.$$

So, it remains to consider only the three following cases, when $\Sigma^{12}=\Sigma^{13}=\infty$:

$$(1.1.0)\ \ (1.2.0)\ \ (1.3.0)$$

By Lemma 5.3 we have

$$D_{2n}\,\eta\,\mathfrak{A}^3 \;\Leftrightarrow\; \Delta(Y_{22},Y_{23},Y_{21})=\infty, \quad D_{3n}\,\eta\,\mathfrak{A}^3 \;\Leftrightarrow\; \Delta(Y_{33},Y_{31},Y_{32})=\infty,$$

where the vectors Y_{rs} for $2\le r\le 3$, $1\le s\le 3$ are defined by (5.12)–(5.13). We can not prove that $\Delta(Y_{22},Y_{23},Y_{21})=\infty$ or $\Delta(Y_{33},Y_{31},Y_{32})=\infty$. Therefore, in order to approximate D_{3n} we are forced to prove Lemma 5.4 and its analogue for D_{2n}, see Remark 4.21 below.

4.4.5. Two technical lemmas

Lemma 4.19. *Let $a_1,a_2\notin l_2(\mathbb{Z})$ and $C_1a_1+C_2a_2\in l_2(\mathbb{Z})$ for some $(C_1,C_2)\in\mathbb{R}^2\setminus\{0\}$, where $a_r=(a_{rk})_{k\in\mathbb{Z}}$, $1\le r\le 2$. Then we have*

$$\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+a_{2k}^2}=\infty. \tag{4.59}$$

Proof. We set $Y_r=a_r$, in the case (1.1.0) when $C_1Y_1+C_2Y_2=h\in l_2(\mathbb{Z})$ with $C_1C_2>0$ (we have $C_1C_2\neq 0$) we should take $a_2=-a_1+h$, in the case when $C_1C_2<0$ we take $a_2=a_1+h$. The series $\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+a_{2k}^2}$ will remain equivalent with the initial one, if we replace (C_1,C_2) with $(\pm1,1)$ in the expression for h. Fix a small $\varepsilon>0$ and a large $N\in\mathbb{N}$. Since $|\pm a+b|\le|a|+|b|$, we get

$$\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+a_{2k}^2}=\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+\big(\pm a_{1k}+h_k\big)^2}\ge\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+a_{1k}^2+2|a_{1k}||h_k|+h_k^2}$$
$$\overset{(2.19)}{\sim}\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2}{1+2|a_{1k}||h_k|+h_k^2}\overset{(*)}{>}\sum_{k\in\mathbb{Z}_N}\frac{a_{1k}^2}{1+2|a_{1k}|\varepsilon+\varepsilon^2}\overset{(2.17)}{\sim}\sum_{k\in\mathbb{Z}_N}a_{1k}^2=\infty,$$

where $\mathbb{Z}_N:=\{n\in\mathbb{Z}\mid |n|>N\}$. The inequality (*) holds, since $h\in l_2$ and we have $\sum_{k\in\mathbb{Z}_N}h_k^2<\varepsilon^2$ for sufficiently large $N\in\mathbb{N}$. ∎

Lemma 4.20. *Let $a_1,a_2,a_3\notin l_2(\mathbb{Z})$ and $C_1a_1+C_2a_2+C_3a_3\in l_2(\mathbb{Z})$ for some $(C_1,C_2,C_3)\in\mathbb{R}^3$, $C_3\neq 0$, where $a_r=(a_{rk})_{k\in\mathbb{Z}}$ for $1\le r\le 3$. Then we have*

$$\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2+a_{2k}^2}{1+a_{3k}^2}=\infty. \tag{4.60}$$

Proof. We set $Y_r=a_r$, in the case (1.3.0), we have $C_1a_1+C_2a_2+C_3a_3=h\in l_2(\mathbb{Z})$ for some $(C_1,C_2,C_3)\in\mathbb{R}^3$, see Remark 4.17. We can take $C_3=1$, then we have $a_3=-C_1a_1-C_2a_2+h$. When $C_1=0$ or $C_2=0$ lemma is reduced to Lemma 4.19. Suppose $C_1C_2\neq 0$. The series $\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2+a_{2k}^2}{1+a_{3k}^2}$ will remain equivalent with the initial one, if we replace (C_1,C_2,C_3) with $(\pm1,\pm1,1)$ in the expression for h. Fix a small $\varepsilon>0$ and a large $N\in\mathbb{N}$. Suppose the opposite, i.e.,

$$\infty>\sum_{k\in\mathbb{Z}}\frac{a_{1k}^2+a_{2k}^2}{1+\big(\pm a_{1k}\pm a_{2k}+h_k\big)^2},\ \text{ then }\ \infty>\sum_{k\in\mathbb{Z}}\frac{(|a_{1k}|+|a_{2k}|)^2}{1+\big(\pm a_{1k}\pm a_{2k}+h_k\big)^2}$$
$$\ge\sum_{k\in\mathbb{Z}}\frac{(|a_{1k}|+|a_{2k}|)^2}{1+a_{1k}^2+a_{2k}^2+2|a_{1k}||a_{2k}|+2|a_{1k}||h_k|+2|a_{2k}||h_k|+h_k^2}$$

$$\overset{(2.19)}{\sim}\sum_{k\in\mathbb{Z}}\frac{(|a_{1k}|+|a_{2k}|)^2}{1+2|a_{1k}||h_k|+2|a_{2k}||h_k|+h_k^2}\overset{(*)}{>}\sum_{k\in\mathbb{Z}_N}\frac{(|a_{1k}|+|a_{2k}|)^2}{1+2\big(|a_{1k}|+|a_{2k}|\big)\varepsilon+\varepsilon^2}$$
$$\overset{(2.17)}{\sim}\sum_{k\in\mathbb{Z}_N}(|a_{1k}|+|a_{2k}|)^2=\infty,$$

where $\mathbb{Z}_N := \{n \in \mathbb{Z} \mid |n| > N\}$, contradiction. The inequality (*) holds, since $h \in l_2(\mathbb{Z})$ and we have $\sum_{k\in\mathbb{Z}_N} h_k^2 < \varepsilon^2$ for sufficiently large $N \in \mathbb{N}$. ∎

Remark 4.21. It is possible to prove an analogue of Lemma 5.4 to approximate D_{2n} with corresponding expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$ and $\Sigma_3(D)$, $\Sigma_3^\vee(D)$. If one of the expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$, $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent for some sequence $s = (s_k)_{k\in\mathbb{Z}}$, we can approximate D_{2k} or D_{3k} and the proof is finished, when $S = (0,1,1)$ in the cases (a) and (b). Suppose that for all sequence $s = (s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_2(D,s)+\Sigma_2^\vee(D,s)+\Sigma_3(D,s)+\Sigma_3^\vee(D,s)<\infty.$$

Then, in particularly, we have for $s^{(r)} = (s_{rk})_{k\in\mathbb{Z}}$, $2 \le r \le 3$ with $\frac{s_{rk}^2}{b_{rk}} \equiv 1$

$$\begin{aligned}\infty &> \Sigma_2(D,s^{(2)})+\Sigma_2^\vee(D,s^{(2)})+\Sigma_3(D,s^{(3)})+\Sigma_3^\vee(D,s^{(3)})\\ &\sim \Sigma_2(D)+\Sigma_2^\vee(D)+\Sigma_3(D)+\Sigma_3^\vee(D) \qquad (4.61)\\ &= \sum_k\frac{\frac{1}{2b_{2k}}+a_{2k}^2+\frac{1}{2b_{3k}}+a_{3k}^2}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}\overset{(2.19)}{\sim}\sum_k\frac{\frac{1}{2b_{2k}}+a_{2k}^2+\frac{1}{2b_{3k}}+a_{3k}^2}{\frac{1}{2b_{1k}}+a_{1k}^2}=:\Sigma_{23}^\vee(D)\\ &\sim\sum_k\frac{\frac{b_{1k}}{b_{2k}}+2b_{1k}a_{2k}^2+\frac{b_{1k}}{b_{3k}}+2b_{1k}a_{3k}^2}{1+2b_{1k}a_{1k}^2}\overset{(4.31)}{\sim}\sum_k\frac{a_{2k}^2+a_{3k}^2}{1+a_{1k}^2}=:\Sigma_{23}^a(D).\end{aligned}$$

Remark 4.22. In case (1.1.0) (resp. case (1.2.0)) we have $\|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} a_{3n}^2 < \infty$ (resp. $\|Y_2\|^2 \sim \sum_{n\in\mathbb{Z}} a_{2n}^2 < \infty$), therefore,

$$\Sigma_{23}^a(D)\sim\sum_k\frac{a_{2k}^2}{1+a_{1k}^2}=\infty,\quad\text{resp.}\quad\Sigma_{23}^a(D)\sim\sum_k\frac{a_{3k}^2}{1+a_{1k}^2}=\infty,$$

by Lemma 4.19, which is contradicting (4.61). In the case (1.3.0) we have four possibilities:

(0) when $C_1C_2C_3 \neq 0$, $C_1a_1 + C_2a_2 + C_3a_3 = h \in l_2(\mathbb{Z})$,

(1) when $C_1 = 0$ hence, $C_2C_3 \neq 0$, $C_2a_2 + C_3a_3 = h \in l_2(\mathbb{Z})$,

(2) when $C_2 = 0$ hence, $C_1C_3 \neq 0$, $C_1a_1 + C_3a_3 = h \in l_2(\mathbb{Z})$,

(3) when $C_3 = 0$ hence, $C_1C_2 \neq 0$ $C_1a_1 + C_2a_2 = h \in l_2(\mathbb{Z})$.

In the case (0) we have $\Sigma_{23}^a(D) = \infty$ by Lemma 4.20, contradicting (4.61). In the cases (2) and (3) we get $\Sigma_{23}^a(D) = \infty$ by Lemma 4.19, contradicting (4.61). Therefore, one of the expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$, $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is convergent hence, we can approximate D_{2n} or D_{3n} and the proof is finished. To study the case (1) we need the following lemma.

Lemma 4.23. *Let $C_2Y_2+C_3Y_3=h_{23}\in l_2$ for some $(C_2,C_3)\in\big(\mathbb{R}\setminus\{0\}\big)^2$ and $C_1Y_1+C_2Y_2\notin l_2$ or $C_1Y_1+C_3Y_3\notin l_2$ for all $(C_1,C_r)\in\big(\mathbb{R}\setminus\{0\}\big)^2$, then*

$$\Delta(Y_1,Y_2,Y_3)=\infty. \tag{4.62}$$

Proof. To prove (4.62) we have by (2.15)

$$\Delta(Y_1,Y_2,Y_3)=\frac{\Gamma(Y_1)+\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)+\Gamma(Y_1,Y_2,Y_3)}{1+\Gamma(Y_2)+\Gamma(Y_3)+\Gamma(Y_2,Y_3)}$$

$$\overset{(*)}{>}\frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{1+(1+c_2)\Gamma(Y_2)+\Gamma(Y_3)}\sim\frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{\Gamma(Y_2)+\Gamma(Y_3)}$$

$$\overset{(4.64)}{\sim}\frac{\Gamma(Y_1,Y_2)+\Gamma(Y_1,Y_3)}{2\Gamma(Y_2)}\overset{(4.64)}{\sim}\frac{\Gamma(Y_1,Y_2)}{\Gamma(Y_2)}+\frac{\Gamma(Y_1,Y_3)}{\Gamma(Y_3)}=\infty, \tag{4.63}$$

$$\Gamma(Y_2)\sim\Gamma(Y_3),\quad \text{since}\quad C_2Y_2+C_3Y_3=h\in l_2. \tag{4.64}$$

The relation (*) holds by the inequality $\Gamma(Y_2,Y_3)\le c_2\Gamma(Y_2)$, since $C_2Y_2+C_3Y_3\in l_2$, the relation (4.63) holds by Lemma 6.3 for $m=2$. To prove (4.64) we get since $Y_2\notin l_2$ and $h\in l_2$

$$\frac{\Gamma(Y_3)}{\Gamma(Y_2)}=\frac{\|Y_3\|^2}{\|Y_2\|^2}=\frac{\|Y_2+h\|^2}{\|Y_2\|^2}\le\Big(\frac{\|Y_2\|+\|h\|}{\|Y_2\|}\Big)^2=1.$$

If $C_1Y_1+C_2Y_2\notin l_2$ for all $(C_1,C_2)\in\big(\mathbb{R}\setminus\{0\}\big)^2$, or if $C_1Y_1+C_3Y_3\notin l_2$ for all $(C_1,C_3)\in\big(\mathbb{R}\setminus\{0\}\big)^2$, by Lemma 4.23 we get $\Delta(Y_1,Y_2,Y_3)=\infty$ hence, we can approximate D_{1n} using Lemma 5.3 and the proof is finished. If $C_1Y_1+C_2Y_2=h_{12}\in l_2$ for some for $(C_1,C_2)\in\big(\mathbb{R}\setminus\{0\}\big)^2$ or $C_1Y_1+C_3Y_3=h_{13}\in l_2$ for some $(C_1,C_3)\in\big(\mathbb{R}\setminus\{0\}\big)^2$, then we have $h_{12}+\alpha h_{23}=C_1Y_1+C_2Y_2+C_3Y_3\in l_2$ or $h_{12}+\beta h_{13}=C_1Y_1+C_2Y_2+C_3Y_3\in l_2$ with $C_1C_2C_3\neq 0$ for an appropriate $\alpha\beta\neq 0$, and we are in the case (0). ■

4.4.6. Case (c)

In this case both $\rho_2(C_2,-C_3)$ and $\rho_3(C_2,-C_3)$ are finite, i.e., we are in the case (2) therefore, we can not approximate $x_{2n}x_{2t}$, $x_{3n}x_{3t}$ by Lemma 5.1. By Lemma 4.13 $\Sigma_{23}(s)<\infty$ and hence, $\Sigma_{23}(C_2,C_3)=\infty$. Indeed, reasoning as in Remark 3.3, we see that

$$\mu^{L_{\tau_{23}(\phi,s)}}\perp\mu,\ \ \phi\in[0,2\pi),\ \ s>0\ \Leftrightarrow\ \Sigma_{23}(s)+\Sigma_{23}(C_2,C_3)=\infty,\ \ s>0, \tag{4.65}$$

for $(C_2,C_3)\in\mathbb{R}^2\setminus\{0\}$, where $\tau_{23}(\phi,s)$, $\Sigma_{23}(s)$ and $\Sigma_{23}(C_2,C_3)$ are defined as follows:

$$\tau_{12}(\phi,s)=\begin{pmatrix}\cos\phi & s^2\sin\phi & 0\\ s^{-2}\sin\phi & -\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix},\quad \tau_{23}(\phi,s)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & s^2\sin\phi\\ 0 & s^{-2}\sin\phi & -\cos\phi\end{pmatrix}, \tag{4.66}$$

$$\Sigma_{ij}(s)=\sum_{n\in\mathbb{Z}}\Big(s^2\sqrt{\frac{b_{in}}{b_{jn}}}-s^{-2}\sqrt{\frac{b_{jn}}{b_{in}}}\Big)^2,\quad s\in\mathbb{R}\setminus\{0\}, \tag{4.67}$$

$$\Sigma_{ij}(C_i,C_j)=\sum_{n\in\mathbb{Z}}(C_i^2b_{in}+C_j^2b_{jn})(C_ia_{in}+C_ja_{jn})^2. \tag{4.68}$$

In this case there are four possibilities for the pair $(\Sigma^{12},\Sigma^{13})$:

(2.1) $(\Sigma^{12},\Sigma^{13})=(0,0)$, i.e., $\Sigma^{12}<\infty$ and $\Sigma^{13}<\infty$,

(2.2) $(\Sigma^{12},\Sigma^{13})=(0,1)$, i.e., $\Sigma^{12}<\infty$, but $\Sigma^{13}=\infty$,

(2.3) $(\Sigma^{12},\Sigma^{13})=(1,0)$, i.e., $\Sigma^{12}=\infty$, but $\Sigma^{13}<\infty$,

(2.4) $(\Sigma^{12},\Sigma^{13})=(1,1)$, i.e., $\Sigma^{12}=\infty$ and $\Sigma^{13}=\infty$.

Lemma 4.24. *In the case* (2.1), *i.e., when* $(\Sigma^{12},\Sigma^{13})=(0,0)$, *we can approximate* D_{rn} *for* $1\le r\le 3$, *hence the representation is irreducible.*

Proof. Let $\Sigma^{12}<\infty$ and $\Sigma^{13}<\infty$ we have by (4.38)

$$\nu(C_1,C_2,C_3)\sim\sum_{k\in\mathbb{Z}}b_{1k}(C_1a_{1k}+C_2a_{2k}+C_3a_{3k})^2$$
$$\overset{(2.1)}{\sim}\sum_{k\in\mathbb{Z}}\Big[\frac{t^2}{4}\frac{b_{1k}}{b_{2k}}+\frac{s^2}{4}\frac{b_{1k}}{b_{3k}}+\frac{b_{1k}}{2}(-2a_{1k}+ta_{2k}+sa_{3k})^2\Big]\overset{(3.28)}{=}S^L_{1,23}(\mu,t,s)=\infty.$$

Hence, D_{1n}, D_{2n}, $D_{3n}\ \eta\ \mathfrak{A}^3$ and the proof is finished. ■

Remark 4.25. The cases (2.2) and (2.3) do not occur.

Indeed, by Lemma 4.13 the three series $\Sigma_{23}(s)$ (defined by (4.67)), $\sum_{n\in\mathbb{Z}}c_n^2$ and $\sum_{n\in\mathbb{Z}}e_n^2$ are equivalent where $\frac{s^4b_{2n}}{b_{3n}}=(1+c_n)$, see Lemma 4.14. In the case (c) we have $\sum_{n\in\mathbb{Z}}c_n^2<\infty$, therefore, $\lim_n c_n=0$ and hence,

$$\lim_n d_n^{-1}=\lim_n\frac{b_{2n}}{b_{3n}}=s^{-4}>0.$$

Recall that $d_n=\frac{d_{2n}}{d_{3n}}=\frac{b_{2n}}{b_{3n}}$. But this contradicts $(\Sigma^{12},\Sigma^{13})=(0,1)$, or $(\Sigma^{12},\Sigma^{13})=(1,0)$, since the two series

$$\Sigma^{12}=\sum_n d_{2n}^{-1}\quad\text{and}\quad\Sigma^{13}=\sum_n d_{3n}^{-1}$$

should be equivalent by $\lim_n\frac{d_{2n}}{d_{3n}}=s^{-4}>0$. In the case (2.4) we have

$$\Sigma_{23}(s)<\infty,\quad\Sigma_{23}(C_2,C_3)=\infty,\quad\Sigma^{12}=\Sigma^{13}=\infty.\tag{4.69}$$

To approximate D_{rn}, we need to estimate $\nu(C_1,C_2,C_3)$ defined by (4.36). By (4.38) we have

$$\nu(C_1,C_2,C_3)\sim\sum_{n\in\mathbb{Z}}b_{1n}\Big(\sum_{r=1}^3C_ra_{rn}\Big)^2.$$

Since $\|Y_1\|^2=\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2\sim S_{11}^L(\mu)=\infty$, in the case (2.4) we have four possibilities for $y_{23}:=(y_2,y_3)\in\{0,1\}^2$:

	(2.4.1)	(2.4.2)	(2.4.3)	(2.4.4)
y_1	1	1	1	1
y_2	0	1	0	1
y_3	0	0	1	1

Remark 4.26. The cases (2.4.1)–(2.4.3) are not compatible with the condition $\Sigma_{23}(C_2, C_3) = \infty$ for all $(C_2, C_3) \in \mathbb{R}^2 \setminus \{0\}$.

So it suffices to consider only the case (2.4.4) when $y_{123} = (1, 1, 1)$. The case (2.4.4) splits into two subcases:

(2.4.4.1) when $\Sigma_{12}(s_{12})<\infty$ (resp. $\Sigma_{13}(s_{13})<\infty$) for some s_{12}, $s_{13} > 0$,

(2.4.4.2) when both $\Sigma_{12}(s_{12}) = \Sigma_{13}(s_{13}) = \infty$ for all s_{12}, $s_{13} > 0$.

The case (2.4.4.1) does not occur. Indeed, we have in this case $\Sigma_{13}(s_{12}s_{23}) < \infty$ (resp. $\Sigma_{12}(s_{13}s_{23}^{-1})<\infty$) since

$$\Sigma_{12}(s_{12}) < \infty \Leftrightarrow \mu_{(s_{12}^4 b_1,0)} \sim \mu_{(b_2,0)}, \quad \Sigma_{23}(s_{23}) < \infty \Leftrightarrow \mu_{(s_{23}^4 b_2,0)} \sim \mu_{(b_3,0)}$$

where $\mu_{(b_r,0)} = \otimes_{n\in\mathbb{Z}}\mu_{(b_{rk},0)}$ for $1\le r \le 3$. Therefore,

$$\mu_{((s_{12}s_{23})^4 b_1,0)} \sim \mu_{(b_3,0)} \quad \Leftrightarrow \quad \Sigma_{13}(s_{12}s_{23})<\infty.$$

Similarly, if $\Sigma_{13}(s_{13})<\infty$ and $\Sigma_{23}(s_{23})<\infty$ we have

$$\mu_{(s_{13}^4 b_1,0)} \sim \mu_{(b_3,0)}, \quad \mu_{(s_{23}^4 b_2,0)} \sim \mu_{(b_3,0)} \Rightarrow \mu_{\left((s_{13}s_{23}^{-1})^4 b_1,0\right)} \sim \mu_{(b_2,0)}$$

hence, $\Sigma_{12}(s_{13}s_{23}^{-1}) < \infty$. But condition $\Sigma_{13}(s_{12}s_{23}) + \Sigma_{12}(s_{13}s_{23}^{-1}) < \infty$ contradicts the first condition of (4.30). Indeed, we have by Lemma 4.14

$$\Sigma_{12}(s) = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{2n}}} - s^{-2}\sqrt{\frac{b_{2n}}{b_{1n}}}\right)^2 \sim \sum_{n\in\mathbb{Z}} c_n^2 < \infty, \quad s^2\sqrt{\frac{b_{1n}}{b_{2n}}} = 1 + c_n,$$
$$\Sigma_{13}(s) = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{b_{1n}}{b_{3n}}} - s^{-2}\sqrt{\frac{b_{3n}}{b_{1n}}}\right)^2 \sim \sum_{n\in\mathbb{Z}} f_n^2 < \infty, \quad s^2\sqrt{\frac{b_{1n}}{b_{3n}}} = 1 + f_n,$$

and $\lim_n c_n = \lim_n f_n = 0$. This contradicts $S_1(3) \sim \sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}} < \infty$. Indeed,

$$\lim_{n\to\infty} \frac{b_{1n}^2}{b_{2n}b_{3n}} = s^{-4} \lim_{n\to\infty} (1 + c_n)^2(1 + f_n)^2 = s^{-4} > 0.$$

Finally, to finish the case $S = (0, 1, 1)$, we need to consider only the case (2.4.4.2) when $\Sigma_{12}(s_{12})=\Sigma_{13}(s_{13}) = \infty$ for all s_{12}, $s_{13} > 0$.

By (4.30) and all the previous considerations we have the conditions:

$$\begin{aligned} &S_1(3)\sim\sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}}<\infty, \quad S_2(3)\sim\sum_n \frac{b_{2n}}{b_{3n}}=\infty, \quad S_3(3)\sim\sum_n \frac{b_{3n}}{b_{2n}}=\infty,\\ &\Sigma_{23}(C_2, C_3) = \infty, \quad \Sigma^{12} = \sum_n \frac{b_{1n}}{b_{2n}} = \infty, \quad \Sigma^{13} = \sum_n \frac{b_{1n}}{b_{3n}} = \infty, \\ &\Sigma_{12}(s_{12})=\Sigma_{13}(s_{13})=\infty \text{ for all } s_{12},\ s_{13} > 0,\ \Sigma_{23}(s_{23})<\infty \text{ for some } s_{23} > 0. \end{aligned} \tag{4.70}$$

Remark 4.27. By (4.17) we can suppose that (b_{1n}, b_{2n}, b_{3n}) is replaced with $(1, d_{2n}, d_{3n})$ without loss of generality. Since $\Sigma_{23}(s) < \infty$, using notations (4.45) and (4.46) of Lemma 4.13

$$\Sigma_{23}(s) = \sum_{n\in\mathbb{Z}} \left(\frac{s^2}{\sqrt{d_n}} - \frac{\sqrt{d_n}}{s^2}\right)^2 = \sum_{n\in\mathbb{Z}} \left(s^2\sqrt{\frac{d_{2n}}{d_{3n}}} - s^{-2}\sqrt{\frac{d_{3n}}{d_{2n}}}\right)^2,$$

and taking into consideration (4.70), we can choose d_{2n} and d_{3n} as follows:

$$d_n = \frac{d_{3n}}{d_{2n}} = s^4(1+c_n), \ \sum_n c_n^2 < \infty, \ \sum_n \frac{1}{d_{2n}^2} < \infty, \ \sum_n \frac{1}{d_n} = \sum_n d_n = \infty. \quad (4.71)$$

Since $\sum_n c_n^2 < \infty$ we have $\sum_n \frac{b_{1n}^2}{b_{2n}b_{3n}} \sim \sum_n \frac{1}{d_{2n}^2}$ and the measures $\mu_{(d_3^{c,s},0)}$ and $\mu_{(d_3^s,0)}$ are equivalent, where

$$\mu_{(d_3^{c,s},0)} = \otimes_n \mu_{(s^4 d_{2n}(1+c_n),0)}, \quad \mu_{(d_3^s,0)} = \otimes_n \mu_{(s^4 d_{2n},0)},$$

hence, we can choose $c_n \equiv 0$ and $s = 1$. So, to finish the case $S = (0,1,1)$ we should prove the irreducibility for $b = (1, d_{2n}, d_{2n})_{n\in\mathbb{Z}}$ with the only condition:

$$\sum_n d_{2n}^{-2} < \infty. \quad \text{Since} \quad d_n \equiv 1, \quad \text{we have} \quad \sum_n d_n^{-1} = \sum_n d_n = \infty. \quad (4.72)$$

Example 4.28. The pairwise conditions

$$\|C_rY_r + C_sY_s\|^2 = \infty \ \text{ for } 1 \le r < s \le 3 \quad \text{do not imply} \quad \|\sum_{r=1}^3 C_rY_r\|^2 = \infty.$$

Let $a_{r,n} = a_{r,-n}$ for $n \in \mathbb{N}$ and $a_{1,0} = 1$, $a_{2,0} = 2$, $a_{3,0} = 3$. We define $a_{r,n}$ for $n \in \mathbb{N}$ as follows

$$a_{1n} = \begin{cases} 2 & n = 2k+1 \\ 1 & n = 2k \end{cases}, \quad a_{2n} = \begin{cases} 1 & n = 2k+1 \\ 2 & n = 2k \end{cases}, \quad a_{3n} \equiv 3. \quad (4.73)$$

Then we have clearly for arbitrary $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$

$$\|C_1a_1 + C_2a_2\|^2 = \infty, \ \|C_1a_1 + C_3a_3\|^2 = \infty, \ \|C_2a_2 + C_3a_3\|^2 = \infty, \quad (4.74)$$

$$\text{but} \quad a_1 + a_2 - a_3 = 0 \quad \text{hence,} \quad \|a_1 + a_2 - a_3\|^2 = 0. \quad (4.75)$$

Example 4.29. Let us consider the measure $\mu^3_{(b,a)}$ with $a = \big(a_{rn}\big)_{r,n}$ from Example 4.28 and $b = (b_{1n}, b_{2n}, b_{3n})$ defined as follows:

$$b_{1n} \equiv 1, \quad d_{2n} = d_{3n} = |n| \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}, \quad d_{20} = d_{30} = 1. \quad (4.76)$$

Lemma 4.30. *In Example* 4.28 *we have* (*for* $n \in \mathbb{N}$ *only*)

$$\Delta(a_1, a_2, a_3) = 2, \quad \Delta(a_2, a_3, a_1) = 2, \quad \Delta(a_3, a_1, a_2) = 2, \quad (4.77)$$

where $a_r = (a_{rn})_{n\in\mathbb{N}}$, $1 \le r \le 3$.

Proof. Set $a_r(n) = (a_{rl})_{l=1}^n$ for $1 \le r \le 3$ and $n \in \mathbb{N}$, then for $1 \le k < r \le 3$

$$\Gamma(a_k(n)) \sim \Gamma(a_1(n) + a_2(n)) \sim n, \ \Gamma(a_k(n), a_r(n)) \sim \frac{n(n-1)}{2}, \ \Gamma(a_1, a_2, a_3) = 0.$$

We observe that $\Gamma(a_k, a_k + a_r) = \Gamma(a_k, a_r)$ for $k \ne r$. Since $a_3 = a_1 + a_2$ we get

$$\begin{aligned}\Delta(a_1, a_2, a_3) &= \frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_3) + \Gamma(a_1, a_2, a_3)}{1 + \Gamma(a_2) + \Gamma(a_3) + \Gamma(a_2, a_3)} \\ &= \frac{\Gamma(a_1) + \Gamma(a_1, a_2) + \Gamma(a_1, a_1 + a_2) + \Gamma(a_1, a_2, a_1 + a_2)}{1 + \Gamma(a_2) + \Gamma(a_1 + a_2) + \Gamma(a_2, a_1 + a_2)} \\ &= \frac{\Gamma(a_1) + 2\Gamma(a_1, a_2)}{1 + \Gamma(a_2) + \Gamma(a_1 + a_2) + \Gamma(a_1, a_2)} = 2,\end{aligned}$$

$$\Delta(a_2,a_3,a_1)=\frac{\Gamma(a_2)+\Gamma(a_2,a_3)+\Gamma(a_2,a_1)+\Gamma(a_2,a_3,a_1)}{1+\Gamma(a_3)+\Gamma(a_1)+\Gamma(a_3,a_1)}$$
$$=\frac{\Gamma(a_2)+\Gamma(a_2,a_1+a_2)+\Gamma(a_2,a_1)+\Gamma(a_2,a_1+a_2,a_1)}{1+\Gamma(a_1+a_2)+\Gamma(a_1)+\Gamma(a_1+a_2,a_1)}$$
$$=\frac{\Gamma(a_2)+2\Gamma(a_2,a_1)}{1+\Gamma(a_1+a_2)+\Gamma(a_1)+\Gamma(a_2,a_1)}=2,$$
$$\Delta(a_3,a_1,a_2)=\frac{\Gamma(a_3)+\Gamma(a_3,a_1)+\Gamma(a_3,a_2)+\Gamma(a_3,a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}$$
$$=\frac{\Gamma(a_1+a_2)+\Gamma(a_1+a_2,a_1)+\Gamma(a_1+a_2,a_2)+\Gamma(a_1+a_2,a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}$$
$$=\frac{\Gamma(a_1+a_2)+2\Gamma(a_1,a_2)}{1+\Gamma(a_1)+\Gamma(a_2)+\Gamma(a_1,a_2)}=2.$$

We use two facts for $1\leq r\leq 2$:

$$\frac{\Gamma(a_1,a_2)}{\Gamma(a_r)}=\infty\quad\text{and}\quad\Gamma(a_1+a_2)\leq\Gamma(a_1)+\Gamma(a_2)+2\sqrt{\Gamma(a_1)\Gamma(a_2)}.$$

The first relation follows from Lemma 6.3 for $m=2$, since $\|C_1a_1+C_2a_2\|^2=\infty$. We get

$$\frac{\Gamma(a_1,a_2)}{\Gamma(a_r)}=\lim_{n\to\infty}\frac{\Gamma(a_1(n),a_2(n))}{\Gamma(a_r(n))}=\infty.$$

Recall that $\Gamma(a)=\|a\|^2$. The inequality follows from $\|a_1+a_2\|\leq\|a_1\|+\|a_2\|$, i.e., $\sqrt{\Gamma(a_1+a_2)}\leq\sqrt{\Gamma(a_1)}+\sqrt{\Gamma(a_2)}$. ∎

By Lemma 3.5 we have

$$(\mu^3_{(b,a)})^{L_t}\perp\mu^3_{(b,a)}\quad\Leftrightarrow\quad\Sigma^{\pm}(t):=\Sigma_1^{\pm}(t)+\Sigma_2(t)=\infty,$$

where $\Sigma_1^{\pm}(t)$ and $\Sigma_2(t^{-1})$ are defined by (3.18) and (3.15). In Example 4.29 we can not approximate x_{2n}, x_{3n}, since in this case we have

$$\Delta(Y_2^{(2)},Y_3^{(2)})=1,\quad\Delta(Y_3^{(3)},Y_2^{(3)})=1.\tag{4.78}$$

Indeed, by (4.34) we have

$$\Delta(Y_2^{(2)},Y_3^{(2)})=\frac{\Gamma(Y_2^{(2)})+\Gamma(Y_2^{(2)},Y_3^{(2)})}{1+\Gamma(Y_3^{(2)})},\ \Delta(Y_3^{(3)},Y_2^{(3)})=\frac{\Gamma(Y_3^{(3)})+\Gamma(Y_3^{(3)},Y_2^{(3)})}{1+\Gamma(Y_2^{(3)})}.$$

In Example 4.29 we have $d_n=\frac{d_{3n}}{d_{2n}}\equiv 1$ and hence, by (4.41) we have

$$\|Y_2^{(2)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{1}{1+2d_n}=\sum_{n\in\mathbb{Z}}\frac{1}{3},\quad\|Y_3^{(2)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{d_n^2}{1+2d_n}=\sum_{n\in\mathbb{Z}}\frac{1}{3},$$
$$\|Y_2^{(3)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_n^2+2d_n}=\sum_{n\in\mathbb{Z}}\frac{1}{3},\quad\|Y_3^{(3)}\|^2\sim\sum_{n\in\mathbb{Z}}\frac{d_n^2}{d_n^2+2d_n}=\sum_{n\in\mathbb{Z}}\frac{1}{3}.$$

Therefore, $\Gamma(Y_2^{(2)},Y_3^{(2)})=\Gamma(Y_3^{(3)},Y_2^{(3)})=0$, and

$$\Delta(Y_2^{(2)},Y_3^{(2)})=\frac{\Gamma(Y_2^{(2)})}{1+\Gamma(Y_3^{(2)})}=1,\quad\Delta(Y_3^{(3)},Y_2^{(3)})=\frac{\Gamma(Y_3^{(3)})}{1+\Gamma(Y_2^{(3)})}=1.$$

Since $b_{1n}\equiv 1$, by (4.42) we get

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}a_{3n}^2,\quad \text{so we have}$$

$$\nu(C_1,C_2,C_3)\overset{(4.38)}{\sim}\sum_{n\in\mathbb{Z}}b_{1n}\Big(\sum_{r=1}^3 C_ra_{rn}\Big)^2=\sum_{n\in\mathbb{Z}}\Big(\sum_{r=1}^3 C_ra_{rn}\Big)^2.$$

But in Example 4.28 there does not exist some $t\in\pm\mathrm{SL}(3,\mathbb{R})\setminus\{e\}$ such that $\nu(C_1,C_2,C_3)=\infty$ for all $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$ to approximate some D_{rn}.

4.4.7. Approximations of $x_{2k}x_{2r}+x_{3k}x_{3r}$ in the case (c)

Since we can not approximate $x_{2n}x_{2t}$, $x_{3n}x_{3t}$ using Lemma 5.1 in the case (c), we shall try to approximate $x_{2k}x_{2r}+s^4x_{3k}x_{3r}$ by an appropriate combination of $A_{kn}A_{rn}$ for $n\in\mathbb{Z}$. Let $s=1$, the general case is similar.

Lemma 4.31. *For any $k,r\in\mathbb{Z}$ one has*

$$(x_{2k}x_{2r}+x_{3k}x_{3r})\mathbf{1}\in\langle A_{kn}A_{rn}\mathbf{1}\mid n\in\mathbb{Z}\rangle\ \Leftrightarrow\ \Delta(Y^{(2)},Y^{(1)})=\infty, \tag{4.79}$$

where $Y^{(r)}=\Big(\frac{b_{rn}}{\sqrt{\lambda_n}}\Big)_{n\in\mathbb{Z}}$, $1\le r\le 2$, $\lambda_n=(b_{1n}+b_{2n}+b_{3n})^2-b_{1n}^2$.

Proof. The proof of Lemma 4.31 is based on Lemma 6.5 for $m=1$. We study when $(x_{2k}x_{2r}+x_{3k}x_{3r})\mathbf{1}\in\langle A_{kn}A_{rn}\mathbf{1}\mid n\in\mathbb{Z}\rangle$. Since

$$\begin{aligned}A_{kn}A_{rn}&=(x_{1k}D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n})(x_{1r}D_{1n}+x_{2r}D_{2n}+x_{3r}D_{3n})\\&=x_{1k}x_{1r}D_{1n}^2+x_{2k}x_{2r}D_{2n}^2+x_{3k}x_{3r}D_{3n}^2+(x_{1k}x_{2r}+x_{2k}x_{1r})D_{1n}D_{2n}\\&\quad+(x_{1k}x_{3r}+x_{3k}x_{1r})D_{1n}D_{3n}+(x_{2k}x_{3r}+x_{3k}x_{2r})D_{2n}D_{3n},\end{aligned}$$

and $MD_{rn}^2\mathbf{1}=-b_{rn}/2$, for $2\le r\le 3$ we take $t=(t_n)_{n=-m}^m$ as follows:

$$(t,b_2)=(t,b_3)=1,\ \text{where } t=(t_n)_{k=-m}^m,\ b_2=-(b_{2n}/2)_{n=-m}^m,\ b_3=-(b_{3n}/2)_{n=-m}^m.$$

We have $\|\big[\sum_{n=-m}^m t_nA_{kn}A_{rn}-\big(x_{2k}x_{2r}+x_{3k}x_{3r}\big)\big]\mathbf{1}\|^2$

$$\begin{aligned}&=\|\sum_{n=-m}^m t_n\Big[x_{1k}x_{1r}D_{1n}^2+x_{2k}x_{2r}\Big(D_{2n}^2+\frac{b_{2n}}{2}\Big)+x_{3k}x_{3r}\Big(D_{3n}^2+\frac{b_{3n}}{2}\Big)\\&+(x_{1k}x_{2r}+x_{2k}x_{1r})D_{1n}D_{2n}+(x_{1k}x_{3r}+x_{3k}x_{1r})D_{1n}D_{3n}\\&+(x_{2k}x_{3r}+x_{3k}x_{2r})D_{2n}D_{3n}\Big]\mathbf{1}\|^2\\&=\sum_{-m\le n,l\le m}(f_n,f_l)t_nt_l=:(A_{2m+1}t,t),\quad\text{where } A_{2m+1}=(f_n,f_l)_{n,l=-m}^m,\end{aligned} \tag{4.80}$$

$$f_n=\sum_{i=1}^3 f_n^i+\sum_{1\le i<j\le 3}f_n^{ij},\quad f_n^i=x_{ik}x_{ir}\Big(D_{in}^2+\frac{b_{in}}{2}(1-\delta_{i1})\Big)\mathbf{1}, \tag{4.81}$$

$$f_n^{ij}=(x_{ik}x_{jr}+x_{jk}x_{ir})D_{in}D_{jn}\mathbf{1},$$

for $1\le i\le 3$, $1\le i<j\le 3$. Since $f_n^{i'}\perp f_n^{ij}$, $f_n^{ij}\perp f_n^{i'j'}$ for different (ij), $(i'j')$, writing

$$c_{kn}=\|x_{kn}\|^2=\frac{1}{2b_{kn}}+a_{kn}^2,$$

we get

$$(f_n,f_n)=\sum_{i=1}^{3}\|f_n^i\|^2+\sum_{1\le i<j\le 3}\|f_n^{ij}\|^2=c_{1k}c_{1r}3\big(\frac{b_{1n}}{2}\big)^2+c_{2k}c_{2r}2\big(\frac{b_{2n}}{2}\big)^2$$
$$+c_{3k}c_{3r}2\big(\frac{b_{3n}}{2}\big)^2+\big(c_{1k}c_{2r}+c_{2k}c_{1r}+2a_{1k}a_{2r}a_{2k}a_{1r}\big)\frac{b_{1n}}{2}\frac{b_{2n}}{2}$$
$$+\big(c_{1k}c_{3r}+c_{3k}c_{1r}+2a_{1k}a_{3r}a_{3k}a_{1r}\big)\frac{b_{1n}}{2}\frac{b_{3n}}{2}$$
$$+\big(c_{2k}c_{3r}+c_{3k}c_{2r}+2a_{2k}a_{3r}a_{3k}a_{2r}\big)\frac{b_{2n}}{2}\frac{b_{3n}}{2}$$
$$\sim(b_{1n}+b_{2n}+b_{3n})^2,\ \ (f_n,f_l)=(f_n^1,f_l^1)=c_{1k}c_{1r}\frac{b_{1n}}{2}\frac{b_{1l}}{2}\sim b_{1n}b_{1l}.$$

Finally, we get $$(f_n,f_n)\sim(b_{1n}+b_{2n}+b_{3n})^2,\quad (f_n,f_l)\sim b_{1n}b_{1l},\quad n\neq l. \tag{4.82}$$

Set $$\lambda_n=(b_{1n}+b_{2n}+b_{3n})^2-b_{1n}^2,\quad g_n=(b_{1n}), \tag{4.83}$$

then $$(f_n,f_n)\sim\lambda_n+(g_n,g_n),\quad (f_n,f_l)\sim(g_n,g_l). \tag{4.84}$$

For $A_{2m+1}=\big((f_n,f_l)\big)_{n,l=-m}^m$ and $b_2=b_3=-(b_{2n}/2)_{n=-m}^m\in\mathbb{R}^{2m+1}$ we have
$$A_{2m+1}=\sum_{n=-m}^{m}\lambda_nE_{nn}+\gamma(g_{-m},\dots,g_0,\dots,g_m).$$

To finish the proof, it suffices to use Lemma 6.5 for $m=1$. ∎

Remark 4.32. In case (c) we can approximate $x_{2k}x_{2r}+x_{3k}x_{3r}$ since we have $\Delta(Y^{(2)},Y^{(1)})=\infty$.

By (4.79) we have
$$\Delta(Y^{(2)},Y^{(1)})=\frac{\Gamma(Y^{(2)})+\Gamma(Y^{(2)},Y^{(1)})}{1+\Gamma(Y^{(1)})}>\frac{\Gamma(Y^{(2)})}{1+\Gamma(Y^{(1)})}=\infty,$$
since $\Gamma(Y^{(2)})=\infty$ by $\sum_n\frac{1}{d_{2n}^2}<\infty$ and $\Gamma(Y^{(1)})<\infty$. Indeed,
$$\Gamma(Y^{(2)})=\sum_{n\in\mathbb{Z}}\frac{b_{2n}^2}{\lambda_n}=\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{(1+2d_{2n})^2-1}\sim\sum_{n\in\mathbb{Z}}\frac{d_{2n}^2}{d_{2n}+d_{2n}^2}\overset{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}d_{2n}=\infty,$$
$$\Gamma(Y^{(1)})\overset{(4.79)}{=}\sum_{n\in\mathbb{Z}}\frac{1}{(1+2d_{2n})^2-1}\sim\sum_{n\in\mathbb{Z}}\frac{1}{d_{2n}+d_{2n}^2}<\sum_{n}\frac{1}{d_{2n}^2}\overset{(4.71)}{<}\infty.$$

Lemma 4.33. *Let $\{r,s\}$ be a cyclic permutation of $\{2,3\}$, then for all $k\in\mathbb{Z}$*
$$x_{rk}\mathbf{1}\in\langle(x_{2k}x_{2n}+x_{3k}x_{3n})\mathbf{1}\mid n\in\mathbb{Z}\rangle\ \Leftrightarrow\ \sigma_r(\mu)=\sum_{n\in\mathbb{Z}}\frac{a_{rn}^2}{\frac{1}{2b_{rn}}+c_{sn}^2}=\infty. \tag{4.85}$$

Proof. Recall the notation $c_{rn}=\dfrac{1}{2b_{rn}}+a_{rn}^2$. Since we have $Mx_{2n}\mathbf{1}=a_{2n}$ we take $t=(t_n)_{n=-m}^m$ as follows: $(t,a_2)=1$, where $a_2=(a_{2n})_{n=-m}^m$. For $r=2$ we get
$$\|\Big[\sum_{n=-m}^{m}t_n\big(x_{2k}x_{2n}+x_{3k}x_{3n}\big)-x_{2k}x_{2n}\Big]\mathbf{1}\|^2=\|\Big[\sum_{n=-m}^{m}t_n\big(x_{2k}(x_{2n}-a_{2n})+x_{3k}x_{3n}\big)\Big]\mathbf{1}\|^2$$
$$=\|x_{2k}\mathbf{1}\|^2\|\sum_{n=-m}^{m}t_n(x_{2n}-a_{2n})\mathbf{1}\|^2+\|x_{3k}\mathbf{1}\|^2\|\sum_{n=-m}^{m}t_nx_{3n}\mathbf{1}\|^2$$
$$=c_{2k}\sum_{n=-m}^{m}t_n^2\frac{1}{2b_{2n}}+c_{3k}\sum_{n=-m}^{m}t_n^2c_{3n}\sim\sum_{n=-m}^{m}t_n^2\big(\frac{1}{2b_{2n}}+c_{3n}^2\big).$$

By (6.1) we get (4.85). ∎

Remark 4.34. Suppose that $\sigma_2(\mu)+\sigma_3(\mu)<\infty$. This contradicts $\Sigma_{23}(C_2,C_3)=\infty$ for $(C_2,C_3)\in\mathbb{R}^2\setminus\{0\}$, where $\Sigma_{23}(C_2,C_3)$ is defined by (4.68).

Proof. Indeed, we have

$$\infty>\sigma_2(\mu)+\sigma_3(\mu)=\sum_{\in\mathbb{Z}}\frac{a_{2n}^2}{\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}+a_{3n}^2}+\sum_{\in\mathbb{Z}}\frac{a_{3n}^2}{\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}+a_{2n}^2}\sim$$
$$\sum_{\in\mathbb{Z}}\frac{a_{2n}^2+a_{3n}^2}{\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}+a_{2n}^2+a_{3n}^2}\sim\sum_{\in\mathbb{Z}}\frac{a_{2n}^2+a_{3n}^2}{\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}\overset{(4.71)}{=}\frac{2}{1+s^{-4}}\sum_{n\in\mathbb{Z}}b_{2n}(a_{2n}^2+a_{3n}^2).$$

This contradicts $\Sigma_{23}(C_2,C_3)=\infty$. Indeed, by $b_{3n}=s^4b_{2n}$ (see (4.71)) we have

$$\Sigma_{23}(C_2,C_3)=\sum_{n\in\mathbb{Z}}\big(C_2^2+C_3^2s^4\big)b_{2n}\big(C_2a_{2n}+C_3a_{3n}\big)^2<\infty. \qquad\blacksquare$$

Finally, we have $\sigma_2(\mu)+\sigma_3(\mu)=\infty$, and therefore we have $x_{rn}\,\eta\,\mathfrak{A}^3$ for some $2\le r\le 3$. Let $x_{3n}\,\eta\,\mathfrak{A}^3$, then we can approximate x_{2n} by combinations of $x_{2n}x_{2k}$, $k\in\mathbb{Z}$ using an analogue of Lemma 4.6. To approximate D_{rn}, $1\le r\le 3$ we again proceed as in Section 4.4.4. As in (4.51) we get

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{n\in\mathbb{Z}}a_{3n}^2.$$

Indeed, for example, by (4.5) we get

$$\|Y_1\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2}+\frac{1}{d_{2n}}}\overset{(4.72)}{\sim}\sum_{n\in\mathbb{Z}}a_{1n}^2.$$

Again, as in (4.52) we have four possibilities: (1.0), (1.1), (1.2) and (1.3). The corresponding expressions in (4.56), (4.57), (4.58) become as follows:

$$\nu_{12}(C_1,C_2):=\|C_1Y_1+C_2Y_2\|^2\sim\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_2a_{2n}\Big)^2,$$
$$\nu_{13}(C_1,C_3):=\|C_1Y_1+C_3Y_3\|^2\sim\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_3a_{3n}\Big)^2,$$
$$\nu(C_1,C_2,C_3)=\sum_{n\in\mathbb{Z}}\Big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\Big)^2.$$

To study the cases (1.1.1)–(1.3.1) we should use Remark 4.17. We can approximate in these cases respectively D_{1n} and D_{2n} in (4.53), D_{1n} and D_{3n} in (4.54) all D_{1n}, D_{2n}, D_{3n} in (4.55). The proof of irreducibilty is finished in these cases because we have $D_{rn},x_{2n},x_{3n}\,\eta\,\mathfrak{A}^3$ for some $1\le r\le 3$. Following Remark 4.21 we can use Lemma 5.4 and its analogue to approximate D_{2n} and D_{3n} with corresponding expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$ and $\Sigma_3(D)$, $\Sigma_3^\vee(D)$. If one of the expressions $\Sigma_2(D,s)$, $\Sigma_2^\vee(D,s)$, $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent for some sequence $s=(s_k)_{k\in\mathbb{Z}}$, we can approximate D_{2k} or D_{3k} and the proof is finished. Suppose that for any sequence $s=(s_k)_{k\in\mathbb{Z}}$ we have

$$\Sigma_2(D,s)+\Sigma_2^\vee(D,s)+\Sigma_3(D,s)+\Sigma_3^\vee(D,s)<\infty.$$

Then, by (4.61) we have

$$\infty > \Sigma_{23}^{\vee}(D) = \sum_k \frac{\frac{1}{2b_{2k}} + a_{2k}^2 + \frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2} = \sum_k \frac{\frac{1}{d_{2k}} + a_{2k}^2 + a_{3k}^2}{\frac{1}{2} + a_{1k}^2} \stackrel{(4.72)}{\sim} \sum_k \frac{a_{2k}^2 + a_{3k}^2}{1 + a_{1k}^2} =: \Sigma_{23}^{a}(D).$$

To study cases (1.1.0)–(1.3.0) we should follow Remark 4.22.

4.5. Case $S = (1, 1, 1)$

Denote by
$$\Sigma_{123}(s) = (\Sigma_{12}(s_1), \Sigma_{23}(s_2), \Sigma_{13}(s_3)), \tag{4.86}$$

where $s = (s_1, s_2, s_3)$ and $\Sigma_{ij}(s)$ are defined by (4.67) for $1 \le i < j \le 3$. In terms of Remark 4.4, we have 2^3 possibilities for $\Sigma_{123}(s) \in \{0, 1\}^3$:

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$\Sigma_{12}(s_1)$	0	0	0	0	1	1	1	1
$\Sigma_{23}(s_2)$	0	0	1	1	0	0	1	1
$\Sigma_{13}(s_3)$	0	1	0	1	0	1	0	1

The cases (1), (2) and (4) and respectively the cases (3), (5) and (6) result from cyclic permutations of three measures $\mu^{(1)}, \mu^{(2)}, \mu^{(3)}$ defined as follows:

$$\mu^{(r)} = \otimes_{n\in\mathbb{Z}} \mu_{(b_{rn}, a_{rn})}, \ 1 \le r \le 3, \quad \mu_0^{(r)} = \otimes_{n\in\mathbb{Z}} \mu_{(b_{rn}, 0)}, \ 1 \le r \le 3. \tag{4.87}$$

The cases (1), (2) and (4) can not be realized. We prove this only in the case (1). By Lemma 6.6 we have $\Sigma_{12}(s_1) < \infty \ \Leftrightarrow \ \mu_0^{(1)} \sim \mu_0^{(2)}$ and $\Sigma_{23}(s_2) < \infty \Leftrightarrow \mu_0^{(2)} \sim \mu_0^{(3)}$ hence, $\mu_0^{(1)} \sim \mu_0^{(3)}$, that contradicts $\Sigma_{13}(s_2) = \infty \Leftrightarrow \mu_0^{(1)} \perp \mu_0^{(3)}$. Finally, we are left with the three cases (0), (3) and (7):

the **case (0)**, i.e., $\Sigma_{123}(s) = (0, 0, 0)$,

the **case (3)**, i.e., $\Sigma_{123}(s) = (0, 1, 1)$,

the **case (7)**, i.e., $\Sigma_{123}(s) = (1, 1, 1)$.

4.5.1. Case $\Sigma_{123}(s) = (0, 0, 0)$

In the case (0), we have for some $s = (s_1, s_2, s_3) \in \big(\mathbb{R}_+\big)^3$

$$\Sigma_{12}(s_1) < \infty, \quad \Sigma_{23}(s_2) < \infty, \quad \Sigma_{13}(s_3) < \infty.$$

In this case we get $\mu_0^{(1)} \sim \mu_0^{(2)} \sim \mu_0^{(3)}$. By (4.17) we can make the following change of the variables:

$$\begin{pmatrix} b_{1n} & b_{2n} & b_{3n} \\ a_{1n} & a_{2n} & a_{3n} \end{pmatrix} \to \begin{pmatrix} b'_{1n} & b'_{2n} & b'_{3n} \\ a'_{1n} & a'_{2n} & a'_{3n} \end{pmatrix} = \begin{pmatrix} 1 & \frac{b_{2n}}{b_{1n}} & \frac{b_{3n}}{b_{1n}} \\ a_{1n}\sqrt{b_{1n}} & a_{2n}\sqrt{b_{1n}} & a_{3n}\sqrt{b_{1n}} \end{pmatrix}.$$

Remark 4.35. By Lemma 6.7, we can suppose that

$$b = (b_{1n}, b_{2n}, b_{3n})_{n\in\mathbb{Z}} = (1, 1 + c_n, 1 + e_n)_{n\in\mathbb{Z}}, \ \sum_n c_n^2 < \infty, \ \sum_n e_n^2 < \infty. \tag{4.88}$$

But the two measures $\mu_{(b,a)}$ and $\mu_{(\mathbb{I},a)}$ are equivalent, where b is defined by (4.88) and

$$\mathbb{I} := (1,1,1)_{n\in\mathbb{Z}}. \tag{4.89}$$

Finally, it is sufficient to consider the measure $\mu_{(\mathbb{I},a)}$.

Example 4.36. Let $b_{1n} = b_{2n} = b_{3n} \equiv 1,\ n \in \mathbb{Z}$.

(a) Take $a_n = (a_{1n}, a_{2n}, a_{3n}),\ n \in \mathbb{Z}$ as it was defined in Example 4.28:

$$a_{1n} = \begin{cases} 2 & n = 2k+1 \\ 1 & n = 2k \end{cases}, \quad a_{2n} = \begin{cases} 1 & n = 2k+1 \\ 2 & n = 2k \end{cases}, \quad a_{3n} \equiv 3.$$

Then $a_1 + a_2 - a_3 = 0$, where $a_r = (a_{rn})_{n\in\mathbb{Z}}$.

(b) Take any $a_r = (a_{rn})_{n\in\mathbb{Z}}$ such that $a_1, a_2, a_3 \notin l_2(\mathbb{Z})$, but $C_1a_1+C_2a_2+C_3a_3 \in l_2(\mathbb{Z})$ for some $(C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}$.

Example 4.37. Let $b_{1n} = b_{2n} = b_{3n} \equiv 1,\ n \in \mathbb{Z}$ and $a = (a_{1n}, a_{2n}, a_{3n})_{n\in\mathbb{Z}}$ such that $a_1, a_2, a_3 \notin l_2(\mathbb{Z})$, but the measure $\mu^3_{(b,a)}$ satisfies the orthogonality conditions. The case $\Sigma_{123}(s) = (0,0,0)$ is reduced to this example.

Remark 4.38. Since the measure $\mu^3_{(b,0)}$ is *standard* in Example 4.36 and 4.37, i.e., it is invariant under rotations $\pm\mathrm{O}(3)$, we have

$$(\mu^3_{(b,0)})^{L_t} = \mu^3_{(b,0)} \quad \text{for all} \quad t \in \pm\mathrm{O}(3). \tag{4.90}$$

By Lemma 3.7, the orthogonality condition $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{O}(3) \setminus \{e\}$, is equivalent to

$$\Sigma_1^{\pm}(t) + \Sigma_2(t) = \infty,$$

where $\Sigma_1^{+}(t),\ \Sigma_1^{-}(t)$ are defined by (3.17) and $\Sigma_2(t)$ is defined by (3.15). By (4.90) we get $\Sigma_1^{\pm}(t) < \infty$ in Example 4.36 and 4.37 hence the orthogonality condition $(\mu^3_{(b,a)})^{L_t} \perp \mu^3_{(b,a)}$ for $t \in \pm\mathrm{O}(3) \setminus \{e\}$ is equivalent to $\Sigma_2(t) = \infty$. Further, to prove the irreducibility in Example 4.36 and 4.37 we should show that $\Sigma_2(t) = \infty$ for all $t \in \pm\mathrm{O}(3) \setminus \{e\}$ implies

$$\|C_1Y_1 + C_2Y_2 + C_3Y_3\|^2 = \infty \quad \text{for all} \quad (C_1, C_2, C_3) \in \mathbb{R}^3 \setminus \{0\}.$$

Lemma 4.39. (1) *The representations corresponding to the measures in Example* 4.36 (a) *and* (b) *are reducible.*

(2) *The representations corresponding to the measures in Example* 4.37 *are irreducible.*

Proof. To prove part (1) of the lemma, by Remark 4.38 and (4.90), we should find for the measure in Example 4.36 an element $t \in \pm\mathrm{O}(3) \setminus \{e\}$ such that $\Sigma_2(t) < \infty$. This will imply $(\mu^3_{(b,a)})^{L_t} \sim \mu^3_{(b,a)}$ hence, the *reducibility.*

Finally, it is sufficient to find $t \in \pm\mathrm{O}(3) \setminus \{e\}$ such that

$$t - 1 = \begin{pmatrix} \lambda_1C_1 & \lambda_1C_2 & \lambda_1C_3 \\ \lambda_2C_1 & \lambda_2C_2 & \lambda_2C_3 \\ \lambda_3C_1 & \lambda_3C_2 & \lambda_3C_3 \end{pmatrix}, \tag{4.91}$$

where $(C_1,C_2,C_3)=(1,1,-1)$, in part (a), or for an arbitrary $(C_1,C_2,C_3)\in\mathbb{R}^3\backslash\{0\}$ in the part (b). Such an element exists by Lemma 4.40 below. For such an element t we get respectively in the cases (a), (b) and Example 4.37 (see (3.15)):

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}(b_{1n}\lambda_1^2+b_{2n}\lambda_2^2+b_{3n}\lambda_3^2)\big(a_{1n}+a_{2n}-a_{3n}\big)^2=0,$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}(b_{1n}\lambda_1^2+b_{2n}\lambda_2^2+b_{3n}\lambda_3^2)\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2<\infty,$$

$$\Sigma_2(t^{-1})=\sum_{n\in\mathbb{Z}}(b_{1n}\lambda_1^2+b_{2n}\lambda_2^2+b_{3n}\lambda_3^2)\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2=\infty. \tag{4.92}$$

Note that the measure in Example 4.36 does not satisfy the orthogonality conditions.

(2) *Irreducibility.* In Example 4.37 we can not approximate x_{rn} by Lemma 5.1, since all the expressions

$$\Delta(Y_1^{(1)},Y_2^{(1)},Y_3^{(1)}),\quad \Delta(Y_2^{(2)},Y_3^{(2)},Y_1^{(2)}),\quad \Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})$$

are bounded. To approximate D_{rn} using Lemma 5.2 we should estimate the following expressions:

$$\Delta(Y_1,Y_2,Y_3),\quad \Delta(Y_2,Y_3,Y_1),\quad \Delta(Y_3,Y_1,Y_2).$$

Following Lemma 6.3 for $m=2$, all these expressions are infinite, if we have for all $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$:

$$\nu(C_1,C_2,C_3):=\|C_1Y_1+C_2Y_2+C_3Y_3\|^2=\sum_{n\in\mathbb{Z}}\frac{\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\infty.$$

In Examples 4.37 we have

$$\nu(C_1,C_2,C_3)=\|C_1Y_1+C_2Y_2+C_3Y_3\|^2\sim\sum_{k\in\mathbb{Z}}b_{1n}(C_1a_{1k}+C_2a_{2k}+C_3a_{3k})^2$$

$$\sim\sum_{n\in\mathbb{Z}}(b_{1n}\lambda_1^2+b_{2n}\lambda_2^2+b_{3n}\lambda_3^2)\big(C_1a_{1n}+C_2a_{2n}+C_3a_{3n}\big)^2=\Sigma_2(t^{-1})=\infty. \quad\blacksquare$$

Lemma 4.40. *For an arbitrary element $(C_1,C_2,C_3)\in\mathbb{R}^3\backslash\{0\}$, and an arbitrary $D_3(s)=\operatorname{diag}(s_1,s_2,s_3)$ with $(s_1,s_2,s_3)\in(\mathbb{R}_+)^3$, there exists a unique element $t\in\pm\mathrm{O}(3)\setminus\{e\}$ and $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3\setminus\{0\}$ such that*

$$D_3(s)tD_3^{-1}(s)-\mathrm{I}=\begin{pmatrix}\lambda_1C_1&\lambda_1C_2&\lambda_1C_3\\ \lambda_2C_1&\lambda_2C_2&\lambda_2C_3\\ \lambda_3C_1&\lambda_3C_2&\lambda_3C_3\end{pmatrix}=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}C_1&0&0\\0&C_2&0\\0&0&C_3\end{pmatrix}. \tag{4.93}$$

Proof. By (4.93) we get

$$\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}:=t=\begin{pmatrix}t_{11}&t_{12}&t_{13}\\t_{21}&t_{22}&t_{23}\\t_{31}&t_{32}&t_{33}\end{pmatrix}=\begin{pmatrix}C_1\lambda_1+1&\frac{s_2}{s_1}C_2\lambda_1&\frac{s_3}{s_1}C_3\lambda_1\\ \frac{s_1}{s_2}C_1\lambda_2&C_2\lambda_2+1&\frac{s_3}{s_2}C_3\lambda_2\\ \frac{s_1}{s_3}C_1\lambda_3&\frac{s_2}{s_3}C_2\lambda_3&C_3\lambda_3+1\end{pmatrix}, \tag{4.94}$$

$$\text{where}\quad \|e_k\|^2=1\quad\text{and}\quad e_k\perp e_r,\quad 1\le k<r\le 3. \tag{4.95}$$

By (4.94) and the first relations in (4.95) we get

$$\lambda_k=-\frac{2s_k^2C_k}{s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2},\quad 1\le k\le 3. \tag{4.96}$$

Then the matrix elements $t=(t_{kr})_{k,r=1}^3$ are defined by (4.94). To verify $e_k\perp e_r$ we need to show that

$$(e_1,e_2)=\frac{(s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2)\lambda_1\lambda_2}{s_1s_2}+\frac{s_1^2C_1\lambda_2+s_2^2C_2\lambda_1}{s_1s_2}=0,$$
$$(e_1,e_3)=\frac{(s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2)\lambda_1\lambda_3}{s_1s_3}+\frac{s_1^2C_1\lambda_3+s_3^2C_3\lambda_1}{s_1s_3}=0,$$
$$(e_2,e_3)=\frac{(s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2)\lambda_2\lambda_3}{s_2s_3}+\frac{s_2^2C_2\lambda_3+s_3^2C_2\lambda_3}{s_2s_3}=0.$$

Indeed, for example, for (e_1,e_2) we have

$$(e_1,e_2)=\frac{(s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2)\lambda_1\lambda_2}{s_1s_2}+\frac{s_1^2C_1\lambda_2+s_2^2C_2\lambda_1}{s_1s_2}$$
$$=\frac{1}{s_1s_2(s_1^2C_1^2+s_2^2C_2^2+s_3^2C_3^2)}\big(4s_1^2s_2^2-(2s_1^2s_2^2+2s_1^2s_2^2)\big)C_1C_2=0.$$

The proofs of $e_1\perp e_3$ and $e_2\perp e_3$ are similar. ■

Similarly, for any $m\geq 2$ we can prove the following lemma:

Lemma 4.41. *For an arbitrary $(C_k)_{k=1}^m\in\mathbb{R}^m\setminus\{0\}$, and $D_m(s)=\operatorname{diag}(s_k)_{k=1}^m$ with $s_k\in\mathbb{R}_+$, $1\leq k\leq m$ there exists a unique element $t\in\pm\mathrm{O}(m)\setminus\{e\}$ and $(\lambda_k)_{k=1}^m\in\mathbb{R}^m\setminus\{0\}$ such that*

$$D_m(s)tD_m^{-1}(s)-\mathrm{I}=\begin{pmatrix}\lambda_1C_1&\lambda_1C_2&\dots&\lambda_1C_m\\ \lambda_2C_1&\lambda_2C_2&\dots&\lambda_2C_m\\ &&\dots&\\ \lambda_mC_1&\lambda_mC_2&\dots&\lambda_mC_m\end{pmatrix}. \tag{4.97}$$

The formulas for the corresponding λ_k are as follows:

$$\lambda_k=-2s_k^2C_k\Big(\sum_{r=1}^m s_r^2C_r^2\Big)^{-1},\quad 1\leq k\leq m. \tag{4.98}$$

4.5.2. Case $\Sigma_{123}(s)=(0,1,1)$

We have for some $s_1\in\mathbb{R}_+$ and all $(s_2,s_3)\in\big(\mathbb{R}_+\big)^2$: $\Sigma_{23}(s_2)=\infty$, $\Sigma_{13}(s_3)=\infty$.

Remark 4.42. Since $\Sigma_{12}(s_1)<\infty$, by (4.17) and Lemma 6.7, we can suppose

$$b=(b_{1n},b_{2n},b_{3n})_{n\in\mathbb{Z}}=(1,s_1^4(1+c_n),b_{3n})_{n\in\mathbb{Z}},\qquad \sum_n c_n^2<\infty.$$

Therefore, we can take $b=(1,1,b_{3n})_{n\in\mathbb{Z}}$, $s=1$, $c_n\equiv 0$.

Since $\Sigma_{13}(s)=\sum_{n\in\mathbb{Z}}\big(\frac{s^2}{\sqrt{b_{3n}}}-\frac{\sqrt{b_{3n}}}{s^2}\big)^2=\infty$, we have as in (4.48) three cases:

$$\lim_n b_{3n}=\begin{cases}(a)&\infty\\(b)&b>0\quad\text{with}\quad\sum_n b_n^2=\infty,\\(c)&0\end{cases} \tag{4.99}$$

where $b_{3n}=b(1+b_n)$ with $\lim_n b_n=0$ in the case (b).
Note that condition $S_3(3)=\infty$, implies $\sum_n b_{3n}^2=\infty$. Indeed, by (4.6) we have

$$S_1(3)=S_2(3)=\sum_{n\in\mathbb{Z}}\frac{1}{1+2b_{3n}}=\infty,\quad \infty=S_3(3)=\sum_{n\in\mathbb{Z}}\frac{b_{3n}^2}{1+2b_{3n}}\overset{(2.17)}{\sim}\sum_{n\in\mathbb{Z}}b_{3n}^2. \tag{4.100}$$

By (4.4) we have $\|Y_r^{(r)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{rk}^2}{b_{rk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})},$

$$\|Y_r^{(s)}\|^2 = \sum_{k\in\mathbb{Z}} \frac{b_{rk}^2}{b_{sk}^2 + 2(b_{1n}b_{2n} + b_{1n}b_{3n} + b_{2n}b_{3n})},\ s \neq r.$$

Let us denote

$$\begin{pmatrix} Y_{1n}^{(1)} & Y_{2n}^{(1)} & Y_{3n}^{(1)} \\ Y_{1n}^{(2)} & Y_{2n}^{(2)} & Y_{3n}^{(2)} \\ Y_{1n}^{(3)} & Y_{2n}^{(3)} & Y_{3n}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{3n}}{\sqrt{3+4b_{3n}}} \\ \frac{1}{\sqrt{3+4b_{3n}}} & \frac{1}{\sqrt{3+4b_{3n}}} & \frac{b_{3n}}{\sqrt{3+4b_{3n}}} \\ \frac{1}{\sqrt{b_{3n}^2+4b_{3n}+2}} & \frac{1}{\sqrt{b_{3n}^2+4b_{3n}+2}} & \frac{b_{3n}}{\sqrt{b_{3n}^2+4b_{3n}+2}} \end{pmatrix} \tag{4.101}$$

We have $\Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) < \infty$, indeed, since $Y_1^{(2)} = Y_2^{(2)}$ we get for example

$$\Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) = \frac{\Gamma(Y_2^{(2)}) + \Gamma(Y_2^{(2)}, Y_3^{(2)})}{1 + \Gamma(Y_3^{(2)}) + \Gamma(Y_1^{(2)}) + \Gamma(Y_3^{(2)}, Y_1^{(2)})} < 1.$$

Lemma 4.43. *In the cases* (a), (b) *and* (c) *given by* (4.99) *we have*

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \infty. \tag{4.102}$$

Proof. In all these cases we have $Y_1^{(3)} = Y_2^{(3)}$ hence, $\Gamma(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = 0$ and $\Gamma(Y_1^{(3)}, Y_2^{(3)}) = 0$. Therefore, by (2.15)

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \frac{\Gamma(Y_3^{(3)}) + \Gamma(Y_3^{(3)}, Y_1^{(3)}) + \Gamma(Y_3^{(3)}, Y_2^{(3)})}{1 + \Gamma(Y_1^{(3)}) + \Gamma(Y_2^{(3)})}$$

$$= \frac{\Gamma(Y_3^{(3)}) + 2\Gamma(Y_3^{(3)}, Y_1^{(3)})}{1 + 2\Gamma(Y_1^{(3)})} \sim \Delta(Y_3^{(3)}, Y_1^{(3)}). \tag{4.103}$$

We have two cases:

(a.1) when $\|Y_1^{(3)}\| < \infty$, and (a.2) when $\|Y_1^{(3)}\| = \infty$. In the case (a.1) we have $\Delta(Y_3^{(3)}, Y_1^{(3)}) \sim \Gamma(Y_3^{(3)}) = \infty$. Therefore, (4.102) holds. In the case (a.2) we should verify that

$$\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \infty \quad \text{for all} \quad (C_1, C_3) \in \mathbb{R}^2 \setminus \{0\}. \tag{4.104}$$

Then this will imply (4.102). We have

$$\|C_1 Y_1^{(3)} + C_3 Y_3^{(3)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{\left(C_1 + C_3 b_{3n}\right)^2}{b_{3n}^2 + 4b_{3n} + 2} =: \sum_{n\in\mathbb{Z}} g_n.$$

If $C_1 = 0$ or $C_3 = 0$ the later expression is divergent since $Y_1^{(3)} = Y_3^{(3)} = \infty$. Let $C_1C_3 \neq 0$. In this case $\lim_n g_n = C_3^2 > 0$ since $\lim_n b_{3n} = \infty$, case (a).

Therefore, $\sum_{n\in\mathbb{Z}} g_n = \infty$. By Lemma 6.3 for $m=1$, this implies $\Delta(Y_3^{(3)}, Y_1^{(3)}) = \infty$, therefore, (4.102). In the case (b) we have by (4.103)

$$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) \sim \Delta(Y_3^{(3)}, Y_1^{(3)}).$$

To prove that $\Delta(Y_3^{(3)}, Y_1^{(3)}) = \infty$ using Lemma 6.3 for $m = 1$ we should verify (4.104). We have $\|Y_3^{(3)}\|^2 = \infty$ since $S = (0, 1, 1)$. By (4.101)

$$\|Y_1^{(3)}\|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{b_{3n}^2 + 4b_{3n} + 2} \sim \sum_{n\in\mathbb{Z}} \frac{1}{b^2 + 4b + 2} = \infty.$$

The expression $\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2$ can be finite only for $(C_1,C_3)=\lambda(b,-1)$. Take $\lambda=1$, we get in the case (b)

$$\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{(b-b_{3n})^2}{b_{3n}^2+4b_{3n}+2}=\sum_{n\in\mathbb{Z}}\frac{b^2b_n^2}{b^2(1+b_n)^2+4b(1+b_n)+2}$$
$$\stackrel{(2.19)}{\sim}\sum_{n\in\mathbb{Z}}\frac{b_n^2}{(4b+2b^2)b_n+b^2+4b+2}\stackrel{(2.17)}{\sim}\sum_{n\in\mathbb{Z}}b_n^2=\infty.$$

In the case (c), we have by (4.103)

$$\Delta(Y_3^{(3)},Y_1^{(3)},Y_2^{(3)})\sim\Delta(Y_3^{(3)},Y_1^{(3)}).$$

To prove that $\Delta(Y_3^{(3)},Y_1^{(3)})=\infty$ using Lemma 6.3 for $m=1$, we should verify (4.104). Again, we have $\|Y_3^{(3)}\|^2=\infty$ since $S=(0,1,1)$. Because of $\lim_n b_{3n}=0$, we have by (4.101)

$$\|Y_1^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{1}{b_{3n}^2+4b_{3n}+2}\sim\sum_{n\in\mathbb{Z}}\frac{1}{2}=\infty,$$

Let $C_1C_3\neq 0$, then since $\lim_n b_{3n}=0$ we get

$$\|C_1Y_1^{(3)}+C_3Y_3^{(3)}\|^2=\sum_{n\in\mathbb{Z}}\frac{\left(C_1+C_3b_{3n}\right)^2}{b_{3n}^2+4b_{3n}+2}=\sum_{n\in\mathbb{Z}}\frac{C_3^2\left(b_{3n}+C_1C_3^{-1}\right)^2}{b_{3n}^2+4b_{3n}+2}=\infty. \qquad \blacksquare$$

By Lemma 4.43 we can approximate x_{3n}. By (4.5) we have

$$\|Y_1\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{1n}^2}{\frac{1}{2b_{1n}}+\frac{1}{2b_{2n}}+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{a_{1n}^2}{1+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{2b_{3n}a_{1n}^2}{1+2b_{3n}},$$

$$\|Y_2\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{2n}^2}{1+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{2b_{3n}a_{2n}^2}{1+2b_{3n}},\quad \|Y_3\|^2=\sum_{n\in\mathbb{Z}}\frac{a_{3n}^2}{1+\frac{1}{2b_{3n}}}=\sum_{k\in\mathbb{Z}}\frac{2b_{3n}a_{3n}^2}{1+2b_{3n}}.$$

Therefore, in the cases (a) and (b) we have

$$\|Y_1\|^2\sim\sum_{k\in\mathbb{Z}}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{k\in\mathbb{Z}}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{k\in\mathbb{Z}}a_{3n}^2,$$

In the case (c) we get

$$\|Y_1\|^2\sim\sum_{k\in\mathbb{Z}}b_{3n}a_{1n}^2,\quad \|Y_2\|^2\sim\sum_{k\in\mathbb{Z}}b_{3n}a_{2n}^2,\quad \|Y_3\|^2\sim\sum_{k\in\mathbb{Z}}b_{3n}a_{3n}^2.$$

Since in the cases (a) and (b)

$$\|Y_1\|^2\sim\sum_{n\in\mathbb{Z}}a_{1n}^2=\sum_{n\in\mathbb{Z}}b_{1n}a_{1n}^2\sim S_{11}^L(\mu)=\infty,$$
$$\|Y_2\|^2\sim\sum_{n\in\mathbb{Z}}a_{2n}^2=\sum_{n\in\mathbb{Z}}b_{2n}a_{2n}^2=S_{22}^L(\mu)=\infty,$$

we have two possibilities for $y_{23}:=(y_2,y_3)\in\{0,1\}^2$, see Section 4.4.4:

	(1.1)	(1.3)
y_1	1	1
y_2	1	1
y_3	0	1

In the case (c) we have $\|Y_3\|^2 \sim \sum_{n\in\mathbb{Z}} b_{3n}a_{3n}^2 \sim S_{33}^L(\mu) = \infty$.

Therefore, we have four possibilities for $y_{12} := (y_1, y_2) \in \{0,1\}^2$, see (4.52),

	(1.0)	(1.1)	(1.2)	(1.3)
y_1	0	1	0	1
y_2	0	0	1	1
y_3	1	1	1	1

Further, in the cases (a) and (b) we have four possibilities: (1.1.1), (1.3.1) and (1.1.0), (1.3.0), see Remark 4.17. In the case (1.1.1) we can approximate D_{1n}, D_{2n}, in the case (1.3.1) we can approximate all D_{rn}, $1 \le r \le 3$. In these cases the proof is finished, since we get respectively $D_{1n}, D_{2n}, x_{3n}\,\eta\,\mathfrak{A}^3$. The cases (a) and (b) subcases (1.1.0) and (1.3.0) are considered below.

In the case (c) subcase (1.0) we can approximate D_{3n} using Lemma 5.2, since $\Delta(Y_3, Y_2, Y_1) \sim \|Y_3\|^2 = \infty$, so we have $D_{3n}, x_{3n}\,\eta\,\mathfrak{A}^3$, and the proof is finished. Further, in the case (c) we have six cases (1.1.1), (1.2.1), (1.3.1) and (1.1.0), (1.2.0), (1.3.0), according to whether the corresponding expressions are divergent, see Remark 4.17. We can approximate in the three first cases by respectively D_{1n} and D_{3n} in the case (1.1.1), D_{2n} and D_{3n} in the case (1.1.2) and all D_{1n}, D_{2n}, D_{3n} in (1.1.3). The proof of irreducibilty is finished in these cases because we have respectively D_{1n}, D_{3n}, $x_{3n}\,\eta\,\mathfrak{A}^3$, D_{2n}, D_{3n}, $x_{3n}\,\eta\,\mathfrak{A}^3$, or D_{1n}, D_{2n}, D_{3n}, $x_{3n}\,\eta\,\mathfrak{A}^3$.

If the opposite holds, in the cases (a), (b) or (c), i.e., we are in the cases (1.1.0), (1.2.0) and (1.3.0) respectively, we try to approximate D_{3n} using Lemma 5.4. If one of the expressions $\Sigma_3(D,s)$ or $\Sigma_3^\vee(D,s)$ is divergent, we can approximate D_{3k} and the proof is finished, since we have x_{3n}, $D_{3n}\ \eta\ \mathfrak{A}^3$. Let us suppose, as in Remark 4.8, that for every sequence $s = (s_k)_{k\in\mathbb{Z}}$

$$\Sigma_3(D,s) + \Sigma_3^\vee(D,s) < \infty.$$

Then, in particularly, we have for $s^{(3)} = (s_k)_{k\in\mathbb{Z}}$ with $\frac{s_k^2}{b_{3k}} \equiv 1$

$$\infty > \Sigma_3(D,s^{(3)}) + \Sigma_3^\vee(D,s^{(3)}) \sim \Sigma_3(D) + \Sigma_3^\vee(D) = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}$$

$$\overset{(2.19)}{\sim} \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{\frac{1}{2b_{1k}} + a_{1k}^2 + \frac{1}{2b_{2k}} + a_{2k}^2} = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} =: \Sigma_3^{\vee,+}(D). \qquad (4.105)$$

In the case (a), (b) and (c) we have respectively

$$\Sigma_3^{\vee,+}(D) \sim \Sigma_3^+(D) = \sum_k \frac{2a_{3k}^2}{1 + 2a_{1k}^2 + 2a_{2k}^2}, \quad \Sigma_3^{\vee,+}(D) = \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2}.$$

In particular, in the case (c) we have by (4.105)

$$\infty > \sum_k \frac{\frac{1}{2b_{3k}} + a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} > \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} \sim \Sigma_3^+(D). \qquad (4.106)$$

The cases (a), subcase (1.1.0), where $\|Y_3\|^2 < \infty$ can not occur, because conditions $\Sigma_{12}(s_1) < \infty$ and $\nu_{12}(C_1, C_2) < \infty$ defined by (4.56), contradict the orthogonality condition for the matrix $\tau_{12}(\phi, s)$ defind by (4.66).

Indeed, by Remark 3.3 (instead of $\mu^2_{(b,a)}$ we can write $\mu^3_{(b,a)}$)

$$\left(\mu^3_{(b,a)}\right)^{L_{\tau_{12}(\phi,s)}} \perp \mu^3_{(b,a)} \Leftrightarrow \Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty,$$

where Σ_{12} is defined by (4.67), and $\Sigma_{12}(C_1, C_2)$ is defined by (4.68).

$$\Sigma_{12}(C_1, C_2) := \sum_{n\in\mathbb{Z}} (C_1^2 b_{1n} + C_2^2 b_{2n})(C_1 a_{1n} + C_2 a_{2n})^2 \sim \nu_{12}(C_1, C_2),$$

$$\infty > \Sigma_{12}(s) + \nu_{12}(C_1, C_2) \sim \Sigma_{12}(s) + \Sigma_{12}(C_1, C_2) = \infty, \tag{4.107}$$

which is a contradiction. In the case (a), (b), subcase (1.3.0) we get $\Sigma_3^+(D) = \infty$ by Lemma 4.20, a contradiction with (4.105) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$. In the case (c), subcases (1.1.0) and (1.2.0) we have respectively $\|Y_2\|^2 < \infty$ and $\|Y_1\|^2 < \infty$ hence,

$$\Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2} = \infty, \quad \Sigma_3^+(D) \sim \sum_k \frac{a_{3k}^2}{1 + a_{2k}^2} = \infty.$$

by Lemma 4.19, one gets again a contradiction with (4.105) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$. In the case (c), subcase (1.3.0) we get

$$\Sigma_3^+(D) = \sum_k \frac{a_{3k}^2}{1 + a_{1k}^2 + a_{2k}^2} = \infty$$

by Lemma 4.20, which is contradictory with (4.105) hence, $D_{3n}\,\eta\,\mathfrak{A}^3$.

4.5.3. Case $\Sigma_{123}(s) = (1, 1, 1)$

We have for all $s = (s_{12}, s_{23}, s_{13}) \in \mathbb{R}^3_+ \setminus \{0\}$

$$\Sigma_{12}(s_{12}) = \infty, \quad \Sigma_{23}(s_{23}) = \infty, \quad \Sigma_{13}(s_{13}) = \infty, \tag{4.108}$$

$$b = (b_{1n}, b_{2n}, b_{3n})_{n\in\mathbb{Z}} \stackrel{(4.17)}{=} (1, d_{2n}, d_{3n})_{n\in\mathbb{Z}}.$$

Recall that, see (4.31), we denote $D_n := d_{2n}^{-1} + d_{3n}^{-1} + 1$ and $d_n = \frac{d_{3n}}{d_{3n}}$. Set

$$\begin{pmatrix} Y_{1n}^{(1)} & Y_{2n}^{(1)} & Y_{3n}^{(1)} \\ Y_{1n}^{(2)} & Y_{2n}^{(2)} & Y_{3n}^{(2)} \\ Y_{1n}^{(3)} & Y_{2n}^{(3)} & Y_{3n}^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+2D_n d_{2n} d_{3n}}} & \frac{d_{2n}}{\sqrt{1+2D_n d_{2n} d_{3n}}} & \frac{d_{3n}}{\sqrt{1+2D_n d_{2n} d_{3n}}} \\ \frac{1}{\sqrt{d_{2n}^2+2D_n d_{2n} d_{3n}}} & \frac{d_{2n}}{\sqrt{d_{2n}^2+2D_n d_{2n} d_{3n}}} & \frac{d_{3n}}{\sqrt{d_{2n}^2+2D_n d_{2n} d_{3n}}} \\ \frac{1}{\sqrt{d_{3n}^2+2D_n d_{2n} d_{3n}}} & \frac{d_{2n}}{\sqrt{d_{3n}^2+2D_n d_{2n} d_{3n}}} & \frac{d_{3n}}{\sqrt{d_{3n}^2+2D_n d_{2n} d_{3n}}} \end{pmatrix}. \tag{4.109}$$

Remark 4.44. For (r, s) such that $1 \le r < s \le 3$ the following equivalence hold:

$$\Sigma_{rs}(s_{rs}) < \infty \ \Leftrightarrow \ \sum_{n\in\mathbb{Z}} c_{rs,n}^2 < \infty \ \Leftrightarrow \ \sum_{n\in\mathbb{Z}} c_{sr,n}^2 < \infty, \text{ where} \tag{4.110}$$

$$\frac{b_{rn}}{b_{sn}} =: s_{rs}^{-4}(1 + c_{rs,n}), \ \frac{b_{sn}}{b_{rn}} = s_{rs}^4(1 + c_{sr,n}), \ \lim_n \frac{b_{rn}}{b_{sn}} \in (0, \infty). \tag{4.111}$$

Proof. By Lemma 6.7 we have

$$\Sigma_{rs}(s_{rs}) = \sum_{n\in\mathbb{Z}} \left(s_{rs}^2 \sqrt{\frac{b_{rn}}{b_{sn}}} - s_{rs}^{-2} \sqrt{\frac{b_{sn}}{b_{rn}}}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{c_{rs,n}^2}{1 + c_{rs,n}} \sim \sum_{n\in\mathbb{Z}} c_{rs,n}^2,$$

$$\Sigma_{sr}(s_{rs}^{-1}) = \sum_{n\in\mathbb{Z}} \left(s_{rs}^{-2} \sqrt{\frac{b_{sn}}{b_{rn}}} - s_{rs}^2 \sqrt{\frac{b_{rn}}{b_{sn}}}\right)^2 = \sum_{n\in\mathbb{Z}} \frac{c_{sr,n}^2}{1 + c_{sr,n}} \sim \sum_{n\in\mathbb{Z}} c_{sr,n}^2.$$

Note also that

$$1 = \frac{b_{rn}}{b_{sn}} \frac{b_{sn}}{b_{rn}} = (1 + c_{rs,n})(1 + c_{sr,n}). \quad \blacksquare \tag{4.112}$$

By Remark 4.44, the condition $\Sigma_{rs}(s_{rs}) = \infty$ holds in the following cases:

$$l_{sr} := \lim_n \frac{b_{sn}}{b_{rn}} = \begin{cases} (a) & \infty \\ (b) & s_{rs}^4 > 0 \quad \text{with } \sum_{n\in\mathbb{Z}} c_{sr,n}^2 = \infty. \\ (c) & 0 \\ (d) & \lim \quad \text{does not exist} \end{cases} \tag{4.113}$$

Remark 4.45. In the case (d) we can use the fact that some *subsequence* of $\left(\frac{b_{sn}}{b_{rn}}\right)_{n\in\mathbb{Z}}$ has property (a), (b) or (c). We can avoid the case (c). Namely, if $l_{sr} = 0$ for some pair (r, s) with $1 \le r < s \le 3$, we can exchange the two lines (b_{sn}, a_{sn}) and (b_{rn}, a_{rn}) to obtain $l_{sr} = \infty$.

Formally, we have $3^3 = \#(A)^{\#(B)}$ possibilities where $A = \{(21), (32), (31)\}$ and $B = \{(a), (b), (d)\}$. Since $l_{32}l_{21} = l_{31}$ we get only the following cases:

$e \setminus (rs)$	(21)	(32)	(31)
(1)	b	b	b
(2)	a	a	a
(3)	a	b	a
(4)	b	a	b

.

To be able to approximate x_{rn} for $1 \le r \le 3$ we should study when the following expressions are infinite:

$$\rho_r(C_1, C_2, C_3) = \|C_1 Y_1^{(r)} + C_2 Y_2^{(r)} + C_2 Y_3^{(r)}\|^2. \tag{4.114}$$

By (4.109) we have

$$\rho_r(C_1, C_2, C_3) =: \sum_n \frac{|C_1 + C_2 d_{2n} + C_3 d_{3n}|^2}{C_{rn}}, \tag{4.115}$$

where $C_{1n} = 1+2D_n d_{2n} d_{3n}$, $C_{2n} = d_{2n}^2+2D_n d_{2n} d_{3n}$, $C_{3n} = d_{3n}^2+2D_n d_{2n} d_{3n}$.

Consider the case (1)=(bbb). We prove the following lemma.

Lemma 4.46. *Assume, that* (4.108) *holds for all* $s = (s_{12}, s_{23}, s_{13}) \in (\mathbb{R}_+)^3$.

Then $$\Delta(Y_3^{(3)}, Y_1^{(3)}, Y_2^{(3)}) = \Delta(Y_2^{(2)}, Y_3^{(2)}, Y_1^{(2)}) = \Delta(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}) = \infty. \tag{4.116}$$

Proof. For $1 \le r < s \le 3$ set

$$\frac{b_{sn}}{b_{rn}} = s_{rs}^4(1+c_{sr,n}) \quad \text{with} \quad \sum_{n\in\mathbb{Z}} c_{sr,n}^2 = \infty, \quad \lim_{n\to\infty} c_{sr,n} = 0.$$

For $b_{1n} \equiv 1$ we have

$$b_{2n} = s_{12}^4(1 + c_{21,n}),\ b_{3n} = s_{13}^4(1 + c_{31,n}),$$

$$\frac{b_{3n}}{b_{2n}} = \frac{s_{13}^4}{s_{12}^4}\frac{1 + c_{31,n}}{1 + c_{21,n}} = s_{23}^4(1 + c_{32,n}), \quad c_{32,n} = \frac{1 + c_{31,n}}{1 + c_{21,n}} - 1,\ s_{23} = \frac{s_{13}}{s_{12}},$$

$$\sum_n c_{32,n}^2 = \sum_n \left(\frac{1 + c_{31,n}}{1 + c_{21,n}} - 1\right)^2 = \sum_n \left(\frac{c_{31,n} - c_{21,n}}{1 + c_{21,n}}\right)^2 \sim \sum_n (c_{21,n} - c_{31,n})^2 = \infty.$$

Finally, we get

$$\sum_n c_{21,n}^2=\infty,\quad \sum_n c_{31,n}^2=\infty,\quad \sum_n (c_{21,n}-c_{31,n})^2=\infty. \tag{4.117}$$

By (4.114) and (4.115) we get

$$\rho_r(C_1,C_2,C_3)=\|C_1Y_1^{(r)}+C_2Y_2^{(r)}+C_2Y_3^{(r)}\|^2=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C_{rn}}$$
$$=\sum_n \frac{|C_1+C_2s_{12}^4(1+c_{21,n})+C_3s_{13}^4(1+c_{31,n})|^2}{C_{rn}}.$$

The latter expression is divergent if $C_1+C_2s_{12}^4+C_3s_{13}^4\neq 0$ since $\lim_{n\to\infty} c_{21,n}=\lim_{n\to\infty} c_{31,n}=0$ and $A_1\leq C_{rn}\leq A_2$. If $C_1+C_2s_{12}^4+C_3s_{13}^4=0$ we get

$$\rho_r(C_1,C_2,C_3)=\sum_{n\in\mathbb{Z}} \frac{|C_2s_{12}^4c_{21,n}+C_3s_{13}^4c_{31,n}|^2}{C_{rn}}=:\rho_r(C_2,C_3). \tag{4.118}$$

The latter expression is divergent by the first two relations in (4.117) when 1) $C_2C_3>0$, 2) $C_2=0$ and $C_3\neq 0$, 3) $C_2\neq 0$ and $C_3=0$. If $C_2C_3<0$ we have by the last relation in (4.117)

$$\sum_{n\in\mathbb{Z}} \frac{|C_2s_{12}^4c_{21,n}-C_3s_{13}^4c_{31,n}|^2}{C_{rn}}\sim\sum_{n\in\mathbb{Z}}|C_2s_{12}^4c_{21,n}-C_3s_{13}^4c_{31,n}|^2=\infty,$$

since $(s_{12},s_{13})=\frac{1}{s_1}(s_2,s_3)\in(\mathbb{R}^*)^2$ are arbitrary.
Consider the case (2)=(aaa). Now, see (4.113), we have

$$l_{21}=\lim_n \frac{b_{2n}}{b_{1n}}=\infty,\quad l_{32}=\lim_n \frac{b_{3n}}{b_{2n}}=\infty,\quad \text{therefore,}\ l_{31}=\lim_n \frac{b_{3n}}{b_{1n}}=\infty. \tag{4.119}$$

Since $b_{1n}\equiv 1$ we conclude that

$$l_{21}=\lim_n d_{2n}=\infty \quad\text{and}\quad l_{31}=\lim_n d_{3n}=\infty. \tag{4.120}$$

Therefore, we get for some $C>0$ and all $n\in\mathbb{Z}$

$$1\leq D_n=1+\big(d_{2n}\big)^{-1}+\big(d_{3n}\big)^{-1}\leq C. \tag{4.121}$$

By (4.114) and (4.115) we obtain

$$\rho_r(C_1,C_2,C_3)=\|C_1Y_1^{(r)}+C_2Y_2^{(r)}+C_2Y_3^{(r)}\|^2=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C_{rn}}$$
$$\sim\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{C'_{rn}}=:\rho'_r(C_1,C_2,C_3),$$
where $\quad C'_{rn}=1+2d_{2n}d_{3n},\quad C'_{rn}=d_{2n}^2+2d_{2n}d_{3n},\quad C'_{rn}=d_{3n}^2+2d_{2n}d_{3n}.$

We should study when $\rho'_r(C_1,C_2,C_3)=\infty$ for some $1\leq r\leq 3$:

$$\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{1+2d_{2n}d_{3n}},\ \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{2n}^2+2d_{2n}d_{3n}},\ \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{3n}^2+2d_{2n}d_{3n}}.$$

Writing as before $d_n = \frac{d_{3n}}{d_{2n}} =: l_n^{-1}$, we get

$$\rho_1'(C_1,C_2,C_3)=\sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{1+2d_{2n}d_{3n}} = \sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{\frac{1}{d_{2n}^2}+2d_n} \sim \sum_n \frac{|C_2+C_3d_n|^2}{2d_n},$$

$$\rho_2'(C_1,C_2,C_3) = \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{2n}^2+2d_{2n}d_{3n}} = \sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{1+2d_n}$$

$$\sim \sum_n \frac{|C_2+C_3d_n|^2}{1+2d_n}, \quad \rho_3'(C_1,C_2,C_3) = \sum_n \frac{|C_1+C_2d_{2n}+C_3d_{3n}|^2}{d_{3n}^2+2d_{2n}d_{3n}}$$

$$= \sum_n \frac{|\frac{C_1}{d_{2n}}+C_2+C_3d_n|^2}{d_n^2+2d_n} \sim \sum_n \frac{|C_2+C_3d_n|^2}{d_n^2+2d_n} = \sum_n \frac{|C_2l_n+C_3|^2}{1+2l_n}.$$

By Lemma 4.13 we get when $C_2C_3^{-1} > 0$

$$\begin{aligned} \rho_1'(C_1,C_2,-C_3) &\sim \sum_n \frac{|C_2-C_3d_n|^2}{2d_n} > \sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s), \\ \rho_2'(C_1,C_2,-C_3) &\sim \sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s), \qquad (4.122) \\ \rho_3'(C_1,C_2,-C_3) &\sim \sum_n \frac{|C_2l_n-C_3|^2}{1+2l_n} \sim \sum_n e_n^2 \sim \Sigma_{23}(s), \end{aligned}$$

where $\quad d_n = C_2C_3^{-1}(1+c_n), \;\; l_n = C_3C_2^{-1}(1+e_n), \quad s^4 = C_2C_3^{-1} > 0.$

But $\Sigma_{23}(s) = \infty$ for all $s > 0$, therefore for $C_2C_3^{-1} > 0$ we have

$$\rho_r(C_1,C_2,-C_3) \sim \Sigma_{23}(s) = \infty. \qquad (4.123)$$

If $C_2C_3^{-1} > 0$, by (4.122) we get

$$\sum_n \frac{|C_2+C_3d_n|^2}{1+2d_n} > \sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s) = \infty,$$

$$\sum_n \frac{|C_2+C_3d_n|^2}{1+2d_n} \sum_n \frac{|C_2-C_3d_n|^2}{1+2d_n} \sim \sum_n c_n^2 \sim \Sigma_{23}(s) = \infty,$$

$$\sum_n \frac{|C_2l_n+C_3|^2}{1+2l_n} > \sum_n \frac{|C_2l_n-C_3|^2}{1+2l_n} \sim \sum_n e_n^2 \sim \Sigma_{23}(s) = \infty.$$

Therefore, $\rho_r(C_1,C_2,C_3)=\infty$ for every $(C_1,C_2,C_3)\in\mathbb{R}^3\setminus\{0\}$.
Consider the case (3)=(aba). Now, see (4.113), we have

$$l_{21}=\lim_n \frac{b_{2n}}{b_{1n}} = \infty, \quad l_{32}=\lim_n \frac{b_{3n}}{b_{2n}} < \infty, \quad \text{therefore,} \quad l_{31}=\lim_n \frac{b_{3n}}{b_{1n}} = \infty.$$

So, we have again, see (4.120)

$$l_{21}=\lim_n d_{2n} = \infty, \quad \text{and} \quad l_{31}=\lim_n d_{3n} = \infty.$$

We are reduced to the case (2).
Consider the case (4)=(baa). Now, see (4.113), we have

$$l_{21}=\lim_n d_{2n} < \infty, \quad \text{and} \quad l_{31}=\lim_n d_{3n} = \infty.$$

Hence (4.121) holds too and we can use all estimations of the case (1). ■

Remark 4.47. In the cases (1)–(4) by Lemma 4.46 and Lemma 5.1 we can approximate x_{rn} for all $1 \le r \le 3$ and $n \in \mathbb{Z}$ and the irreducibility is proved.

5. Approximation of D_{kn} and x_{kn}

5.1. Approximation of x_{kn} by $A_{nk}A_{tk}$

For $m=3$, consider three rows as follows

$$\begin{pmatrix} ... & b_{11} & b_{12} & ... & b_{1n} & ... \\ ... & b_{21} & b_{22} & ... & b_{2n} & ... \\ ... & b_{31} & b_{32} & ... & b_{3n} & ... \end{pmatrix}.$$

Set $\lambda_k^{(r)}=(b_{1k}+b_{2k}+b_{3k})^2-(b_{1k}^2+b_{2k}^2+b_{3k}^2-b_{rk}^2),\ r=1,2,3,\ k\in\mathbb{Z}.$ (5.1)

Denote by $Y_r^{(s)}$ the following vectors:

$$x_{rk}^{(s)}=b_{rk}/\sqrt{\lambda_k^{(s)}},\ k\in\mathbb{Z},\quad Y_r^{(s)}=(x_{rk}^{(s)})_{k\in\mathbb{Z}}. \tag{5.2}$$

Lemma 5.1. *For any $n,t\in\mathbb{Z}$ and $1\leq r\leq 3$ one has*

$$x_{rn}x_{rt}\mathbf{1}\in\langle A_{nk}A_{tk}\mathbf{1}\mid k\in\mathbb{Z}\rangle\ \Leftrightarrow\ \Delta(Y_r^{(r)},Y_s^{(r)},Y_l^{(r)})=\infty,$$

where $\{r,s,l\}$ is a cyclic permutation of $\{1,2,3\}$.

Proof. The proof of Lemma 5.1 for $r=1$ is also based on Lemma 6.5 for $m=2$. We study when $x_{1n}x_{1t}\mathbf{1}\in\langle A_{nk}A_{tk}\mathbf{1}\mid k\in\mathbb{Z}\rangle$. Since

$$\begin{aligned}&A_{nk}A_{tk}=(x_{1n}D_{1k}+x_{2n}D_{2k}+x_{3n}D_{3k})(x_{1t}D_{1k}+x_{2t}D_{2k}+x_{3t}D_{3k})\\&=x_{1n}x_{1t}D_{1k}^2+x_{2n}x_{2t}D_{2k}^2+x_{3n}x_{3t}D_{3k}^2+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+\\&(x_{1n}x_{3t}+x_{3n}x_{1t})D_{1k}D_{3k}+(x_{2n}x_{3t}+x_{3n}x_{2t})D_{2k}D_{3k}\end{aligned}$$

and $MD_{1k}^2\mathbf{1}=-\frac{b_{1k}}{2}$, we take $t=(t_k)$ as follows:

$$-\sum_{k=-m}^{m}t_k\frac{b_{1k}}{2}=(t,b')=1,$$

where $t=(t_k)_{k=-m}^m$ and $b'=-(\frac{b_{1k}}{2})_{k=-m}^m\sim b=(b_{1k})_{k=-m}^m$.

We have

$$\begin{aligned}&\|[\sum_{k=-m}^{m}t_kA_{nk}A_{tk}-x_{1n}x_{1t}]\mathbf{1}\|^2=\|\sum_{k=-m}^{m}t_k\Big[x_{1n}x_{1t}\Big(D_{1k}^2+\frac{b_{1k}}{2}\Big)+x_{2n}x_{2t}D_{2k}^2\\&+x_{3n}x_{3t}D_{3k}^2+(x_{1n}x_{2t}+x_{2n}x_{1t})D_{1k}D_{2k}+(x_{1n}x_{3t}+x_{3n}x_{1t})D_{1k}D_{3k}\\&+(x_{2n}x_{3t}+x_{3n}x_{2t})D_{2k}D_{3k}\Big]\mathbf{1}\|^2=\sum_{-m\leq k,r\leq m}(f_k,f_r)t_kt_r=:(A_{2m+1}t,t),\end{aligned}$$

where $A_{2m+1}=\big((f_k,f_r)\big)_{k,r=-m}^m$ and $f_k=\sum_{r=1}^3f_k^r+\sum_{1\leq i<j\leq 3}f_k^{ij}$,

$$f_k^r=x_{rn}x_{rt}\Big(D_{rk}^2+\frac{b_{rk}}{2}\delta_{1r}\Big)\mathbf{1},\quad f_k^{ij}=(x_{in}x_{jt}+x_{jn}x_{it})D_{ik}D_{jk}\mathbf{1} \tag{5.3}$$

for $1\leq r\leq 3,\ 1\leq i<j\leq 3$.

Since $f_k^r \perp f_k^{ij}$, $\quad f_k^{ij} \perp f_k^{i'j'}$ for different (ij), $(i'j')$, writing

$$c_{kn} = \|x_{kn}\|^2 = \frac{1}{2b_{kn}} + a_{kn}^2,$$

we get

$$(f_k, f_k) = \sum_{r=1}^{3} \|f_k^r\|^2 + \sum_{1\leq i<j\leq 3} \|f_k^{ij}\|^2 = c_{1n}c_{1t}2\Big(\frac{b_{1k}}{2}\Big)^2 + c_{2n}c_{2t}3\Big(\frac{b_{2k}}{2}\Big)^2 + c_{3n}c_{3t}3$$
$$\times \Big(\frac{b_{3k}}{2}\Big)^2 + \big(c_{1n}c_{2t} + c_{1t}c_{2n} + 2a_{1n}a_{2t}a_{1t}a_{2n}\big)\frac{b_{1k}}{2}\frac{b_{2k}}{2} + \big(c_{1n}c_{3t} + c_{3t}c_{1n} + 2a_{1n}a_{3t}a_{3t}a_{1n}\big)$$
$$\times \frac{b_{1k}}{2}\frac{b_{3k}}{2} + \big(c_{2n}c_{3t} + c_{3t}c_{2n} + 2a_{2n}a_{3t}a_{3t}a_{2n}\big)\frac{b_{2k}}{2}\frac{b_{3k}}{2} \sim (b_{1k} + b_{2k} + b_{3k})^2,$$
$$(f_k, f_r) = (f_k^2, f_r^2) + (f_k^3, f_r^3) = c_{2n}c_{2t}\frac{b_{2k}}{2}\frac{b_{2r}}{2} + c_{3n}c_{3t}\frac{b_{3k}}{2}\frac{b_{3r}}{2} \sim b_{2k}b_{2r} + b_{3k}b_{3r}.$$

Finally, we have

$$(f_k, f_k) \sim (b_{1k} + b_{2k} + b_{3k})^2, \quad (f_k, f_r) \sim b_{2k}b_{2r} + b_{3k}b_{3r}, \quad k \neq r. \tag{5.4}$$

Set
$$\lambda_k = (b_{1k} + b_{2k} + b_{3k})^2 - (b_{2k}^2 + b_{3k}^2), \quad g_k = (b_{2k}, b_{3k}), \tag{5.5}$$
then
$$(f_k, f_k) \sim \lambda_k + (g_k, g_k), \quad (f_k, f_r) \sim (g_k, r_r). \tag{5.6}$$

For $A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^m$, and $b = -(b_{1k}/2)_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^{m} \lambda_k E_{kk} + \gamma(g_{-m}, \ldots, g_0, \ldots, g_m).$$

To finish the proof, it suffices to invoke Lemma 6.5 for $m = 2$. ■

5.2. Approximation of D_{rn} by A_{kn}

We will formulate several useful lemmas for the approximation of the independent variables x_{kn} and operators D_{kn} by combinations of the generators A_{kn}. The generators A_{kn} have the following form:

$$A_{kn} = x_{1k}D_{1n} + x_{2k}D_{2n} + x_{3k}D_{3n}, \; k, n \in \mathbb{Z}.$$

For $m = 3$, consider three rows as follows

$$\begin{pmatrix} \ldots & a_{11} & a_{12} & \ldots & a_{1n} & \ldots \\ \ldots & a_{21} & a_{22} & \ldots & a_{2n} & \ldots \\ \ldots & a_{31} & a_{32} & \ldots & a_{2n} & \ldots \end{pmatrix} \quad \text{and set} \quad \lambda_k = \frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}}. \tag{5.7}$$

Denote by Y_1, Y_2 and Y_3 the three following vectors:

$$x_{rk} = \frac{a_{rk}}{\sqrt{\lambda_k}}, \; k \in \mathbb{Z}, \quad Y_r = (x_{rk})_{k\in\mathbb{Z}}. \tag{5.8}$$

The proofs of Lemma 5.2 and 5.1 are based on Lemma 6.5 for $m = 2$.

Lemma 5.2. *For any $l \in \mathbb{Z}$ we have*

$$D_{rl}\mathbf{1} \in \langle A_{kl}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_r, Y_s, Y_t) = \infty,$$

where $\{r, s, t\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Proof. Without loss of generality we may assume that $r=1$. We determine when the inclusion

$$D_{1n}\mathbf{1}\in\langle A_{kn}\mathbf{1}=(x_{1k}D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n})\mathbf{1}\mid k\in\mathbb{Z}\rangle$$

holds. Fix $m\in\mathbb{N}$, since $Mx_{1k}=a_{1k}$, we put $\sum_{k=-m}^{m}t_ka_{1k}=(t,b)=1$, where $t=(t_k)_{k=-m}^{m}$ and $b=(a_{1k})_{k=-m}^{m}$. We have

$$\begin{aligned}&\|\big[\sum_{k=-m}^{m}t_k(x_{1k}D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n})-D_{1n}\big]\mathbf{1}\|^2\\ =\ &\|\sum_{k=-m}^{m}t_k[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n}]\mathbf{1}\|^2\\ =\ &\sum_{-m\le k,r\le m}(f_k,f_r)t_kt_r=:(A_{2m+1}t,t),\end{aligned}$$

where $A_{2m+1}=\big((f_k,f_r)\big)_{k,r=-m}^{m}$, and $f_k=[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n}]\mathbf{1}$.

We get $(f_k,f_k)=\|\,[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n}]\,\mathbf{1}\|^2=\frac{1}{2b_{1k}}\frac{b_{1n}}{2}$

$$+\left(\frac{1}{2b_{2k}}+a_{2k}^2\right)\frac{b_{2n}}{2}+\left(\frac{1}{2b_{3k}}+a_{3k}^2\right)\frac{b_{3n}}{2}\sim\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{2k}^2+a_{3k}^2,$$

$$\begin{aligned}(f_k,f_r)=&\Big(\big[(x_{1k}-a_{1k})D_{1n}+x_{2k}D_{2n}+x_{3k}D_{3n}\big]\mathbf{1},\big[(x_{1r}-a_{1r})D_{1n}\\&+x_{2r}D_{2n}+x_{3r}D_{3n}\big]\mathbf{1}\Big)=(x_{2k}\mathbf{1},x_{2r}\mathbf{1})(D_{2n}\mathbf{1},D_{2n}\mathbf{1})+(x_{3k}\mathbf{1},x_{3r}\mathbf{1})\\&\times(D_{3n}\mathbf{1},D_{3n}\mathbf{1})=a_{2k}a_{2r}\frac{b_{2n}}{2}+a_{3k}a_{3r}\frac{b_{3n}}{2}\sim a_{2k}a_{2r}+a_{3k}a_{3r}.\end{aligned}$$

Finally, we have

$$(f_k,f_k)\sim\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}}+a_{2k}^2+a_{3k}^2,\quad (f_k,f_r)\sim a_{2k}a_{2r}+a_{3k}a_{3r},\ k\ne r.\tag{5.9}$$

If we denote
$$\lambda_k=\frac{1}{2b_{1k}}+\frac{1}{2b_{2k}}+\frac{1}{2b_{3k}},\quad g_k=(a_{2k},a_{3k}),\tag{5.10}$$

then we have
$$(f_k,f_k)\sim\lambda_k+(g_k,g_k),\quad (f_k,f_r)\sim(g_k,g_r).\tag{5.11}$$

For $A_{2m+1}=\big((f_k,f_r)\big)_{k,r=-m}^{m}$, and $b=(a_{1k})_{k=-m}^{m}\in\mathbb{R}^{2m+1}$ we have

$$A_{2m+1}=\sum_{k=-m}^{m}\lambda_kE_{kk}+\gamma(g_{-m},\dots,g_0,\dots,g_m).$$

To finish the proof, it suffices to apply Lemma 6.5 for $m=2$. ■

5.3. Approximation of D_{kn} by $x_{rk}A_{kn}$

Set for $\{r,s,t\}$ a cyclic permutation of $\{1,2,3\}$:

$$\lambda_k^{(r)}=\left(\frac{1}{2b_{rk}}+a_{rk}^2\right)\Big(C_k+\sum_{l=1,l\ne r}^{3}a_{lk}^2\Big)-a_{rk}^2\Big(\sum_{l=1,l\ne r}^{3}a_{lk}^2\Big),\ C_k=\sum_{l=1}\frac{1}{2b_{lk}},\tag{5.12}$$

$$Y_{rr}=\left(\frac{\frac{1}{2b_{rk}}+a_{rk}^2}{\sqrt{\lambda_k^{(r)}}}\right)_{k\in\mathbb{Z}},Y_{rs}=\left(\frac{a_{rk}a_{sk}}{\sqrt{\lambda_k^{(r)}}}\right)_{k\in\mathbb{Z}},\ Y_{rt}=\left(\frac{a_{rk}a_{tk}}{\sqrt{\lambda_k^{(r)}}}\right)_{k\in\mathbb{Z}}.\tag{5.13}$$

Lemma 5.3. *For any $n \in \mathbb{Z}$ and $1 \le r \le 3$ we have*

$$D_{rn}\mathbf{1} \in \langle x_{1k}A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \quad \Leftrightarrow \quad \Delta(Y_{rr}, Y_{rs}, Y_{rt}) = \infty,$$

where $\{r, s, t\}$ is a cyclic permutation of $\{1, 2, 3\}$.

Proof. We prove for $r = 1$. We determine when the following

$$D_{1n}\mathbf{1} \in \langle x_{1k}A_{kn}\mathbf{1} = (x_{1k}^2 D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n})\mathbf{1} \mid k \in \mathbb{Z}\rangle$$

holds. Fix $m \in \mathbb{N}$, since $Mx_{1k}^2 = \frac{1}{2b_{1k}} + a_{1k}^2 =: c_{1k}$, we put $(t, b) = \sum_{k=-m}^{m} t_k c_{1k} = 1$, where $t = (t_k)_{k=-m}^{m}$ and $b = (c_{1k})_{k=-m}^{m}$. We have

$$\begin{aligned}
& \| \Big[\sum_{k=-m}^{m} t_k (x_{1k}^2 D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n}) - D_{1n} \Big] \mathbf{1} \|^2 \\
= \; & \| \sum_{k=-m}^{m} t_k \Big[\Big(x_{1k}^2 - c_{1k} \Big) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \Big] \mathbf{1} \|^2 \\
= \; & \sum_{-m \le k, r \le m} (f_k, f_r) t_k t_r =: (A_{2m+1}t, t), \text{ where } A_{2m+1} = \Big((f_k, f_r) \Big)_{k,r=-m}^{m},
\end{aligned}$$

and $$f_k = \Big[\Big(x_{1k}^2 - c_{1k} \Big) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \Big] \mathbf{1}.$$

Since $M|\psi - M|\psi||^2 = M\psi^2 - |M\psi|^2$ we have

$$M \left| x_{1k}^2 - c_{1k} \right|^2 = Mx_{1k}^4 - c_{1k}^2 = \frac{3}{(2b_{1k})^2} + 6\frac{1}{2b_{1k}} a_{1k}^2 + a_{1k}^4 - c_{1k}^2 = \frac{1}{2b_{1k}} \Big(\frac{2}{2b_{1k}} + 4a_{1k}^2 \Big),$$

hence, we get

$$\begin{aligned}
& (f_k, f_k) = \| \Big[(x_{1k}^2 - c_{1k}) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \Big] \mathbf{1} \|^2 = \\
& \frac{1}{2b_{1k}} \Big(\frac{2}{2b_{1k}} + 4a_{1k}^2 \Big) \frac{b_{1n}}{2} + c_{1k}c_{2k}\frac{b_{2n}}{2} + c_{1k}\frac{b_{3n}}{2}c_{3k} \sim c_{1k}\Big(C_k + a_{2k}^2 + a_{3k}^2 \Big), \\
& (f_k, f_r) = \Big(\Big[\Big(x_{1k}^2 - \Big(\frac{1}{2b_{1k}} + a_{1k}^2 \Big) \Big) D_{1n} + x_{1k}x_{2k}D_{2n} + x_{1k}x_{3k}D_{3n} \Big] \mathbf{1}, \\
& \Big[\Big(x_{1r}^2 - \Big(\frac{1}{2b_{1r}} + a_{1r}^2 \Big) \Big) D_{1n} + x_{1r}x_{2r}D_{2n} + x_{1r}x_{3r}D_{3n} \Big] \mathbf{1} \Big) = \\
& (x_{1k}\mathbf{1}, x_{1r}\mathbf{1})(x_{2k}\mathbf{1}, x_{2r}\mathbf{1})(D_{2n}\mathbf{1}, D_{2n}\mathbf{1}) + (x_{1k}\mathbf{1}, x_{1r}\mathbf{1})(x_{3k}\mathbf{1}, x_{3r}\mathbf{1})(D_{3n}\mathbf{1}, D_{3n}\mathbf{1}) \\
& = a_{1k}a_{1r}a_{2k}a_{2r}\frac{b_{2n}}{2} + a_{1k}a_{1r}a_{3k}a_{3r}\frac{b_{3n}}{2} \simeq a_{1k}a_{1r}(a_{2k}a_{2r} + a_{3k}a_{3r}).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
& (f_k, f_k) \sim \Big(\frac{1}{2b_{1k}} + a_{1k}^2 \Big) \Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2 \Big), \\
& (f_k, f_r) \sim a_{1k}a_{1r}(a_{2k}a_{2r} + a_{3k}a_{3r}), \; k \ne r.
\end{aligned} \tag{5.14}$$

Set

$$\begin{aligned}
& \lambda_k^{(1)} = \Big(\frac{1}{2b_{1k}} + a_{1k}^2 \Big) \Big(\frac{1}{2b_{1k}} + \frac{1}{2b_{2k}} + \frac{1}{2b_{3k}} + a_{2k}^2 + a_{3k}^2 \Big) - a_{1k}^2(a_{2k}^2 + a_{3k}^2), \\
& g_k = a_{1k}(a_{2k}, a_{3k}), \quad \text{then} \\
& (f_k, f_k) = \lambda_k^{(1)} + (g_k, g_k) \quad (f_k, f_r) \sim (g_k, g_r), \; k \ne r.
\end{aligned} \tag{5.15}$$

For $A_{2m+1} = \big((f_k, f_r)\big)_{k,r=-m}^m$, and $b = (a_{1k})_{k=-m}^m \in \mathbb{R}^{2m+1}$ we have

$$A_{2m+1} = \sum_{k=-m}^{m} \lambda_k E_{kk} + \gamma(g_{-m}, \dots, g_0, \dots, g_m).$$

To finish the proof, it suffices to apply Lemma 6.5 for $m = 2$. ■

5.4. Approximation of D_{rn} by $\exp\big(is_k(x_{rk} - a_{rk})\big)A_{kn}$

Lemma 5.4. *We have*

$$D_{3k}\mathbf{1} \in \langle \sin\big(s_k(x_{3k} - a_{3k})\big)A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Sigma_3(D, s) = \infty, \tag{5.16}$$

$$D_{3k}\mathbf{1} \in \langle \cos\big(s_k(x_{3k} - a_{3k})\big)A_{kn}\mathbf{1} \mid k \in \mathbb{Z}\rangle \ \Leftrightarrow\ \Sigma_3^{\vee}(D, s) = \infty, \tag{5.17}$$

$$\text{where}\ \ \Sigma_3(D, s) = \sum_{k\in\mathbb{Z}} \frac{|M\eta_{3k}(s_k)|^2}{\|g_k(s_k)\|^2},\ \ \Sigma_3^{\vee}(D, s) = \sum_{k\in\mathbb{Z}} \frac{|M\eta_{3k}^{\vee}(s_k)|^2}{\|g_k^{\vee}(s_k)\|^2}, \tag{5.18}$$

$$\text{moreover,}\ \ \Sigma_3(D, s^{(3)}) \sim \Sigma_3(D) := \sum_k \frac{\frac{1}{2b_{3k}}}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}, \tag{5.19}$$

$$\text{and}\ \ \Sigma_3^{\vee}(D, s^{(3)}) \sim \Sigma_3^{\vee}(D) := \sum_k \frac{a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2}, \tag{5.20}$$

where $s^{(3)} = (s_{3k})_k$ *with* $\frac{s_{3k}^2}{b_{3k}} \equiv 1$, $k \in \mathbb{Z}$ *and* $\eta_{3k}(s_k)$, $\eta_{3k}^{\vee}(s_k)$, $g_k(s_k)$, $g_k^{\vee}(s_k)$ *are defined by* (5.26)–(5.28).

Proof. We shall try to obtain separately the real part and imaginary part of $M\xi_{3k}(s)$, where $\xi_{3k}(s_k) = ix_{3k}\exp\big(is_k(x_{3k} - a_{3k})\big)$. Setting

$$F_b(s) = \int_{\mathbb{R}} \exp\big(is(x - a)\big)d\mu_{(b,a)}(x),$$

we obtain

$$F_b(s) = \int_{\mathbb{R}} \exp\big(isx\big)d\mu_{(b,0)}(x) = \exp\Big(-\frac{s^2}{4b}\Big), \tag{5.21}$$

where $d\mu_{(b,a)}(x)$ and $d\mu_{(b,0)}(x)$ are defined by

$$d\mu_{(b,a)}(x) = \sqrt{\frac{b}{\pi}}e^{-b(x-a)^2}dx \quad \text{and} \quad d\mu_{(b,0)}(x) = \sqrt{\frac{b}{\pi}}e^{-bx^2}dx. \tag{5.22}$$

Therefore,

$$H_{a,b}(s) = \int_{\mathbb{R}} ixe^{is(x-a)}d\mu_{(b,a)}(x) = \int_{\mathbb{R}} i(x+a)e^{isx}d\mu_{(b,0)}(x) \tag{5.23}$$

$$= \frac{dF_b(s)}{ds} + iaF_b(s) = \Big(-\frac{s}{2b} + ia\Big)\exp\Big(-\frac{s^2}{4b}\Big). \tag{5.24}$$

Recall the Euler formulas

$$e^{it} = \cos t + i\sin t, \quad e^{-it} = \cos t - i\sin t, \tag{5.25}$$
$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

More precisely, we denote for $1 \le r \le 3$

$$\eta_{rk}(s) = x_{rk}\cos\big(s_k(x_{3k} - a_{3k})\big), \quad \eta_{rk}^{\vee}(s) = x_{rk}\cos\big(s_k(x_{3k} - a_{3k})\big). \tag{5.26}$$

We determine when the inclusion holds:

$$D_{3k}\mathbf{1}\in\langle\sin\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1}=\Big(x_{1k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{1n}$$
$$+x_{2k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{2n}+x_{3k}\sin\big(s_k(x_{3k}-a_{3k})\big)D_{3n}\Big)\mathbf{1}\mid k\in\mathbb{Z}\rangle,$$
$$D_{3k}\mathbf{1}\in\langle\cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn}\mathbf{1}=\Big(x_{1k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{1n}$$
$$+x_{2k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{2n}+x_{3k}\cos\big(s_k(x_{3k}-a_{3k})\big)D_{3n}\Big)\mathbf{1}\mid k\in\mathbb{Z}\rangle.$$

Set

$$g_k(s_k)=\Big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1},\tag{5.27}$$
$$g_k^\vee(s_k)=\Big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\Big)\mathbf{1},\tag{5.28}$$

We show that

$$M\eta_{3k}(s)=-\frac{1}{2}\Big(H_{a,b}(s)+\overline{H_{a,b}(s)}\Big)=\frac{s}{2b_{3k}}\exp\Big(-\frac{s^2}{4b_{3k}}\Big),\tag{5.29}$$
$$M\eta_{3k}^\vee(s)=\frac{1}{2i}\Big(H_{a,b}(s)-\overline{H_{a,b}(s)}\Big)=a_{3k}\exp\Big(-\frac{s^2}{4b_{3k}}\Big).\tag{5.30}$$

Using the function $F_b(s)$ defined by (5.21) we get

$$M\eta(s)=\int_{\mathbb{R}}x\sin\big(s(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}}(x+a)\sin(sx)d\mu_{(b,0)}(x)$$
$$=\int_{\mathbb{R}}(x+a)\frac{e^{isx}-e^{-isx}}{2i}d\mu_{(b,0)}(x)=-\frac{1}{2}\int_{\mathbb{R}}i(x+a)\big(e^{isx}-e^{-isx}\big)d\mu_{(b,0)}(x)$$
$$=-\frac{1}{2}\Big(H_{a,b}(s)+\overline{H_{a,b}(s)}\Big)=\frac{s}{2b}\exp\Big(-\frac{s^2}{4b}\Big),$$

implying (5.29). Similarly we get

$$M\eta^\vee(s)=\int_{\mathbb{R}}x\cos\big(s(x-a)\big)d\mu_{(b,a)}(x)=\int_{\mathbb{R}}(x+a)\cos(sx)d\mu_{(b,0)}(x)$$
$$=\frac{1}{2i}\int_{\mathbb{R}}i(x+a)\Big(e^{isx}+e^{-isx}\Big)d\mu_{(b,0)}(x)=\frac{1}{2i}\Big(H_{a,b}(s)-\overline{H_{a,b}(s)}\Big)=ae^{-\frac{s^2}{4b}},$$

implying (5.29).

Fix $m\in\mathbb{N}$, we put $\sum_{k=-m}^m t_kM\eta_{3k}(s_k)=(t,b)=1$, where $t=(t_k)_{k=-m}^m$ and $b=(M\eta_{3k}(s_k))_{k=-m}^m$. We have

$$\|\Big[\sum_{k=-m}^m t_k\sin\big(s_k(x_{3k}-a_{3k})\big)A_{kn}-D_{3n}\Big]\mathbf{1}\|^2$$
$$=\|\sum_{k=-m}^m t_k\Big(\eta_{1k}(s_k)D_{1n}+\eta_{2k}(s_k)D_{2n}+\big[\eta_{3k}(s_k)-M\eta_{3k}(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2$$
$$=\sum_{k=-m}^m t_k^2\|g_k(s_k)\|^2,\ \text{ since }\Big(D_{rn}\mathbf{1},D_{ln}\mathbf{1}\Big)=0,\ \ 1\le r<l\le 3,\tag{5.31}$$

where the $g_k(s_k)$ are defined by (5.27). In order to calculate $\|g_k(s_k)\|^2$ note that

$$\|g_k(s_k)\|^2 = (g_k(s_k), g_k(s_k))$$
$$= \Big(\big(\eta_{1k}(s_k)D_{1n} + \eta_{2k}(s_k)D_{2n} + \big[\eta_{3k}(s_k) - M\eta_{3k}(s_k)\big]D_{3n}\big)\mathbf{1},$$
$$\big(\eta_{1k}(s_k)D_{1n} + \eta_{2k}(s_k)D_{2n} + \big[\eta_{3k}(s_k) - M\eta_{3k}(s_k)\big]D_{3n}\big)\mathbf{1}\Big)$$
$$= \|x_{1k}\mathbf{1}\|^2\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|^2 + \|x_{2k}\mathbf{1}\|^2\|\sin\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{2k}\mathbf{1}\|^2$$
$$+ \Big(M|\eta_{kn}(s_k)|^2 - |M\eta_{kn}(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2 = \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)I_3\frac{b_{1n}}{2}$$
$$+ \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)I_3\frac{b_{2n}}{2} + \Big(M|\eta_{kn}(s_k)|^2 - |M\eta_{kn}(s_k)|^2\Big)\frac{b_{3n}}{2}. \tag{5.32}$$

We need to calculate $I_3 = \|\sin\big(s_k(x_{3k} - a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}(s_k)|^2$ and $|M\eta_{kn}(s_k)|^2$. On setting $a := a_{3k}$, $b := b_{3k}$, we get

$$I_3 = \|\sin\big(s_k(x_{3k} - a_{3k})\big)\mathbf{1}\|^2 = \int_{\mathbb{R}} \frac{e^{isx} - e^{-isx}}{2i}\,\frac{e^{-isx} - e^{isx}}{-2i}\,d\mu_{(b,0)}(x)$$
$$= \frac{1}{2}\int_{\mathbb{R}}\Big(1 - \frac{e^{2isx} + e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x) \overset{(5.21)}{=} \frac{1 - e^{-\frac{s^2}{b}}}{2}, \tag{5.33}$$
$$|M\eta_{kn}(s_k)|^2 = \frac{s_k^2}{4b_{3k}^2}\exp\Big(-\frac{s_k^2}{2b_{3k}}\Big), \tag{5.34}$$
$$M|\eta_{kn}(s_k)|^2 = \int_{\mathbb{R}}(x^2 + 2xa + a^2)\frac{e^{isx} - e^{-isx}}{2i}\,\frac{e^{-isx} - e^{isx}}{-2i}\,d\mu_{(b,0)}(x)$$
$$= \frac{1}{2}\int_{\mathbb{R}}(x^2 + 2xa + a^2)\Big(1 - \frac{e^{2isx} + e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x)$$
$$= \frac{1}{2}\Big[\int_{\mathbb{R}}(x^2 + a^2)d\mu_{(b,0)}(x) - \int_{\mathbb{R}}(x^2 + a^2)\frac{e^{2isx} + e^{-2isx}}{2}\,d\mu_{(b,0)}(x)\Big]$$
$$= \frac{1}{2}\Big[\frac{1}{2b} + a^2 - \frac{d^2F_b(2s)}{ds^2} - a^2F_b(2s)\Big] \overset{(5.21)}{=} \frac{1}{2}\Big[\frac{1}{2b} + a^2 - \frac{1}{(2i)^2}\Big[\Big(\frac{2s}{b}\Big)^2 - \frac{2}{b}\Big]$$
$$\times e^{-\frac{s^2}{b}} - a^2e^{-\frac{s^2}{b}}\Big] = \frac{1}{2}\Big[\Big(\frac{1}{2b} + a^2\Big)(1 - e^{-\frac{s^2}{b}}) + \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big]. \tag{5.35}$$

Finally, we have

$$M|\eta_{kn}(s_k)|^2 - |M\eta_{kn}(s_k)|^2 = \frac{1}{2}\Big[\Big(\frac{1}{2b} + a^2\Big)(1 - e^{-\frac{s^2}{b}}) + \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big] - \frac{s^2}{4b^2}e^{-\frac{s^2}{2b}}. \tag{5.36}$$

By (5.39),(5.32), (5.33), (5.36) and (6.2) we prove (5.16), where

$$\Sigma_3(D,s) = \sum_{k\in\mathbb{Z}}\frac{|M\eta_{kn}(s_k)|^2}{\|g_k(s_k)\|^2}$$
$$= \sum_{k\in\mathbb{Z}}\frac{\frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}{\Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)I_3\frac{b_{1n}}{2} + \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)I_3\frac{b_{2n}}{2} + \Big(M|\eta_{kn}(s_k)|^2 - |M\eta_{kn}(s_k)|^2\Big)\frac{b_{3n}}{2}}$$
$$\sim \sum_{k\in\mathbb{Z}}\frac{\frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}{\frac{1-e^{-\frac{s_k^2}{b_{3k}}}}{2}\Big(c_{1k} + c_{2k}\Big) + \frac{1}{2}\Big[c_{3k}(1 - e^{-\frac{s_k^2}{b_{3k}}}) + \frac{s_k^2}{b_{3k}^2}e^{-\frac{s_k^2}{b_{3k}}}\Big] - \frac{s_k^2}{4b_{3k}^2}e^{-\frac{s_k^2}{2b_{3k}}}}$$
$$= \sum_{k\in\mathbb{Z}}\frac{\frac{x_k^2}{4b_{3k}}e^{-\frac{x_k^2}{2}}}{\frac{1-e^{-x_k^2}}{2}\Big(c_{1k} + c_{2k}\Big) + \frac{1}{2}\Big[c_{3k}(1-e^{-x_k^2}) + \frac{x_k^2}{b_{3k}}e^{-x_k^2}\Big] - \frac{x_k^2}{4b_{3k}}e^{-\frac{x_k^2}{2}}} =: \Sigma_3(D,x),$$

where $x_k^2=\frac{s_k^2}{b_{3k}}$ and $c_{rk}=\frac{1}{2b_{rk}}+a_{rk}^2$. For $x^{(3)}=(x_k)_k$ with $x_k\equiv 1$ we get

$$\Sigma_3(D,x^{(3)})=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}{\frac{1-e^{-1}}{2}\Big(c_{1k}+c_{2k}\Big)+\frac{1}{2}\Big[c_{3k}(1-e^{-1})+\frac{1}{b_{3k}}e^{-1}\Big]-\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}$$
$$=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{4b_{3k}}e^{-\frac{1}{2}}}{\frac{1-e^{-1}}{2}\Big(c_{1k}+c_{2k}+c_{3k}\Big)+\frac{1}{2b_{3k}}\Big(e^{-1}-\frac{1}{2}e^{-\frac{1}{2}}\Big)}$$
$$\overset{(2.11)}{\sim}\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{3k}}}{c_{1k}+c_{2k}+c_{3k}}=\sum_{k\in\mathbb{Z}}\frac{\frac{1}{2b_{3k}}}{C_k+a_{1k}^2+a_{2k}^2+a_{3k}^2}=\Sigma_3(D).$$

So, we have proved (5.19) for $x=(x_k)_k$ with $x_k\equiv 1$. To approximate D_{3n} in terms of functions involving cosine, fix $m\in\mathbb{N}$, and put $\sum_{k=-m}^m t_kM\eta_{3k}^\vee(s_k)=(t,b)=1$, where $t=(t_k)_{k=-m}^m$ and $b=(M\eta_{3k}^\vee(s_k))_{k=-m}^m$. We have

$$\|\Big[\sum_{k=-m}^m t_k\cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn}-D_{3n}\Big]\mathbf{1}\|^2$$
$$=\|\sum_{k=-m}^m t_k\Big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2$$
$$=\sum_{k=-m}^m t_k^2\|g_k^\vee(s_k)\|^2,\ \text{ since }\Big(D_{rn}\mathbf{1},D_{ln}\mathbf{1}\Big)=0,\ \ 1\le r<l\le 3,\tag{5.37}$$

where the $g_k^\vee(s_k)$ are defined by (5.27). To calculate $\|g_k^\vee(s_k)\|^2$ we have

$$\|g_k^\vee(s_k)\|^2=(g_k^\vee(s_k),g_k^\vee(s_k))$$
$$=\Big(\big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1},$$
$$\big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1}\Big)$$
$$=\|x_{1k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{1k}\mathbf{1}\|^2+\|x_{2k}\mathbf{1}\|^2\|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2\|D_{2k}\mathbf{1}\|^2$$
$$+\Big(M|\eta_{kn}^\vee(s_k)|^2-|M\eta_{kn}^\vee(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2=\Big(\frac{1}{2b_{1k}}+a_{1k}^2\Big)I_3^\vee\frac{b_{1n}}{2}$$
$$+\Big(\frac{1}{2b_{2k}}+a_{2k}^2\Big)I_3^\vee\frac{b_{2n}}{2}+\Big(M|\eta_{kn}^\vee(s_k)|^2-|M\eta_{kn}^\vee(s_k)|^2\Big)\frac{b_{3n}}{2}.\tag{5.38}$$

We need to calculate $I_3^\vee=\|\cos\big(is_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}^\vee(s_k)|^2$ and $|M\eta_{kn}^\vee(s_k)|^2$. Let us set $b:=b_{3k}$. To approximate D_{3n} in terms of functions involving the cosine, fix $m\in\mathbb{N}$, and put $\sum_{k=-m}^m t_kM\eta_{3k}^\vee(s_k)=(t,b)=1$, where $t=(t_k)_{k=-m}^m$ and $b=(M\eta_{3k}^\vee(s_k))_{k=-m}^m$. We have

$$\|\Big[\sum_{k=-m}^m t_k\cos\big(s_k(x_{3k}-a_{3k})\big)A_{kn}-D_{3n}\Big]\mathbf{1}\|^2$$
$$=\|\sum_{k=-m}^m t_k\Big(\eta_{1k}^\vee(s_k)D_{1n}+\eta_{2k}^\vee(s_k)D_{2n}+\big[\eta_{3k}^\vee(s_k)-M\eta_{3k}^\vee(s_k)\big]D_{3n}\Big)\mathbf{1}\|^2$$
$$=\sum_{k=-m}^m t_k^2\|g_k^\vee(s_k)\|^2,\ \text{ since }\Big(D_{rn}\mathbf{1},D_{ln}\mathbf{1}\Big)=0,\ \ 1\le r<l\le 3,\tag{5.39}$$

where the $g_k^\vee(s_k)$ are defined by (5.27). To calculate $\|g_k^\vee(s_k)\|^2$ we have

$$\begin{aligned}
&\|g_k^\vee(s_k)\|^2 = (g_k^\vee(s_k), g_k^\vee(s_k))\\
&= \Big(\big(\eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + \big[\eta_{3k}^\vee(s_k) - M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1},\\
&\quad \big(\eta_{1k}^\vee(s_k)D_{1n} + \eta_{2k}^\vee(s_k)D_{2n} + \big[\eta_{3k}^\vee(s_k) - M\eta_{3k}^\vee(s_k)\big]D_{3n}\big)\mathbf{1}\Big)\\
&= \|x_{1k}\mathbf{1}\|^2 \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2 \|D_{1k}\mathbf{1}\|^2\\
&\quad + \|x_{2k}\mathbf{1}\|^2 \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2 \|D_{2k}\mathbf{1}\|^2\\
&\quad + \Big(M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2\Big)\|D_{3k}\mathbf{1}\|^2 = \Big(\frac{1}{2b_{1k}} + a_{1k}^2\Big)I_3^\vee \frac{b_{1n}}{2}\\
&\quad + \Big(\frac{1}{2b_{2k}} + a_{2k}^2\Big)I_3^\vee \frac{b_{2n}}{2} + \Big(M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2\Big)\frac{b_{3n}}{2}. \qquad (5.40)
\end{aligned}$$

We need to calculate $I_3^\vee = \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2$, $M|\eta_{kn}^\vee(s_k)|^2$ and $|M\eta_{kn}^\vee(s_k)|^2$. Setting $a := a_{3k}$, $b := b_{3k}$, we get

$$\begin{aligned}
&I_3^\vee = \|\cos\big(s_k(x_{3k}-a_{3k})\big)\mathbf{1}\|^2 = \int_{\mathbb{R}} \frac{e^{isx}+e^{-isx}}{2}\,\frac{e^{-isx}+e^{isx}}{2}\,d\mu_{(b,0)}(x)\\
&= \frac12\int_{\mathbb{R}}\Big(1+\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x) \overset{(5.21)}{=} \frac{1+e^{-\frac{s^2}{b}}}{2}, \qquad (5.41)\\
&|M\eta_{kn}^\vee(s_k)|^2 = a_{3k}^2\exp\Big(-\frac{s_k^2}{2b_{3k}}\Big), \qquad (5.42)\\
&M|\eta_{kn}^\vee(s_k)|^2 = \int_{\mathbb{R}}(x^2+2xa+a^2)\frac{e^{isx}+e^{-isx}}{2}\,\frac{e^{-isx}+e^{isx}}{2}\,d\mu_{(b,0)}(x)\\
&= \frac12\int_{\mathbb{R}}(x^2+2xa+a^2)\Big(1+\frac{e^{2isx}+e^{-2isx}}{2}\Big)d\mu_{(b,0)}(x)\\
&= \frac12\Big[\int_{\mathbb{R}}(x^2+a^2)d\mu_{(b,0)}(x) + \int_{\mathbb{R}}(x^2+a^2)\frac{e^{2isx}+e^{-2isx}}{2}\,d\mu_{(b,0)}(x)\Big]\\
&= \frac12\Big[\frac{1}{2b}+a^2+\frac{d^2F_b(2s)}{ds^2}+a^2F_b(2s)\Big] \overset{(5.21)}{=} \frac12\Big[\frac{1}{2b}+a^2+\frac{1}{(2i)^2}\Big[\Big(\frac{2s}{b}\Big)^2-\frac{2}{b}\Big]\\
&\times e^{-\frac{s^2}{b}} + a^2e^{-\frac{s^2}{b}}\Big] = \frac12\Big[\Big(\frac{1}{2b}+a^2\Big)(1+e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big]. \qquad (5.43)
\end{aligned}$$

Finally, we get

$$M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2 = \frac12\Big[\Big(\frac{1}{2b}+a^2\Big)(1+e^{-\frac{s^2}{b}}) - \frac{s^2}{b^2}e^{-\frac{s^2}{b}}\Big] - e^{-\frac{s^2}{2b}}. \qquad (5.44)$$

By (5.37), (5.38), (5.41), (5.44) and (6.2) we prove (5.17), where

$$\begin{aligned}
&\Sigma_3^\vee(D,s) = \sum_{k\in\mathbb{Z}} \frac{|M\eta_{kn}^\vee(s_k)|^2}{\|g_k^\vee(s_k)\|^2}\\
&= \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2\exp(-s_k^2/2b_{3k})}{\big(\frac{1}{2b_{1k}}+a_{1k}^2\big)I_3^\vee\frac{b_{1n}}{2} + \big(\frac{1}{2b_{2k}}+a_{2k}^2\big)I_3^\vee\frac{b_{2n}}{2} + \big(M|\eta_{kn}^\vee(s_k)|^2 - |M\eta_{kn}^\vee(s_k)|^2\big)\frac{b_{3n}}{2}}\\
&\sim \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2\exp(-\frac{s_k^2}{2b_{3k}})}{\frac{1+\exp(-\frac{s_k^2}{b_{3k}})}{2}\Big(c_{1k}+c_{2k}\Big)+\frac12\Big[c_{3k}(1+\exp(-\frac{s_k^2}{b_{3k}})) - \frac{s_k^2}{b_{3k}^2}\exp(-\frac{s_k^2}{b_{3k}})\Big] - a_{3k}^2\exp(-\frac{s_k^2}{2b_{3k}})}\\
&= \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2\exp(-x_k^2/2)}{\frac{1-e^{-x_k^2}}{2}\Big(c_{1k}+c_{2k}\Big)+\frac12\Big[c_{3k}(1+e^{-x_k^2}) - \frac{x_k^2}{b_{3k}}e^{-x_k^2}\Big] - a_{3k}^2e^{-\frac{x_k^2}{2}}} = \Sigma_3(D,x), \qquad (5.45)
\end{aligned}$$

where $x_k^2 = \frac{s_k^2}{b_{3k}}$ and $c_{rk} = \frac{1}{2b_{rk}} + a_{rk}^2$. For $x^{(3)} = (x_k)_k$ with $x_k \equiv 1$ we get

$$\Sigma_3^\vee(D, x^{(3)}) = \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}\Big(c_{1k} + c_{2k}\Big) + \frac{1}{2}\Big[c_{3k}(1+e^{-1}) - \frac{1}{b_{3k}}e^{-1}\Big] - a_{3k}^2 e^{-\frac{1}{2}}}$$

$$= \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2 e^{-\frac{1}{2}}}{\frac{1+e^{-1}}{2}\Big(c_{1k} + c_{2k} + c_{3k}\Big) - \Big(\frac{1}{2b_{3k}}e^{-1} + a_{3k}^2 e^{-\frac{1}{2}}\Big)}$$

$$\overset{(2.11)}{\sim} \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2}{c_{1k} + c_{2k} + c_{3k}} = \sum_{k\in\mathbb{Z}} \frac{a_{3k}^2}{C_k + a_{1k}^2 + a_{2k}^2 + a_{3k}^2} = \Sigma_3^\vee(D). \tag{5.46}$$

So, we have proved (5.20) for $x = (x_k)_k$ with $x_k \equiv 1$. ■

6. Appendix

6.1. Some estimates and the generalized characteristic polynomial

Lemma 6.1 ([23]). *For a strictly positive operator A (i.e., $(Af, f) > 0$ for $f \neq 0$) acting on $\mathbb{R}^n$ and a vector $b \in \mathbb{R}^n \setminus \{0\}$, we have*

$$\min_{x\in\mathbb{R}^n} \Big((Ax, x) \mid (x, b) = 1\Big) = \frac{1}{(A^{-1}b, b)}. \tag{6.1}$$

The minimum is obtained for $x = \dfrac{A^{-1}b}{(A^{-1}b, b)}$. In the particular case $A = \mathrm{diag}(a_k)_{k=1}^n$ we get

$$\min_{x\in\mathbb{R}^n} \Big(\sum_{k=1}^n a_k x_k^2 \mid \sum_{k=1}^n x_k b_k = 1\Big) = \Big(\sum_{k=1}^n \frac{b_k^2}{a_k}\Big)^{-1}. \tag{6.2}$$

Definition 6.2. For a matrix $C \in \mathrm{Mat}(n, \mathbb{C})$ and $\lambda = (\lambda_k)_{k=1}^n \in \mathbb{C}^n$ define the *generalization of the characteristic polynomial* $p_C(t) = \det(tI - C)$, $t \in \mathbb{C}$ as follows, see [26]:

$$P_C(\lambda) = \det C(\lambda), \quad \text{where} \quad C(\lambda) = \mathrm{diag}\big(\lambda_1, \dots, \lambda_n\big) + C. \tag{6.3}$$

We calculated in [30] $P_C(\lambda)$, $C^{-1}(\lambda)$ and $(C^{-1}(\lambda)a, a)$ for an arbitrary n.

Fix two natural numbers $n, m \in \mathbb{N}$ with $m \leq n$, two matrices A_{mn} and X_{mn}, vectors $g_k \in \mathbb{C}^{m-1}$, $1 \leq k \leq n$ and $a \in \mathbb{C}^n$ as follows

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

$$g_k = \begin{pmatrix} a_{2k} \\ a_{3k} \\ \dots \\ a_{mk} \end{pmatrix} \in \mathbb{C}^{m-1}, \quad a = (a_{1k})_{k=1}^n \in \mathbb{C}^n. \tag{6.4}$$

Set

$$C = \gamma(g_1, g_2, \dots, g_n) = \begin{pmatrix} (g_1, g_1) & (g_1, g_2) & \dots & (g_1, g_n) \\ (g_2, g_1) & (g_2, g_2) & \dots & (g_2, g_n) \\ & & \dots & \\ (g_n, g_1) & (g_n, g_2) & \dots & (g_n, g_n) \end{pmatrix}. \tag{6.5}$$

Consider the matrix

$$X_{mn}=\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ & & \dots & \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix}, \quad \text{where} \quad x_{rk}=\frac{a_{rk}}{\sqrt{\lambda_k}}, \tag{6.6}$$

$$\bar{x}_k := (x_{rk})_{r=2}^m = \frac{g_k}{\sqrt{\lambda_k}} \in \mathbb{C}^{m-1}, \quad y_r=(x_{rk})_{k=1}^n \in \mathbb{R}^n. \tag{6.7}$$

6.2. Properties of m infinite vectors

In fact, the statements below hold for arbitrary m, see [29, 30].

Lemma 6.3 ([29]). *Let f_0, f_1, f_2 be three infinite real vectors $f_r = (f_{rk})_{k\in\mathbb{N}}$, where $0 \le r \le 2$. Then for all r, s with $0 \le r < s \le 2$ one has*

$$\frac{\Gamma(f_0, f_1, f_2)}{\Gamma(f_r, f_s)} := \lim_{n\to\infty} \frac{\Gamma(f_0^{(n)}, f_1^{(n)}, f_2^{(n)})}{\Gamma(f_r^{(n)}, f_s^{(n)})} = \infty, \tag{6.8}$$

if and only if for all $(C_0, C_1, C_2) \in \mathbb{R}^3 \setminus \{0\}$ one has $\sum_{r=0}^2 C_r f_r \notin l_2(\mathbb{N})$ and $C_r f_r + C_s f_s \notin l_2(\mathbb{N})$ for all $0 \le r < s \le 2$ and all $(C_r, C_s) \in \mathbb{R}^2 \setminus \{0\}$, where $f_r^{(n)} = (f_{rk})_{k=1}^n$.

Lemma 6.4 ([30], Theorem 5.3). *For $m = 3$ we have*

$$(C^{-1}(\lambda)a, a) = \Delta(y_1, y_2, y_3) \overset{(2.13)}{=} \frac{\det\big(I_3 + \gamma(y_1, y_2, y_3)\big)}{\det\big(I_2 + \gamma(y_2, y_3)\big)} - 1, \tag{6.9}$$

where $a = (a_{1k})_{k\in\mathbb{Z}}$, $y_r = (x_{rk})_{k\in\mathbb{Z}}$ are defined by (6.6) and $\lambda = (\lambda_k)_{k\in\mathbb{Z}}$.

Lemma 6.5 ([29]). *Let $(y_k)_{k=1}^3$ be 3 real vectors such that $\sum_{k=1}^3 C_k y_k \notin l_2(\mathbb{Z})$ for* any nontrivial combination *$(C_k)_{k=1}^3 \in \mathbb{R}^3 \setminus \{0\}$, then*

$$\frac{\det\big(I_3 + \gamma(y_1, y_2, y_2)\big)}{\det\big(I_2+\gamma(y_2, y_3)\big)} = \lim_{n\to\infty} \frac{\det\big(I_3 + \gamma(y_1^{(n)}, y_2^{(n)}, y_3^{(n)})\big)}{\det\big(I_2 + \gamma(y_2^{(n)}, y_3^{(n)})\big)} = \infty, \tag{6.10}$$

where $y_r^{(n)} = (x_{rk})_{k=-n}^n \in \mathbb{R}^{2n+1}$.

Proof. The proof follows from Lemma 6.3 and (2.8). ■

6.3. Comparison of two Gaussian measures

For two centered Gaussian measures $\mu_{(b,0)}$ and $\mu_{(b',0)}$ on the real line $\mathbb{R}$ defined by (2.5) it is well known that

$$H(\mu_{(b,0)}, \mu_{(b',0)}) = \Big(\frac{4bb'}{(b+b')^2}\Big)^{1/4}. \tag{6.11}$$

By Kakutani's criterion for product measures on $\mathbb{R}^{\mathbb{N}}$ [14], and (6.11) we see that the following lemma holds true.

Lemma 6.6. *Two Gaussian measures* $\mu_{(b,0)}=\otimes_{n\in\mathbb{Z}}\mu_{(b,0)}$ *and* $\mu_{(b',0)}=\otimes_{n\in\mathbb{Z}}\mu_{(b',0)}$ *are equivalent if and only if the product*

$$\prod_{n\in\mathbb{Z}}\frac{4b_nb'_n}{(b_n+b'_n)^2} \tag{6.12}$$

does not converge to 0. *An equivalent condition is*

$$\sum_{n\in\mathbb{Z}}\Big(\sqrt{\frac{b_n}{b'_n}}-\sqrt{\frac{b'_n}{b_n}}\Big)^2<\infty. \tag{6.13}$$

Consider two measures: $\mu_{(\mathbb{I},0)}=\otimes_{n\in\mathbb{Z}}\mu_{(1,0)}$ and $\mu_{(\mathbb{I}+c,0)}=\otimes_{n\in\mathbb{Z}}\mu_{(1+c_n,0)}$ on the space X_1, where the measure $\mu_{(b,a)}$ on the real line $\mathbb{R}$ is defined by (2.5).

Lemma 6.7. *The two measures* $\mu_{(\mathbb{I},0)}$ *and* $\mu_{(\mathbb{I}+c,0)}$ *are equivalent if and only if* $\sum_{n\in\mathbb{Z}}c_n^2<\infty$.

Proof. By Lemma 6.6 and (6.13),

$$\mu_{(\mathbb{I},0)}\sim\mu_{(\mathbb{I}+c,0)}\ \Leftrightarrow\ \sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{1+c_n}}-\sqrt{1+c_n}\right)^2=\sum_{n\in\mathbb{Z}}\frac{c_n^2}{1+c_n}<\infty.$$

By Lemma 2.10, two series $\sum_{n\in\mathbb{Z}}\frac{c_n^2}{1+c_n}$ and $\sum_{n\in\mathbb{Z}}c_n^2$ are equivalent. ■

The next lemma is also a consequence of Kakutani's criterion [14].

Lemma 6.8. *Two Gaussian measures* $\mu^m_{(b,0)}$ *and* $\mu^m_{(b',0)}$ *are equivalent if and only if the product*

$$\prod_{r=1}^m\prod_{n\in\mathbb{Z}}\frac{4b_{rn}b'_{rn}}{(b_{rn}+b'_{rn})^2} \tag{6.14}$$

does not converge to 0. *An equivalent condition is*

$$\sum_{r=1}^m\sum_{n\in\mathbb{Z}}\left(\sqrt{\frac{b_{rn}}{b'_{rn}}}-\sqrt{\frac{b'_{rn}}{b_{rn}}}\right)^2<\infty. \tag{6.15}$$

Lemma 6.9. *For* $t\in\mathrm{GL}(m,\mathbb{R})\backslash\{e\}$ *we have* $(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$ *if and only if*

$$(\mu^m_{(b,0)})^{L_t}\perp\mu^m_{(b,0)}\quad or\quad \mu^m_{(b,L_ta)}\perp\mu^m_{(b,a)}. \tag{6.16}$$

7. Conclusion

To prove the irreducibility of the representation $T^{R,\mu,m}$ defined by (2.6) for general $m\in\mathbb{N}$ we need:

1. to know the minimal generating set of conditions for the orthogonality

$$(\mu^m_{(b,a)})^{L_t}\perp\mu^m_{(b,a)}$$

   for all $t\in\mathrm{GL}(m,\mathbb{R})\setminus\{e\}$, see Section 3. In fact it is sufficient to replace the group $\mathrm{GL}(m,\mathbb{R})$ by $\pm\mathrm{SL}(m,\mathbb{R})$, see Remark 3.6. These conditions will be expressed in terms of some divergent series $\big(S_\beta(\mu)\big)_{\beta\in B}$;

2. to find an appropriate combinations of generators A_{kn} of all one-parameter subgroups or an appropriate functions of generators. This combination will be expressed in terms of a divergent series $\big(\Sigma_\alpha(A)\big)_{\alpha\in A}$;

3. to show that 1 implies 2.

What we know now is the following.

1. If we have some continuous finite-dimensional group G acting on an infinite-dimensional space X with a measure μ and we are interested in the "admissible" action, i.e., such that $\mu^{\alpha_t}\sim\mu$ for every $t\in G$, the problem is much easier, here $\alpha: G\to \mathrm{Aut}(X)$. To find a minimal set it is sufficient to verify the equivalence only on the one-parameter subgroups $g_k(t)$ generating the group G. This follows from the *transitivity* of the relation of the equivalence on the sets of a probability measures $\mu\sim\nu$. But the relation of the orthogonality on the sets of measures is not *transitive*. That is why the description of the minimal set is so complicated. When $m=1$ the minimal subset is reduced to $-1\in \mathrm{GL}(1,\mathbb{R})$.

   When $m=2$ the description of the minimal generating set is given in Remark 3.2. The families (3.2) are one-parameter subgroups, the families (3.3) are just reflections of (3.2) and the family (3.4) depends on two parameters. All elements are of order 2 except the elements in subgroups given in (3.2).

   When $m=3$ the description of the minimal generating set is given in Lemma 3.8 and it involves a families (3.19)–(3.31) depending respectively on two, three and five parameters, see remarks after Lemma 3.7.

   When $m=4$ we still do not know the answer.

2. When $m=1$ and $m=2$ it was sufficient for the approximation of x_{kr} or D_{kr} to use the linear combinations of products of two generators $A_{kn}A_{rn}$ for $n\in\mathbb{Z}$. When $m=3$ we were not able to use only quadratic functions in generators. As Lemma 5.4 demonstrates, we were forced to use $\exp\big(is_k(x_{rk}-a_{rk})\big)A_{kn}$ in order to approximate D_{rn}.

3. This relies mainly on the properties of the generalized characteristic polynomial, on the explicit expression for the quadratic form on a hyperplane, and on a theorem regarding the height of an infinite parallelotope. Everything is done for general m.

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