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# Tie knots, random walks and topology

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**Abstract**

Necktie knots are inherently topological structures; what makes them tractable is the particular manner in which they are constructed. This observation motivates a map between tie knots and persistent walks on a triangular lattice. The topological structure embedded in a tie knot may be determined by appropriately manipulating its projection; we derive corresponding rules for tie knot sequences. We classify knots according to their size and shape and quantify the number of knots in a class. Aesthetic knots are characterised by the conditions of symmetry and balance. Of the 85 knots which may be tied with conventional tie, we recover the four traditional knots and introduce nine new aesthetic ones. For large (though impractical) half-winding number, we present some asymptotic results. © 2000 Published by Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The most commonly used tie knot, the four-in-hand, originated in late 19th century England: drivers are thought to have used it to tie their scarves round their necks lest they lose the reigns of their four-in-hand carriages. The Duke of Windsor has been credited with introducing what is now known as the Windsor knot, whence its smaller [1] derivative, the half-Windsor, evolved. More recently, in 1989, the Pratt knot was published in broadsheets across the world, the first new knot to appear in 50 years.

Tie knots, as history suggests, are not often discovered by chance. Rather than wait another half-century for the next knot to appear, we present in this paper a more rigorous approach. Our aim is to predict all aesthetic tie knots, the four knots mentioned above as well as any others.

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[1] Though not, as its name suggests, by half; the half-Windsor is three-fourths the size of the Windsor.


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## 2. Definition of tie knots

A tie knot is a slip knot: the tie is placed around the neck and the wide (active) end is manipulated around the narrow (passive) end such that the latter is free to slip through the resulting knot. Not all slip knots, however, are tie knots. In this section we outline the differences.

A tie knot is initiated by wrapping the active end to the left and either over or under the passive end, forming the triagonal basis Y and dividing the space into right, centre and left $(R, C, L)$ regions (Fig. 1). Knots beginning to the right are identical upon reflection to their left-hand counterparts and are omitted from the discussion.

A knot is continued by wrapping the active end around the triagonal basis; this process may be considered a sequence of half-turns from one region to another. The location and orientation of the active end are represented by one of the six states $R_\odot$, $R_\otimes$, $C_\odot$, $C_\otimes$, $L_\odot$ and $L_\otimes$, where $R$, $C$ and $L$ indicate the regions from which the active end emanates and $\odot$ and $\otimes$ denote the directions of the active end as viewed from in front, viz., out of the page (shirt) and into the page (shirt), respectively.

The notational elements $R_\odot$, $R_\otimes$, $C_\odot$, etc., initially introduced as states, may be considered moves in as much as each represents the half-turn necessary to place the active end into the corresponding state (Fig. 2). This makes the successive moves $R_\odot L_\odot$, for instance, impossible, and implies that $R_\otimes$ is the inverse of $R_\odot$. Accordingly, the move direction must oscillate between $\odot$ and $\otimes$ and no two consecutive move regions may be identical.

To complete a knot, the active end must be wrapped over the front, i.e., either $R_\odot L_\otimes$ or $L_\odot R_\otimes$, then underneath to the centre, $C_\odot$, and finally through (denoted $T$ but not considered a move) the front loop just made (Fig. 3).

We can now formally define a tie knot as a sequence of moves chosen from the move set $\{R_\odot, R_\otimes, C_\odot, C_\otimes, L_\odot, L_\otimes\}$, initiated by $L_\odot$ or $L_\otimes$ and terminating with the subsequence $R_\odot L_\otimes C_\odot T$ or $L_\odot R_\otimes C_\odot T$. The sequence is constrained such that no two consecutive moves indicate the same region or direction. The complete sequence for the four-in-hand, for example, is shown in Fig. 4.

## 3. Tie knots as random walks

Our knot notation introduced above allows us to represent knot sequences by walks on a triangular lattice (Fig. 5). The three axes $r, c$ and $l$ represent the move regions $R$, $C$ and $L$ and the unit vectors $\hat{\mathbf{r}}, \hat{\mathbf{c}}$ and $\hat{\mathbf{l}}$ indicate the corresponding moves. Because the direction of consecutive moves alternates between $\odot$ and $\otimes$ and the final direction must be $\odot$, walks of even length $n$ begin with $\otimes$ and walks of odd length begin with $\odot$. We consequently drop the directional notation $\odot$ and $\otimes$ in the context of walks without introducing ambiguity.

The three-fold symmetry of the move regions implies that only steps along the positive lattice axes are acceptable and, as in the case for moves, no consecutive

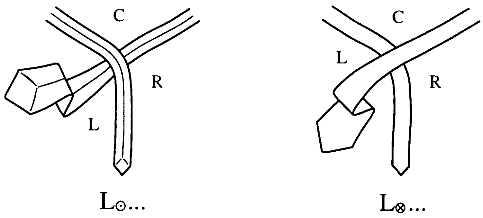


Fig. 1. The two ways of beginning a knot. Both give rise to the triagonal basis Y and divide the space into the three regions through which the active end can subsequently pass. For knots beginning with $L_{\odot}$, the tie must begin inside out. (This and all other figures are drawn in the frame of reference of a mirror image of the actual tie.)

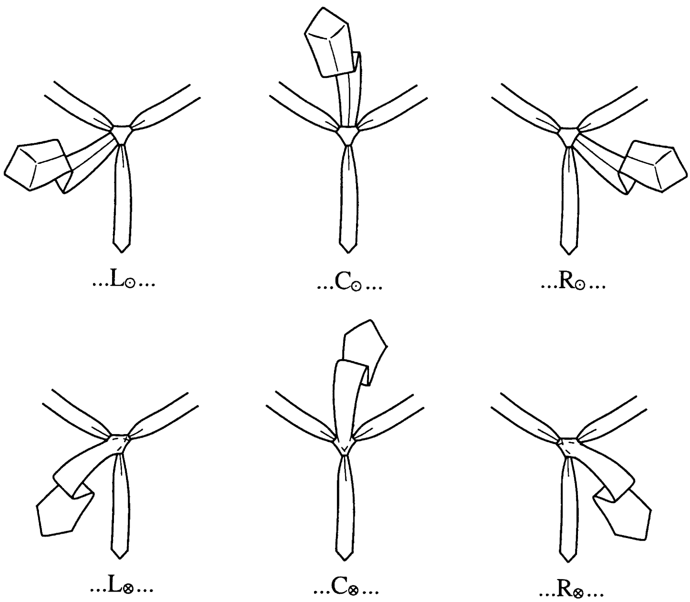


Fig. 2. The six moves with which a tie knot is tied. The move $L_{\odot}$, for instance, indicates the move which places the active end into the left region and directed out of the page.

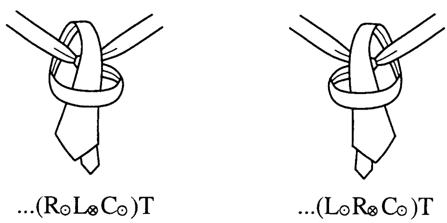


Fig. 3. The two ways of terminating a knot. The active end is finally put through (denoted $T$) the front loop constructed by the last three moves.

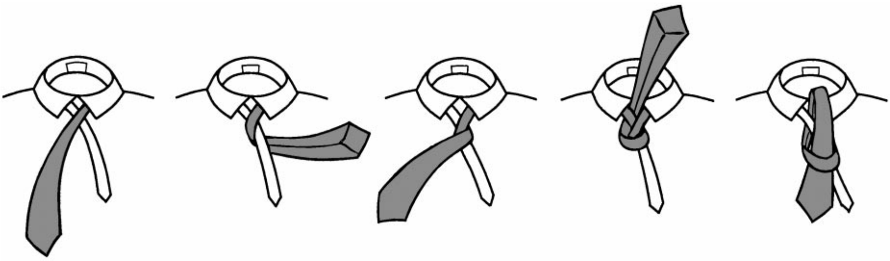

Fig. 4. The four-in-hand, represented by the sequence $L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$.

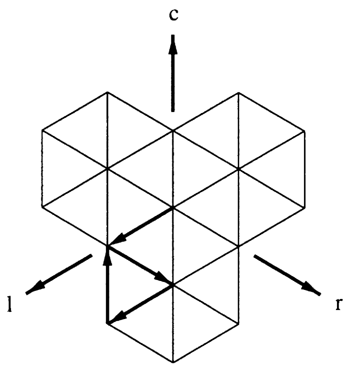


Fig. 5. A tie knot may be represented by a persistent random walk on a triangular lattice, beginning with $\hat{\mathbf{l}}$ and ending with $\hat{\mathbf{l}}\,\hat{\mathbf{r}}\,\hat{\mathbf{c}}$ or $\hat{\mathbf{r}}\,\hat{\mathbf{l}}\,\hat{\mathbf{c}}$. Only steps along the positive $r$-, $c$- and $l$-axes are permitted and no two consecutive steps may be the same. Shown here is the four-in-hand, indicated by the walk $\hat{\mathbf{l}}\,\hat{\mathbf{r}}\,\hat{\mathbf{l}}\,\hat{\mathbf{c}}$.

steps can be identical; the latter condition makes our walk a second-order Markov, or persistent, random walk [3]. Nevertheless, every site on the lattice can be reached since, e.g., $-\hat{\mathbf{c}} = \hat{\mathbf{r}} + \hat{\mathbf{l}}$ and $2\hat{\mathbf{c}} = \hat{\mathbf{c}} + \hat{\mathbf{l}} + \hat{\mathbf{c}} + \hat{\mathbf{r}} + \hat{\mathbf{c}}$.

The evolution equations for the persistent walk are written as

$$\begin{aligned}
k_{n+1}(r,c,l) &= \tfrac{1}{2} p_n(r-1,c,l) + \tfrac{1}{2} q_n(r-1,c,l)\,,\\
p_{n+1}(r,c,l) &= \tfrac{1}{2} k_n(r,c-1,l) + \tfrac{1}{2} q_n(r,c-1,l)\,,\\
q_{n+1}(r,c,l) &= \tfrac{1}{2} k_n(r,c,l-1) + \tfrac{1}{2} p_n(r,c,l-1)\,. \qquad (1)
\end{aligned}$$

where $k_n(r,c,l)$ is the conditional probability that the walker is at point $(r,c,l)$ on the $n$th step, having just taken a step along the positive $r$-axis, $p$ is conditioned on a step along the positive $c$-axis, etc. The unconditional probability of occupation of a site, $U$, may be written as

$$U_n(r,c,l) = k_n(r,c,l) + p_n(r,c,l) + q_n(r,c,l)\,. \qquad (2)$$

## 4. Size of knots

We classify knots according to size, defined as the number of moves in a knot sequence; in the context of walks, size is equal to the number of steps. In both cases, we denote size by the half-winding number $h$. Concatenating the initial and terminal subsequences implies that the smallest knot is given by the sequence $L_{\odot}R_{\otimes}C_{\odot}T$, with $h=3$. The finite length of the tie, as well as aesthetic considerations, suggests an upper bound on knot size; we limit out exact results to $h \leqslant 9$.

The number of knots as a function of size, $K(h)$, is equal to the number of walks of length $h$ satisfying the initial and terminal conditions. We derive $K(h)$ by first considering all walks of length $n$ beginning with $\hat{\mathbf{l}}$, our initial constraint. Let $F_{\hat{\mathbf{r}}}(n)$ be the number of walks beginning with $\hat{\mathbf{l}}$ and ending with $\hat{\mathbf{r}}$, $\hat{F}_{\hat{\mathbf{c}}}(n)$ the number of walks beginning with $\hat{\mathbf{l}}$ and ending with $\hat{\mathbf{c}}$, etc. Accordingly, since at any given site the walker chooses between two steps,

$$F_{\hat{\mathbf{r}}}(n) + F_{\hat{\mathbf{c}}}(n) + F_{\hat{\mathbf{l}}}(n) = 2^{n-1}\,. \tag{3}$$

Because the only permitted terminal sequences are $\hat{\mathbf{r}}\,\hat{\mathbf{l}}\,\hat{\mathbf{c}}$ and $\hat{\mathbf{l}}\,\hat{\mathbf{r}}\,\hat{\mathbf{c}}$, we are interested in the number of walks of length $n = h-2$ ending with $\hat{\mathbf{r}}$ or $\hat{\mathbf{l}}$, after which the respective remaining two terminal steps may be concatenated.

We begin by considering $F_{\hat{\mathbf{l}}}(n)$. Now, $\hat{\mathbf{l}}$ can only follow from $\hat{\mathbf{r}}$ and $\hat{\mathbf{c}}$ upon each additional step, that is,

$$F_{\hat{\mathbf{l}}}(n+2) = F_{\hat{\mathbf{r}}}(n+1) + F_{\hat{\mathbf{c}}}(n+1)\,, \tag{4}$$

from which it follows that

$$F_{\hat{\mathbf{l}}}(n+2) = F_{\hat{\mathbf{r}}}(n) + F_{\hat{\mathbf{c}}}(n) + 2F_{\hat{\mathbf{l}}}(n)\,. \tag{5}$$

Combining (3) and (5) gives rise to the recursion relation

$$F_{\hat{\mathbf{l}}}(n+2) = F_{\hat{\mathbf{l}}}(n) + 2^{n-1}\,. \tag{6}$$

With initial conditions $F_{\hat{\mathbf{l}}}(1) = 1$ and $F_{\hat{\mathbf{l}}}(2) = 0$, (6) is satisfied by

$$F_{\hat{\mathbf{l}}}(n) = \tfrac{2}{3}(2^{n-2} + (-1)^{n-1})\,. \tag{7}$$

The recursion relation for $F_{\hat{\mathbf{r}}}(n)$ is identical to (6), but with initial conditions $F_{\hat{\mathbf{r}}}(1)=0$ and $F_{\hat{\mathbf{r}}}(2) = 1$. Accordingly,

$$F_{\hat{\mathbf{r}}}(n) = \tfrac{1}{3}(2^{n-1} + (-1)^{n-1})\,. \tag{8}$$

The number of knots of size $h$ is equal to the number of walks of length $h-2$ beginning with $\hat{\mathbf{l}}$ and ending with $\hat{\mathbf{r}}$ or $\hat{\mathbf{l}}$, that is,

$$K(h) = F_{\hat{\mathbf{r}}}(h-2) + F_{\hat{\mathbf{l}}}(h-2) = \tfrac{1}{3}(2^{h-2} - (-1)^{h-2})\,, \tag{9}$$

where $K(1) = 0$, and the total number of knots is

$$\sum_{i=1}^{9} K(i) = 85\,. \tag{10}$$

## 5. Shape of knots

While the half-winding number characterises knot size, it says little about the shape of a knot. This depends on the relative number of right, centre and left moves. Since symmetry considerations (see Section 6) suggest an equal number of left and right moves, the shape of a knot is characterised by the number of centre moves, $\gamma$. We use it to classify knots of equal size; knots with identical $h$ and $\gamma$ belong to the same class. A low centre fraction $\gamma/h$ indicates a narrow knot (e.g., the four-in-hand), whereas a high centre fraction suggests a broad knot (e.g., the Windsor).

For a knot of half-winding number $h$, the number of centre moves $\gamma$ is an integer between 1 and $\frac{1}{2}(h-1)$. Accordingly, for large $h$, the range of the centre fraction $\gamma/h$ tends toward $[0, \frac{1}{2}]$. Some centre fractions, however, preclude aesthetic knots; knots with $\gamma/h < \frac{1}{6}$ are too cylindrical and unbalanced (see Section 7). We consequently limit our attention to centre fractions $[\frac{1}{6}, \frac{1}{2}]$, i.e., $\gamma \in [\frac{1}{6}h, \frac{1}{2}(h-1)]$.

Along with our size constraint, this means the knot classes of interest (canonical knot classes) are

$$\begin{aligned}\{\{h,\gamma\}\} = \{&\{3,1\},\{4,1\},\{5,1\},\{5,2\},\{6,1\},\{6,2\},\{7,2\},\{7,3\},\\ &\{8,2\},\{8,3\},\{9,2\},\{9,3\},\{9,4\}\}.\end{aligned} \tag{11}$$

The number of knots in a class, $K(h,\gamma)$, corresponds to the number of walks of length $h$ containing $\gamma$ steps $\hat{\mathbf{c}}$, beginning with $\hat{\mathbf{l}}$ and ending with $\hat{\mathbf{r}}\,\hat{\mathbf{l}}\,\hat{\mathbf{c}}$ or $\hat{\mathbf{l}}\,\hat{\mathbf{r}}\,\hat{\mathbf{c}}$. The sequence of steps may be considered a coarser sequence of $\gamma$ groups, each group composed of $\hat{\mathbf{r}}$s and $\hat{\mathbf{l}}$s and separated from other groups by a $\hat{\mathbf{c}}$ on the right; the Windsor knot, for example, contains three groups, $\hat{\mathbf{l}}\,\hat{\mathbf{c}}\,\hat{\mathbf{r}}\hat{\mathbf{l}}\,\hat{\mathbf{c}}\,\hat{\mathbf{r}}\hat{\mathbf{l}}\,\hat{\mathbf{c}}$, of lengths $1,2,2$ respectively. We refer to a particular assignment of the centre steps as a centre structure.

Let $n_1$ be the number of groups of length 1 in a given sequence, $n_2$ the number of length $2,\ldots,n_{h-2\gamma+1}$ the number of length $h-2\gamma+1$. These group numbers satisfy

$$n_1 + n_2 + \cdots + h_{h-2\gamma+1} = \gamma\,, \tag{12}$$

$$n_1 + 2n_2 + \cdots + (h-2\gamma+1)n_{h-2\gamma+1} = h-\gamma\,. \tag{13}$$

We desire the number of ordered non-negative integer solutions $n_1, n_2, \ldots, n_{h-2\gamma+1}$ to $(12,13)$, that is, the number of ordered ways of partitioning the integer $h-\gamma$ into $\gamma$ positive integers. Call this function $P(h-\gamma,\gamma)$; it is given by

$$P(h-\gamma,\gamma) = \binom{h-\gamma-1}{\gamma-1}\,. \tag{14}$$

The number of centre structures is equivalent to $P(h-\gamma,\gamma)$ subject to the terminal condition, which requires that the final group cannot be of length one. The latter condition reduces the possible centre structures by $\binom{h-\gamma-2}{\gamma-2}$.

Since the steps within each group must alternate between $\hat{\mathbf{r}}$ and $\hat{\mathbf{l}}$, the steps of each group may be ordered in two ways, beginning with $\hat{\mathbf{r}}$ or beginning with $\hat{\mathbf{l}}$, except for the first, which, by assumption, begins with $\hat{\mathbf{l}}$. Accordingly, for a centre structure of $\gamma$ groups, the number of walks is $2^{\gamma-1}$.

It follows that the number of knots in a class is

$$K(h,\gamma) = 2^{\gamma-1}\left(\binom{h-\gamma-1}{\gamma-1} - \binom{h-\gamma-2}{\gamma-2}\right) = 2^{\gamma-1}\binom{h-\gamma-2}{\gamma-1}\ . \tag{15}$$

## 6. Symmetry

The symmetry of a knot, and our first aesthetic constraint, is defined as the difference between the number of moves to the right and the number of moves to the left, i.e.,

$$s = \left|\sum_{i=1}^{h} x_i\right|\ , \tag{16}$$

where $x_i = 1$ if the $i$th step is $\hat{\mathbf{r}}, -1$ if the $i$th step is $\hat{\mathbf{l}}$ and 0 otherwise. We limit our attention to those knots from each class which minimise $s$. For $h-\gamma$ even, the optimal symmetry $s = 0$; otherwise, optimal $s = 1$.

The move composition, and hence the symmetry, of a knot sequence corresponds to the terminal coordinates of the analogous persistent walk. As with other random walks, it is natural to inquire about the distribution of these coordinates. We take particular interest in the end-to-end distance of walks in the class $\{h,\gamma\}$. It turns out that this distance is Gaussianly distributed about a point near the origin. The derivation of this distribution is provided in Appendix A.

## 7. Balance

Whereas the center number $\gamma$ and the symmetry $s$ indicate the move composition of a knot, balance relates to the distribution of these moves. It inversely corresponds to the extent to which the moves are well mixed. A balanced knot is tightly bound and keeps it shape. We use it as our second aesthetic constraint.

Let $\sigma_i$ represent the $i$th step of the walk. The winding direction $\omega_i(\sigma_i,\sigma_{i+1})$ is equal to 1 if the transition from $\sigma_i$ to $\sigma_{i+1}$ is, say, clockwise and $-1$ otherwise. (By clockwise we mean in the frame of reference of the mirror, *viz.*, $\hat{\mathbf{c}}\,\hat{\mathbf{r}}$, $\hat{\mathbf{r}}\,\hat{\mathbf{l}}, \hat{\mathbf{l}}\,\hat{\mathbf{c}}$, which is counter-clockwise in the frame of the shirt. Such distinctions, however, need not concern us.) The balance $b$ may then be expressed

$$b = \frac{1}{2}\sum_{i=2}^{h-1} |\omega_i - \omega_{i-1}|\ . \tag{17}$$

Note that $b$ is simply the number of reversals of the winding direction.

Of those knots which are optimally symmetric, we seek that knot which minimises $b$. Only knots with half-winding number $3i$ and $3i+2$ can have zero balance, where $i$ is a positive integer; half-winding numbers $3i+1$ have optimal balance 1.

## 8. Untying

A tie knot is most easily untied by pulling the passive end out through the knot. It may be readily observed that the resulting conformation, when pulled from both ends, yields either the straightened tie or a subsequent smaller knot. More formally, when the passive end is removed and the two tie ends joined, the tie may either be knotted or unknotted, where any conformation that can be continuously deformed to a standard ring (the canonical unknot) is said to be unknotted.

To determine the topological structure of such configurations, we first note that a knot tied up to but not including the terminal sequence corresponds, upon removing the passive end, to a string wound in a ball with the interior and exterior ends protruding. Since the ball can be undone by pulling the exterior end, all such conformations are unknotted. The terminal sequence, in particular the action $T$, is responsible for any remaining knot.

This can best be observed diagrammatically by projecting the knot onto the plane (Fig. 6). The solid spheres represent the non-terminal sequences (which cannot give rise to a knot), with the terminal sequences drawn explicitly. The dotted lines represent imaginary connections of the tie ends. The left diagram, with the terminal sequence $R_{\odot}L_{\otimes}C_{\odot}T$, can be continuously deformed to a loop and is hence unknotted. No amount of deformation of the right diagram, with terminal sequence $L_{\odot}R_{\otimes}C_{\odot}T$, will reduce the number of intersections below three. It is the simplest knotted diagram, a trefoil knot. The knotted status of all aesthetic tie knots is included in Table 1.

## 9. Topology

We began this paper by considering tie knots as combinatoric constructs in light of the special manner in which they are formed. Here we examine the topological structure of tie knots. As in the previous section, we imagine the tie ends to be connected, this time before removing the passive end.

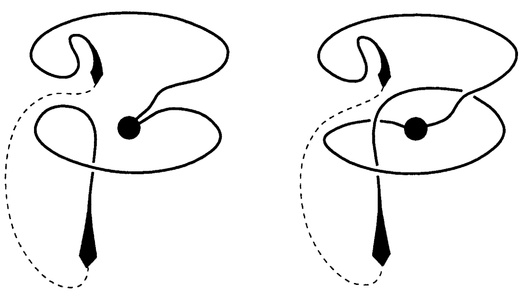

Fig. 6. The left diagram, with terminal sequence $\ldots R_{\odot}L_{\otimes}C_{\odot}T$, is unknotted, while the right, with terminal sequence $\ldots L_{\odot}R_{\otimes}C_{\odot}T$, form a trefoil knot.

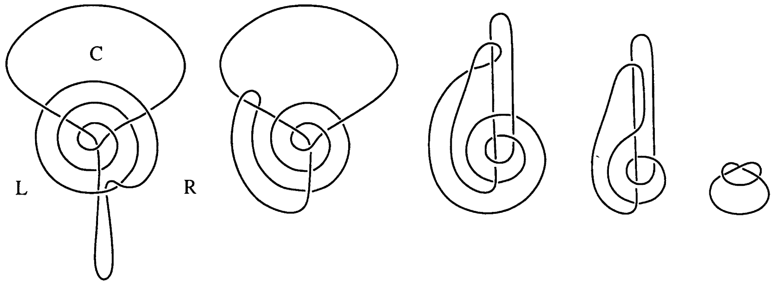


Fig. 7. The Windsor knot (left) has the topological structure of a trefoil (right).

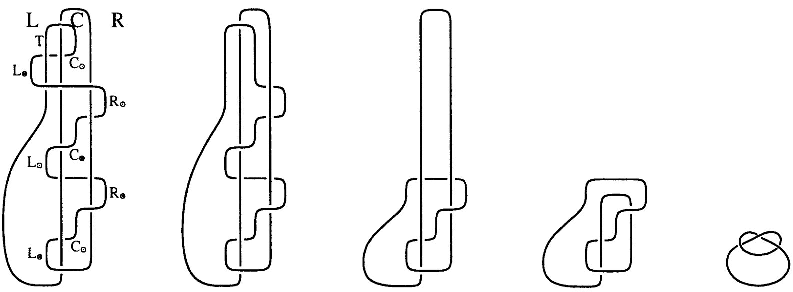


Fig. 8. Alternative projection of the Windsor knot (left). Changes in sequence associated with reduction of intersections are now apparent.

Fig. 7 shows the Windsor knot, for example, projected onto the plane. Let the projected diagram take precedence. By manipulating the diagram such that the corresponding knot is continuously deformed,[2] we see that the Windsor knot is topologically equivalent to a trefoil. Other tie knots give rise to more complicated knots.

The topological complexity of any knot may be characterised by its crossing index [2], the minimum number of intersections allowed by its projection. The standard knot table is arranged by crossing index, for each of which there may be more than one knot. The number of knots per index appears to grow rapidly, but little is known about this number. Demonstrating equivalence between an arbitrary knot projection and its reduced (standard knot table) form by geometric manipulation is a tedious and often nontrivial task. We wish instead to determine the crossing index of tie knots by grammatically manipulating knot sequences.

The grammatical rules associated with diagrammatic reduction become apparent by considering a more tractable diagrammatic representation, applied to the Windsor knot in Fig. 8. The new projection may be derived from the projection used in Fig. 7 by contracting the top two arms of the triagonal basis Y and sliding the windings of the

[2] The diagrammatic manipulations associated with continuous deformation of a knot are called the Reidemeister moves. For an excellent introduction to knot theory, see [2].

active end onto them. Preserving the old regions and directions, the new projection allows immediate recognition of the sequence which generates it.[3]

It may be verified by constructing appropriate projections that the rules of the grammar are

$$\ldots RLCT \text{ and } \ldots LRCT \rightarrow \ldots C\,, \tag{18}$$

$$\ldots Cf(R,L) \rightarrow \ldots C\,, \tag{19}$$

$$\ldots CC \rightarrow \ldots \emptyset\,, \tag{20}$$

$$C \text{ and } f(R,L) \rightarrow \emptyset\,, \tag{21}$$

where $f(R,L)$ is an alternating sequence of $R$s and $L$s (of any length) and ellipsis marks indicate that other moves may precede. Upon applying these rules to exhaustion, the reduced sequence indicates the topological structure of a tie knot. Knots with reduced sequence length of zero may be deformed to the unknot; all other knots have crossing index equal to the minimal sequence length plus one. The standard knot table label of all aesthetic knots is listed in Table 1, where the crossing index is the non-subscripted number.

## 10. Conclusion

The thirteen canonical knot classes and the corresponding most aesthetic knots appear in Table 1. The four named knots are the only ones, to our knowledge, that are widely recognised (although we have learnt that the first entry, $L_{\odot}R_{\otimes}C_{\odot}T$, is extensively used

Table 1
Aesthetic tie knots, characterised, from left, by half-winding number $h$, centre number $\gamma$, centre fraction $\gamma/h$, knots per class $K(h,\gamma)$, symmetry $s$, balance $b$, name, sequence, knotted status and standard knot table label

| $h$ | $\gamma$ | $\gamma/h$ | $K(h,\gamma)$ | $s$ | $b$ | Name | Sequence | KS | SKT |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 0.33 | 1 | 0 | 0 | | $L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $0_1$ |
| 4 | 1 | 0.25 | 1 | 1 | 1 | Four-in-hand | $L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $3_1$ |
| 5 | 1 | 0.20 | 1 | 0 | 2 | | $L_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $4_1$ |
| 5 | 2 | 0.40 | 2 | 1 | 0 | Pratt knot | $L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $0_1$ |
| 6 | 1 | 0.17 | 1 | 1 | 3 | | $L_{\otimes}R_{\odot}L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $5_2$ |
| 6 | 2 | 0.33 | 4 | 0 | 0 | Half-Windsor | $L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $0_1$ |
| 7 | 2 | 0.29 | 6 | 1 | 1 | | $L_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $0_1$ |
| 7 | 3 | 0.43 | 4 | 0 | 1 | | $L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $3_1$ |
| 8 | 2 | 0.25 | 8 | 0 | 2 | | $L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $7_4$ |
| 8 | 3 | 0.38 | 12 | 1 | 0 | Windsor | $L_{\otimes}C_{\odot}R_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $3_1$ |
| 9 | 2 | 0.22 | 10 | 1 | 3 | | $L_{\odot}R_{\otimes}L_{\odot}R_{\otimes}C_{\odot}L_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $8_4$ |
| 9 | 3 | 0.33 | 24 | 0 | 0 | | $L_{\odot}R_{\otimes}C_{\odot}L_{\otimes}R_{\odot}C_{\otimes}L_{\odot}R_{\otimes}C_{\odot}T$ | $y$ | $4_1$ |
| 9 | 4 | 0.44 | 8 | 1 | 2 | | $L_{\odot}C_{\otimes}R_{\odot}C_{\otimes}L_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ | $n$ | $5_2$ |

[3] While all of the diagrams in Fig. 8 are topologically equivalent to the first, only the first diagram, with terminal sequence intact, corresponds to a tie knot sequence.

by the communist youth organisation in China). The remaining nine have recently been introduced by the authors [1].

The first four columns of Table 1 describe the knot classes and the remaining five characterise the corresponding knots. The half-winding number $h$ indicates knot size and the center fraction $\gamma/h$ is a guide to knot shape, the higher fractions corresponding to wider knots. Both should be considered when selecting a knot.

It has come to our attention that certain knot names are used to describe more than one sequence. These sequences differ by the transposition of one or more $RL$ (or $LR$) group; for instance, $L_{\otimes}R_{\odot}C_{\otimes}R_{\odot}L_{\otimes}C_{\odot}T$ is also known as the Windsor. Indeed, it may be argued that the transposition of the last $RL$ group of the sequences for some knots are aesthetically favourable. This ambiguity results from the variable width of conventional ties; a left move, for example, gives greater emphasis to the left than a preceding right move to the right. We do not try to distinguish between these knots and their counterparts this much we leave to the sartorial discretion of the reader.

## Appendix A. Distribution of end-to-end distance of walks in class $\{h, \gamma\}$

Here we derive the probability density distribution of the terminal coordinates of walks in the class $\{h, \gamma\}$. Since, by assumption, $\gamma$ steps are taken along the $c$ axis, our distribution will be a function of one variable only (moves to the left may be considered negative moves to the right).

We begin by rewriting the evolution equations for the persistent walk from (1) *given* that, for large $h$, the fraction of steps along the $c$-axis tends toward $\gamma/h$,

$$k_{n+1}\left(r,c,l\,\middle|\,\frac{\gamma}{h}\right) = \frac{\gamma}{h-\gamma}p_n\left(r-1,c,l\,\middle|\,\frac{\gamma}{h}\right) + \frac{h-2\gamma}{h-\gamma}q_n\left(r-1,c,l\,\middle|\,\frac{\gamma}{h}\right),$$

$$p_{n+1}\left(r,c,l\,\middle|\,\frac{\gamma}{h}\right) = \frac{1}{2}k_n\left(r,c-1,l\,\middle|\,\frac{\gamma}{h}\right) + \frac{1}{2}q_n\left(r,c-1,l\,\middle|\,\frac{\gamma}{h}\right),$$

$$q_{n+1}\left(r,c,l\,\middle|\,\frac{\gamma}{h}\right) = \frac{h-2\gamma}{h-\gamma}k_n\left(r,c,l-1\,\middle|\,\frac{\gamma}{h}\right) + \frac{\gamma}{h-\gamma}p_n\left(r,c,l-1\,\middle|\,\frac{\gamma}{h}\right). \tag{22}$$

Projecting the two-dimensional walk on to the perpendicular to the $c$-axis, say, the $x$-axis, reduces the problem to a symmetric one-dimensional persistent walk of $m=h-\gamma$ steps of $\hat{\mathbf{r}}$ and $\hat{\mathbf{l}}$. In this simplified walk, a step to the left is followed with probability $\gamma/(2h-2\gamma)$ by another step to the left and $(2h-3\gamma)/(2h-2\gamma)$ by a step to the right; a step to the right is similarly biased toward the left.

With $x_i = 1$ if the $i$th step is $\hat{\mathbf{r}}$ and $x_i = -1$ if the $i$th step is $\hat{\mathbf{l}}$, the resulting evolution equations may be written

$$u_{n+1}(x) = \frac{\gamma}{2h-2\gamma}u_n(x-1) + \frac{2h-3\gamma}{2h-2\gamma}v_n(x-1),$$

$$v_{n+1}(x) = \frac{2h-3\gamma}{2h-2\gamma}u_n(x+1) + \frac{\gamma}{2h-2\gamma}v_n(x+1), \tag{23}$$

where $u_n(x)$ is the conditional probability that the walker is at $x$ on the $n$th step, having just taken a step along the positive $x$-axis, and $v$ is conditioned on a step along the negative $x$-axis.

The terminal coordinate of the projected random walk is equivalent to the symmetry $s$, which is now written

$$s = \sum_{i=1}^{m} x_i \ . \tag{24}$$

Since the projected walk is a finite-order Markov chain, the central limit theorem provides that the distribution of $s$ approaches a gaussian for large $m$. Accordingly, we desire the projected walk's mean and variance.

The evolution equations (23) are symmetric about 0 apart from the initial step, which is $\hat{\mathbf{I}}$. Observing the possible paths taken in the first few steps of the unprojected walk, it is evident that the first moment $\langle s \rangle$ satisfies

$$\langle s \rangle = \frac{\gamma}{h-\gamma}(-1) + \frac{h-2\gamma}{h-\gamma}\left(\frac{\gamma}{h-\gamma}(0) + \frac{h-2\gamma}{h-\gamma}\langle s \rangle\right) , \tag{25}$$

and, accordingly, the mean $\mu_s$ is

$$\mu_s = \langle s \rangle = \frac{h-\gamma}{3\gamma-2h} \ . \tag{26}$$

In what follows, we make use of the local correlation function, $\langle x_i x_{i+k} \rangle$. It may be observed that

$$\langle x_i x_{i+1} \rangle \equiv \langle x_i x_{i+1} \rangle_{i+1} = \frac{2\gamma-h}{h-\gamma} x_i x_i = \frac{2\gamma-h}{h-\gamma} , \tag{27}$$

where $\langle \ldots x_{i+1} \rangle_{i+1}$ denotes the average over $x_{i+1}$. By considering the general average $\langle x_i x_{i+k} \rangle$ as successive averages over $x_{i+k}, x_{i+k-1}$, etc., we have

$$\langle x_i x_{i+k} \rangle = \frac{2\gamma-h}{h-\gamma}\langle x_i x_{i+k-1} \rangle = \cdots = \left(\frac{2\gamma-h}{h-\gamma}\right)^k \ . \tag{28}$$

The second moment may be expressed in terms of the local correlation function as

$$\langle s^2 \rangle = \sum_{i,j=1}^{m} \langle x_i x_j \rangle \ . \tag{29}$$

Separating the sum into $i=j$ and $i \neq j$ terms, we have

$$\langle s^2 \rangle = m + 2\sum_{j>i=1}^{m} \langle x_i x_j \rangle = m + 2\sum_{i=1}^{m-1}\sum_{k=1}^{m-i}\langle x_i x_{i+k} \rangle \ . \tag{30}$$

Substituting in (28), it follows that

$$\langle s^2 \rangle \simeq \frac{\gamma}{2h-3\gamma} m = \frac{\gamma(h-\gamma)}{2h-3\gamma} \ . \tag{31}$$

Since the mean $\mu_s$ is always bounded by $[-1, 0]$, we approximate the variance as

$$\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2 \simeq \frac{\gamma(h-\gamma)}{2h-3\gamma} \ . \tag{32}$$

Eqs. (26) and (32) specify the distribution of the terminal coordinate of walks in $\{h,\gamma\}$. For $\gamma = h/3$, the distribution of the terminal coordinates of the two-dimensional persistent random walk (1) readily follows.

## References

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