Exactly solvable model of memristive circuits: Lyapunov functional and mean field theory

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We construct an exactly solvable circuit of interacting memristors and study its dynamics and fixed points. This simple circuit model interpolates between decoupled circuits of isolated memristors, and memristors in series for which exact fixed points can be obtained. We introduce a Lyapunov functional that is found to be minimized along the non-equilibrium dynamics and which resembles a long-range Ising model with non-linear self-interactions. We use the Lyapunov function as an Hamiltonian to calculate, in the mean field theory approximation, the average asymptotic behavior of the circuit given a random initialization, yielding exact predictions for the case of decay to the lower resistance state, and reasonable predictions for the case of a decay to the higher resistance state.

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INTRODUCTION

Neuromorphic circuits are a promising technology to implement at the hardware level the computational power of analog computation inspired by the mammal brain. As for the case of neural networks, the amount and the type of computation performed by neuromorphic circuits requires a still lacking general theoretical framework to allow controllability and interpretability as well as performance. Memristors are becoming the most promising technology for the analogue implementation of artificial intelligence, and their dynamics is known to display memory effects [1–7] being very sensible to initial conditions and, more generally, path-dependent [8–11]. Memristive circuits are also a new direction of study [12, 13] from a statistical physics standpoint, as these show critical behavior [14, 15] and can be connected to the solution of optimization problems [16–18]. A memristor is a 2-port device behaving as a resistance that changes its value as a function of the passing current. In this paper we restrict to ideal memristors with zero-crossing in the Voltage-Current diagram [20–22], though more recently RRAM devices have further generalized this type of behavior [23].

In the paper we study the non-equilibrium properties of a specific memristive circuit, sketching a general methodology for studying its simple phase diagram. This analysis could constitute a baseline for modelling and analysing more complex interacting memristive circuits. We introduce a simple circuit whose asymptotic dynamics we show to be governed by a Lyapunov functional. As for the case of the Hopfield Hamiltonian, such a functional can be casted into a spin-like model with long range interactions but with non-linear self-energy. The model we introduce interpolates in fact between a set of non-interacting memristive circuits and a single mesh memristive models. In a recent paper [24], it has been shown however the interaction strength between memristors is controlled by the Hamming distance on the dual graph of the circuit. In this paper we consider instead the case in which the Hamming distance between the memristors is one, and thus it represents a long-range model.

The present paper is structured as follows: in section two we revise the standard model of a simple circuit with one memristor, in section three we define the model of interacting memristors where many a memristor are coupled with a central loop characterized by a given conductance regulating the coupling between memristors. In section four we analytically and numerically characterize the circuit both in the case of deterministic and random initializations. Finally, in section five we discuss the results and their implications both on real implementations of memristive circuits and on their theoretical modelling.

THE SINGLE MEMRISTOR MODEL

As first observed in [25], physical memristors slowly relax to a limiting resistance even when a voltage is not applied. This observation is key to understand the behavior of the circuit as the physical parameters change.

Let us consider the time evolution of a simple Ag+ memristor (atomic switch)[26]:

$$\partial_t w(t) = \alpha w(t) - \frac{R_{on}}{\gamma} I = \alpha w(t) - \frac{R_{on}}{\gamma} \frac{S}{R(w)} \qquad (1)$$

where $0 \leq w(t) \leq 1$ is the internal memory parameter of the memristor, $R(w) = R_{on}(1-w) + R_{off} w$ is the resistance and I and S are the current and applied voltage respectively. Using this parametrization, R_{on} and R_{off} are the limiting resistances for $w = 0$ and $w = 1$ respectively, and α and γ are constants which set the timescales for the relaxation and excitation of the memristor respec-

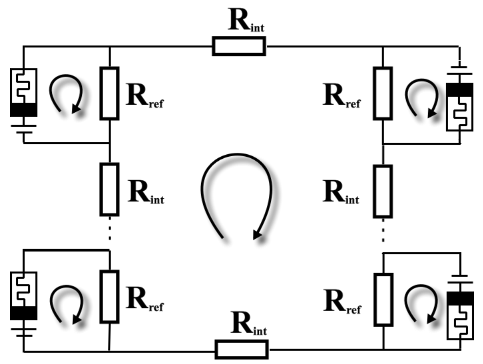

FIG. 1: More general case with n memristors arranged on a loop. Since the circuit is planar, all the loops can be chosen with the same orientation.

tively[1]. The fixed points w^* can be obtained by setting $\partial_t w = 0$, from which we find the equation

$$\frac{R(w^*)}{R_{on}} w^* = \left((1 - w^*) + \frac{R_{off}}{R_{on}} w^* \right) w^* = \frac{S}{\alpha\gamma}. \quad (2)$$

We immediately observe that this equation is quadratic in w^* and that none, one or two solutions can be obtained depending on the values of the parameters. Physical memristors relax to the state of highest resistance R_{off} at zero voltage, i.e. $\alpha > 0$.

THE INTERACTING MODEL

Here we generalize the dynamic model in the case of a circuit composed of N memristors in series to a voltage source, as in Fig. 1: this is a simple modification of the one studied in [27] for machine learning purposes. We introduce control resistances to introduce long range interaction between memristors.

For the derivation of the equations for the circuit of Fig. 1, we first apply the mesh current method, in which we assign to each memristive loop a current i_k, $k = 1, \cdots, n$, and we define the central loop current i_0. By construction, in the central loop there are no memristors but only resistances. Once we have assigned an orientation to the mesh current, the current on each resistance is clear: on the R_{ref} in parallel to the kth memristor, we will have a current $i_0 - i_k$, meanwhile in any R_{int} resistance will flow a current i_0.
On each kth memristor the current is given simply by i_k. The final step is to write the Kirchhoff voltage conservation law for each mesh. Since we have $n+1$ currents and $n+1$ meshes, we have a complete set of equations given by:

Central loop equation:

$$n R_{int} i_0(t) + \sum_{k=1}^{n} R_{ref}(i_0(t) - i_k(t)) = 0, \quad (3)$$

Memristive loop:

$$R(w_k(t)) i_k + R_{ref}(i_k(t) - i_0(t)) = S_k(t), \quad (4)$$

Memory evolution:

$$kT\xi_i(t) + \alpha w_k(t) - \frac{R_{on}}{\gamma} i_k(t) = \frac{d}{dt} w_k. \quad (5)$$

This set of equations completely determines the circuit dynamics and can be used to explicitly derive (see the Appendix) the memristors's dynamics:

$$\frac{dw_k}{dt} = \alpha w_k - \sum_{j=1}^{n} \frac{R_{on}}{\gamma} (\mathcal{I} - \frac{1}{n} M \mathcal{J})^{-1}_{kj} \frac{S_j}{R_{ref} + R(w_j)} \quad (6)$$

Let us note that the dynamics of the circuit is of the form:

$$\partial_t w_i = \alpha w_i - f_i(\vec{w}). \quad (7)$$

and a direct calculation of the partial derivatives, it is easy to see that the dynamics does not derive from a potential, as $\partial_{w_i} f_j(\vec{w}) \neq \partial_{w_j} f_i(\vec{w})$. Both this and the fact that the asymptotic fixed point for $\alpha > 0$ is unstable means that we cannot strictly interpret the behavior of memristive circuits as an Hamiltonian dynamics. The problem can be rather stated in non-equilibrium terms, i.e.:

Given a probability distribution $P(w_i(t=0))$, is it possible to estimate $P(w_i(t=\infty))$?

Mean field theory

The two control resistances are R_{int} and R_{ref}, and intuitively one should expect the ratio R_{ref}/R_{int} to control the interaction: if R_{ref} is zero, the current will prefer to short-circuit and close on its own generator, thus having decoupled memristive circuits. On the other hand, if we consider the case $\frac{R_{ref}}{R_{int}} \to \infty$, the circuit is a unique mesh, with all the memristors in series: the current flowing in *each* memristor is given by $i = \frac{\sum_{i=1}^n S_i}{\sum_{i=1}^n R(w_i)}$ which if $S_i \equiv S$ can be written as $i = \frac{S}{\sum_{i=1}^n R(\langle w \rangle)}$, with $\langle w \rangle = \frac{1}{n}\sum_{i=1}^n w_i$, assuming that the resistance is linear in w. Thus we have an exact equation for the mean memory of the circuit $\langle w \rangle$:

$$\frac{d}{dt}\langle w \rangle = \alpha \langle w \rangle - \frac{R_{on}}{\gamma} \frac{S}{R(\langle w \rangle)} \quad (8)$$

which is the one of a simple memristic circuit with only one internal memory parameter given by the mean field. The circuit interpolates, through the ratio R_{ref}/R_{int}, between a mean field interaction and the case of simple

[1] In particular, meanwhile α has the dimension of an inverse time, γ has the dimension of time and voltage. In previous papers γ was called β, which in the present one could be confused with an inverse temperature.

decoupled memristive circuits. We note that the fixed points for the general case of this circuit can be obtained from eqns. (5) if we set $\frac{d}{dt}w_k = 0$. We derive in fact a direct relationship between equilibrium currents and the internal memory w_k:

$$\begin{aligned} i_k &= \frac{\alpha\gamma}{R_{on}} w_k \\ i_0 &= \frac{\alpha\gamma}{R_{on}} \frac{R_{ref}}{R_{int}+R_{ref}} \frac{1}{n} \sum_{k=1}^{n} w_k \equiv \frac{\alpha\gamma}{R_{on}} \frac{R_{ref}}{R_{int}+R_{ref}} \langle w_k \rangle \end{aligned} \tag{9}$$

and then find a fixed point equation for the internal memory parameters, which are the solution of the following fixed point equations:

$$\frac{S_k}{\alpha\gamma} = \frac{R(w_k)+R_{ref}}{R_{on}} w_k - \frac{R_{ref}^2}{R_{on}(R_{int}+R_{ref})}\langle w_k \rangle.$$

Considering that for an ideal memristor $R(w_k) = R_{off} w_k + (1-w_k)R_{on} = R_{on} + (R_{off}-R_{on})w_k$, we can rewrite this equation in terms of adimensional quantities only:

$$\begin{aligned} \frac{S_k}{\alpha\gamma} &= \frac{R_{off}-R_{on}}{R_{on}} w_k^2 \\ &+ \frac{R_{on}+R_{ref}}{R_{on}} w_k - \frac{R_{ref}^2}{R_{on}(R_{int}+R_{ref})}\langle w_k \rangle \\ &= \xi w_k^2 + \chi w_k - \rho \langle w \rangle, \end{aligned} \tag{10}$$

where we defined $\xi = \frac{R_{off}-R_{on}}{R_{on}}$, $\chi = \frac{R_{on}+R_{ref}}{R_{on}}$ and $\rho = \frac{R_{ref}^2}{R_{on}(R_{int}+R_{ref})}$. We note that the mean internal memory $\langle w_k \rangle$ acts as an effective voltage source for the circuit. We see that already for this rather simple circuit, long range interactions can affect the position of the fixed points. In the mean field case, in which $\vec{w}(t) = w(t)\vec{1}$, a solution of the differential equations of eqn. (5) can be formally found as a function inverse. This allows to calculate exactly formal derivatives of the function $w(t)$ as a function of the external mean field parameter $\langle S(t) \rangle$ if this is piecewise linear. In the present paper we are interested in calculating the equilibrium properties of the circuit, and for case in which the memristances have different initial conditions it is necessary to use a different approach.

First order phase transition

We now can solve the dynamical system for the case $S_k = S$ and initial condition $w_i(0) = 1$, i.e. switched off memristors. The critical value for the voltage S can be worked out and reads:

$$S_c = \alpha\gamma \left(\frac{R_{off}+R_{ref}}{R_{on}} - \frac{R_{ref}^2}{R_{on}(R_{int}+R_{ref})} \right). \tag{11}$$

We can also derive the asymptotic susceptibility from the mean field exact solution in the approximation that $S(t)$ is stepwise constant. The result is given by:

$$\partial_s w(t \gg 1) \approx \frac{R_{on}}{2\gamma\sqrt{\alpha(R_{off}-R_{on})}} e^{-\alpha t} \tag{12}$$

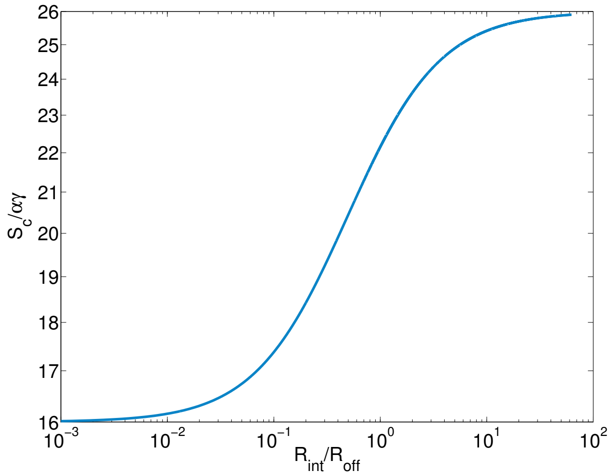

FIG. 2: Critical line of eqn. (11) of the non-equilibrium dynamics starting from switched off memristors, $w_i(0) = 1$. Parameters of the model: $R_{ref} = 1000$, $R_{on} = 100$, $R_{off} = 1600$, and $\alpha = \gamma = 1$.

The case in which the initial state of the system is not homogeneous will be studied in the next section.

Let us consider first with the approximation in which the memristors are non-interacting, i.e. we assume that $\rho = 0$. Using this parametrization, the non-interacting estimate (NIE) becomes:

$$\frac{S_k}{\alpha\gamma} = \xi w_k^2 + \chi w_k \tag{13}$$

where ξ is a adimensional parameter dependent on the physical properties of the memristors, $\frac{S}{\alpha\gamma}$ can be tuned using the external voltage sources, and χ depends on the interaction between the memristive loops. We note that ξ is typically positive as $R_{off} \gg R_{on}$, meanwhile it is important to note that χ cannot be negative for any positive values of the resistances. Let us thus consider for simplicity $\rho = 0$ (in which $\chi = 1$). The solution of this equation is:

$$\begin{aligned} w_{k\pm}^* &= -\frac{\chi}{2\xi} \pm \sqrt{\frac{\chi^2}{4\xi^2} + \frac{S_k}{\xi\alpha\gamma}} \\ &= -\frac{1}{2\xi}\left(1 \pm \sqrt{1 + 4\frac{S_k \xi}{\alpha\gamma}}\right) \end{aligned} \tag{14}$$

We observe that necessarily one root of eqn. (14) falls below zero, and thus only one solution is feasible. In Fig. 3 (top) we plot the numerical solutions obtained for $\alpha < 0$ and $\alpha > 0$. One important fact that we need to stress is that the dynamics of the circuits greatly depends on the signs of S_i and α. The case in which S and α have identical signs is, as the fixed points will not fall in the interval $[0, 1]$, and thus the asymptotic state for memristors is binary, either $\{0, 1\}$. We observe that meanwhile for $\alpha < 0$ the asymptotic fixed point is stable, in the case $\alpha > 0$ (which is the physical case) it is unstable. A simple calculation of the Jacobian confirms this fact. Fig. 3 (bottom) shows the position of the fixed points as a function of α and S for both values of α.

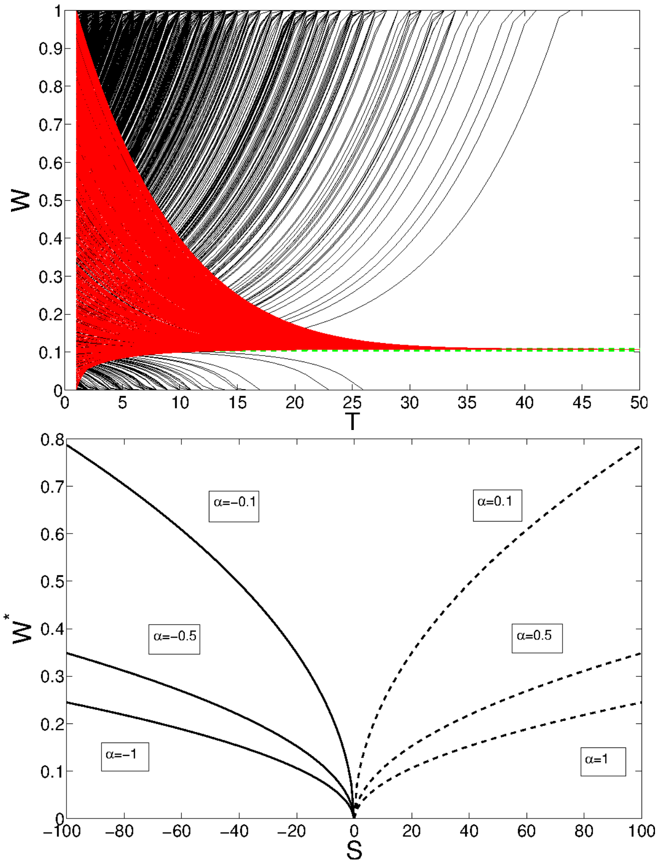

FIG. 3: *Top:* Dynamics for the interacting case in the case $R_{ref} = 100 = R_{int} = 100$, $R_{on} = 100$, $R_{off} = 16000$, $\gamma = 10$. The case of $S = 20$ and $\alpha = 1$ is shown with black curves, meanwhwile $S = -20$, $\alpha = -1$ in red. These equations have been obtained solving numerically eqns. (5) with an integration step $dt = 0.1$ and $n = 1000$ memristors. The green dashed line is the threshold calculated from eqn. (13). We note that the same fixed point can describe an attractive or a repelling fixed point, depending on the signs of S and α. *Bottom:* Fixed points as functions of $\alpha, S > 0$ are dashed curves (unstable) and $\alpha, S < 0$ are continuous curves (stable).

In the case $\alpha < 0, S > 0$, the asymptotic fixed point can be described as the minimum of a functional. In fact, eqn. (10) can be obtained from $\partial_{w_i} H = 0$, where H is given by:

$$H(w_i) = \frac{\rho}{2n}\sum_i w_i^2 - \frac{\rho}{n}\sum_{i,j} w_i w_j - \sum_i \frac{S_i}{\alpha\gamma} w_i + \sum_i E(w_i) \tag{15}$$

and where $E_i(w_i) = \frac{\xi}{3} w_i^3 + \frac{\chi}{2} w_i^2$.

It is important to note that the functional in eqn. (15) is a Lyapunov function, i.e. as we show in the Supplementary Material, $\frac{d}{dt} H(w_i) < 0$ for $\alpha < 0$. Thus, for $\alpha < 0$ the asymptotic states $\langle w(t = \infty)\rangle$ are directly connected to fixed points of a functional which can serve as a Hamiltonian. For $\alpha > 0$ the Lyapunov functional is given by $L = -H$ in eqn. (15); in which the evolution of the Lyapunov functional is shown in Fig. 4. A statistical mechanics interpretation is viable, as for instance in the case of neural networks with temporal delays [28]. Although, the Hamiltonian of eqn. (15) is reminiscent of an Ising model with long-range interactions but with

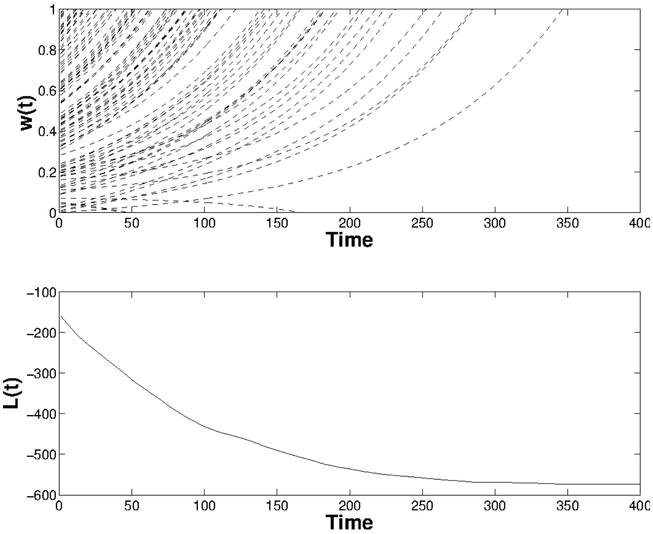

FIG. 4: The evolution over time of the Lyapunov function as the memristors converge to their asymptotic values.

non-linear self-energy [29, 30], it is worth to mention that here the parameters w_i take values in $[0, 1]$. We perform all the calculations for a non-zero temperature, but then take the limit $T \to 0$ at the end.

The first question we aim to answer is whether we can use the calculation for $S > 0, \alpha < 0$ to make any statement regarding the behavior of the system for $S < 0, \alpha > 0$. As mentioned before, $\alpha > 0$ corresponds to a relaxation into an insulating phase, which is the physical case observed in $Ag+$ memristors [25, 26]. We note however that we can use a simple approximation: we can use the position of the unstable fixed point to determine which fixed point each memristor will reach. For instance, in Fig. 3 (top), if $w_{in} > w^*$, the derivative is positive, which implies $w(t = \infty) = 1$; if on the other hand $w_{in} < w^*$, then $w(t = \infty) = 0$, up to a set of measure zero, $w_{in} = w^*$. This also shows that there is a duality between the cases $S < 0, \alpha > 0$, and $S > 0, \alpha < 0$, which is also evident by the fact that since the fixed points depend only on the ratio S/α, the position of the fixed point will be unaffected. We can then use the following rule of thumb which connects the probability on the asymptotic states for $\alpha < 0$ to the ones of the initial states for $\alpha > 0$:

$$\begin{aligned} P(w(t=\infty)=1) &= P(w_{in} > w^*) \\ P(w(t=\infty)=0) &= P(w_{in} < w^*) \\ P(w_{in} < w^*) &= 1 - P(w_{in} > w^*). \end{aligned} \tag{16}$$

The Hamiltonian of eqn. (15) can in fact be used to predict $\psi = \langle w(t = \infty)\rangle$ with the assumption of random initialized memristors, by using a Curie-Weiß approach. Using standard mean field theory techniques and after some straightforward calculations, we find the following mean field theory equation at zero temperature:

$$\psi = \arg\sup_{w\in[0,1]} \left(\left(\rho\psi + \frac{S}{\alpha\gamma} \right) w - E(w) \right). \tag{17}$$

Eqn. (17) can be exactly inverted as a function of $\psi = \langle w \rangle$. This gives the same result as the one we

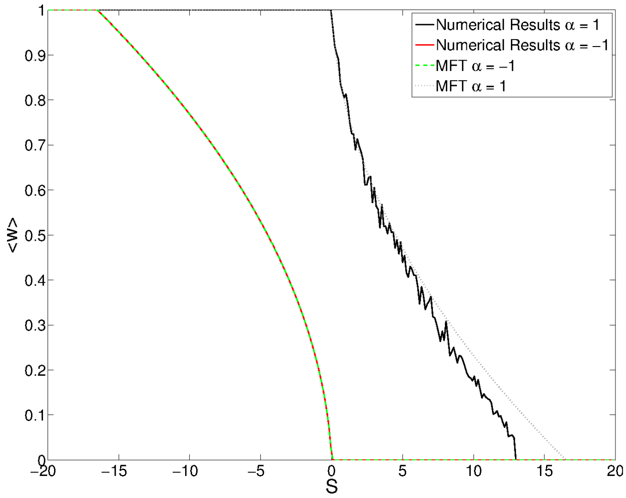

FIG. 5: Asymptotic fixed points as a function of S for $\alpha = \pm 1$, $\gamma = 1$, $R_{ref} = R_{int} = R_{on} = 100$, $R_{off} = 16000$ and $n = 1000$ memristors. We compare the numerical results obtained by simulating the system and the theoretical estimate from the non-interacting assumption and the mean field theory, using the relation of eqn. (16). This figure has been obtained without averaging over the inital condition. For each point, the memristor memories were initiated randomly in $[0, 1]$.

would obtain if we substituted $w_k \to \langle w \rangle$ in eqn. (10). This is simply a correction to the physical parameter χ, as in fact we obtain the same effective equation as the non-interacting approximation with the substitution $\chi \to \chi - \rho$, which is the correction due to the interaction between the memristors. Since the memristor memory is bounded between 0 and 1, we consider the function $\langle w \rangle = \max(0, \max(\psi(S), 1))$.

In Fig. 5 we plot the numerical results on the mean field $\langle w(t = \infty) \rangle$ for $\alpha > 0$ and compare these, as a function of S, to the non-interacting estimate obtained using eqn. (16). We observe that such approximation fails for larger values of S, but yet it provides nonetheless a good estimate for the asymptotic dynamics. Few comments are in order. First we note that for $\alpha = -1$ the mean field theory calculation exactly reproduces the behavior of $\langle w \rangle$. This approximation suggest a second order phase transition at $S = 0$, as in fact one has $\partial_S \langle w \rangle$ given by

$$\partial_S \left(\max\left(0, w^*(S)\right) \right) = \begin{cases} \frac{1}{\xi\alpha\gamma} \frac{1}{2\sqrt{\frac{(\chi-\rho)^2}{4\xi^2} + \frac{S}{\xi\alpha\gamma}}} & S < 0 \\ 0 & S > 0 \end{cases} \tag{18}$$

which is not a differentiable function.

For $\alpha > 0$ we use the approximation of eqns. (16) to calculate the behavior of the system. The validity of this approximation is shown on the right hand side of Fig. 5. We observe that for $S \approx 0$ this approximation is valid. For larger values of S however, such approximation is less valid. We also observe stronger fluctuations around the mean field theory calculation, which we attribute to the effective instability of the fixed point. Nonetheless, such simple approximation provides a good estimate of the mean value of the behavior of the internal memory also for larger values of the external applied voltage. The discrepancy is due to the fact that for larger values of S, because of the instability aforementioned, there can be trajectories which can invert. Since we observe that the real curve lies below the one obtained from the mean field theory, this implies that some memristors whose initial condition lies above the fixed point can invert and reach the asymptotic state $w = 0$, rather than $w = 1$. Since the approximation is valid for $S \approx 0$, we observe that the discontinuity of eqn. (18) applies also for $\alpha > 0$, but is inverted. Also, we note that possibily a divergence would occur if $\chi = \rho$ in eqn.(18), but this does not happen for any positive values of the resistances.

CONCLUSIONS

The results presented in this paper suggests a connection between the Hamiltonian of interacting spin systems with the Lyapunov function of memristive circuits, thus between the equilibrium states of a statistical system of spins and the asymptotic states of memristors in a circuit. It has been insofar hard to obtain an analytical control of the dynamics of memristive circuits. In this paper we made the important step of introducing a Lyapunov functional which can serve as the Hamiltonian of the system. From the point of view of statistical mechanics, the interaction between memristors is long range as every memristor is interacting with other memristors through is passing current. This is simply due to the fact that each memristor is arranged on a large mesh that couples currents in the circuit. Although simple, such model interpolates between the case of memristors in series and the simpler case of non-interacting memristors and a slight modification has been employed for machine learning applications in [27]. We introduced a mean field theory for this non-equilibrium system via a mapping between the equilibrium states and a suitable Lyapunov function. Thus, we have shown, both analytically and numerically, that a first order phase transition occurs for positive α, when memristors are initialized to the high-resistance state, and that a second order phase transition occurs when the initial conditions are chosen uniformly at random between the high and low resistance states. Our results on the mapping between non-equilibrium dynamics and equilibrium states could be extended to the more general case of purely memristive circuits. In future works we will also consider the case of noisy memristive dynamics and random voltages, where complex non-equilibrium spin-glass like behaviour is expected.

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[1] M. Di Ventra, Y. V. Pershin, Nature Phys., 9:200 (2013)
[2] Y. V. Pershin, M. Di Ventra. Solving mazes with memristors: a massively-parallel approach, Phys. Rev. E, 84:046703 (2011)
[3] A. Adamatzky, G. Chen, Chaos, CNN, Memristors and Beyond, World Scientific (2013)
[4] Y. V. Pershin, M. Di Ventra. Self-organization and solution of shortest-path optimization problems with memristive networks, Phys. Rev. E, 88:013305 (2013)
[5] Y. V. Pershin, S. La Fontaine, M. Di Ventra. Memristive model of amoeba learning, Phys. Rev. E, 80:021926 (2009)
[6] D. R. Chialvo, Emergent complex neural dynamics, Nature Physics 6, 744-750 (2010)
[7] Y. V. Pershin, M. Di Ventra. Neural Networks, 23:881 (2010)
[8] G. Indiveri, S.-C. Liu, Proceedings of IEEE, 103:(8) 1379-1397 (2015)
[9] F. L. Traversa, Y. V. Pershin, M. Di Ventra, IEEE Trans. Neural Netw. Learn. Syst., 24:1437 – 1448, 2013.
[10] A.V. Avizienis et al., PLoS ONE 7(8): e42772. (2012)
[11] A. Z. Stieg, A. V. Avizienis et al., Adv. Mater., 24: 286-293 (2012)
[12] F. Caravelli, A. Hamma, M. Di Ventra, Eur. Phys. Lett. 109, 2 (2015)
[13] F. Caravelli, Front. Robot. AI 3, 18 (2016), arXiv:1511.07135
[14] F. Caravelli, F. L. Traversa, M. Di Ventra, arXiv: 1608:08651, Phys. Rev. E 95, 2 (2017)
[15] F. C. Sheldon, M. Di Ventra, Phys. Rev. E 95, 012305 (2017)
[16] F. L. Traversa, M. Di Ventra. IEEE Trans. Neural Netw. Learn. Syst., (DOI: 10.1109/TNNLS.2015.2391182, preprint arXiv:1405.0931) (2015)
[17] F. L. Traversa, C. Ramella, F. Bonani, M. Di Ventra, Science Advances 1, 6 (2015)
[18] F. Caravelli, IJEPD 1-17 (2017), Advances in Memristive Networks, ed. Adamatzky et al.
[19] Y. V. Pershin, M. Di Ventra. Advances in Physics, 60:145–227 (2011)
[20] L. O. Chua, S. M. Kang. Proc. IEEE, 64:209–223, 1976.
[21] D.B. Strukov, G. Snider, D.R. Stewart, and R.S. Williams, Nature 453, pp. 80-83 (2008)
[22] J. J. Yang, D. B. Strukov, D. R. Stewart, Nature Nano. 8 (2013)
[23] I. Valov et al, Nature Comm. 4, 1771 (2013)
[24] F. Caravelli, arXiv:1705.00244 (2017)
[25] Ohno et al., App. Phys. Lett. 99, 203108 (2011);
[26] Wang et al, , Nat. Mat. 16 (2017)
[27] J. P. Carbajal et al., Neur. Comp. 3, vo. 27 (2015)
[28] A. V. M. Herz, Z. Li, J. L. van Hemmen, Phys. Rev. Lett. 66, 10 (1991)
[29] A. Campa, T. Dauxois, S. Ruffo, Physics Reports 480 (2009), pp. 57-159
[30] Campa et al., Phys. Rep. 480 (2009), pp. 57-159
[31] T. Schneider, E. Pytte, Phys. Rev. B 15, 3 (1977)

Supplementary Material

SINGLE MESH TOY MODEL

Simple derivation

From the mesh circuit equations we have:

$$i_k = \frac{S_k}{R_{ref} + R(w_k)} + \frac{R_{ref}}{R_{ref} + R(w_k)} i_0 \tag{19}$$

And,

$$i_0 = \frac{R_{ref}}{R_{ref} + R_{int}} \frac{1}{n} \sum_k^n i_k \tag{20}$$

So that we have:

$$i_k = \frac{S_k}{R_{ref} + R(w_k)} + \frac{R_{ref}}{R_{ref} + R(w_k)} \frac{R_{ref}}{R_{ref} + R_{int}} \frac{1}{n} \sum_j^n i_j \tag{21}$$

That leads to:

$$i_k = \sum_{j=1}^n (\mathcal{I} - \frac{1}{n} M \mathcal{J})^{-1}_{kj} \frac{S_j}{R_{ref} + R(w_j)} \tag{22}$$

where $\mathcal{J}$ is the all-ones matrix and $M_{ij} = \delta_{ij} \frac{R_{ref}}{R_{ref}+R(w_i)} \frac{R_{ref}}{R_{ref}+R_{int}}$. And to the dynamics:

$$\frac{dw_k}{dt} = \alpha w_k - \sum_{j=1}^n \frac{R_{on}}{\gamma} (\mathcal{I} - \frac{1}{n} M \mathcal{J})^{-1}_{kj} \frac{S_j}{R_{ref} + R(w_j)} \tag{23}$$

Alternative derivation and formula

An alternative derivation of the equation above can be obtained via a direct calculation of the inverse. The relation between currents and voltages can be can be written as:

$$\begin{pmatrix} n(R_{int}+R_{ref}) & -R_{ref} & -R_{ref} & \cdots & -R_{ref} \\ -R_{ref} & R(w_1)+R_{ref} & 0 & \cdots & 0 \\ -R_{ref} & 0 & R(w_2)+R_{ref} & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -R_{ref} & 0 & \cdots & 0 & R(w_n)+R_{ref} \end{pmatrix} \begin{pmatrix} i_0 \\ i_1 \\ \vdots \\ \vdots \\ i_n \end{pmatrix} = \begin{pmatrix} 0 \\ S_1 \\ \vdots \\ \vdots \\ S_n \end{pmatrix} \tag{24}$$

which means we need to invert a matrix of the form:

$$M = \begin{pmatrix} a_0 & -b & -b & \cdots & -b \\ -b & a_1 & 0 & \cdots & 0 \\ -b & 0 & a_2 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b & 0 & \cdots & 0 & a_n \end{pmatrix} \tag{25}$$

which a special case of an arrowhead matrix. The inverse of this matrix is rather complicated, but can be easily obtained by means of a cofactor formula: $(A^{-1})_{ij} = \frac{1}{\det(A)} C_{ji}$ where $C_{ij} = (-1)^{i+j}\det(A^{\tilde{i}\tilde{j}})$ is the determinant of the matrix A where the row i and the column j has been removed. We note that because of the properties of A, C is a symmetric matrix. Let us also note that the determinant of matrices of the form as in eqn. (25), $D = \det(M) = \prod_{k=0}^{n} a_k - b^2 \sum_{k=1}^{n} \prod_{j\neq k, j>0} a_j$.

Thus:

$$(M^{-1})_{ij} = \frac{C_{ij}}{D} \tag{26}$$

We that the cofactor of the matrix in the case in which $i = j > 1$ has the same form. We can thus already say that C_{ij} for $i = j > 1$ is of the form: $C_{ii} = \det(M^{\tilde{i}\tilde{i}}) = \prod_{k=0,k\neq i}^{n} a_k - b^2 \sum_{k=1,k\neq i}^{n} \prod_{j\neq k,j>0} a_j$. The special cases $i = 1, j > 1$ and $j = 1, i > 1$, take the form $C_{1j} = C_{j1} = -b^2 \prod_{k\neq 1,k\neq j} a_k$. The case C_{ij} with $i \neq j, i, j > 1$ has to be calculated on its own. We note that for instance the matrix C^{12} is of the form:

$$C_{12} = C_{21} = -det \begin{pmatrix} a_0 & -b & -b & -b & -b & -b \\ -b & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & a_3 & 0 & 0 & 0 \\ -b & 0 & 0 & a_4 & 0 & 0 \\ -b & 0 & 0 & 0 & a_5 & 0 \\ -b & 0 & 0 & 0 & 0 & a_6 \end{pmatrix} \tag{27}$$

and thus introduces a zero on the diagonal. For this reason, the determinant of these matrices are of the form $C_{ij} = -b^2 \prod_{k>0,k\neq i,k\neq j} a_k$. We thus have:

$$\begin{pmatrix} i_0 \\ i_1 \\ \vdots \\ \vdots \\ i_n \end{pmatrix} = \frac{1}{D} C \begin{pmatrix} 0 \\ S_1 \\ \vdots \\ \vdots \\ S_n \end{pmatrix} \tag{28}$$

and thus, since $S_0 = 0$ and we are also interested in $i_{k\geq 1}$, we can write the equation directly for the side loop

$$\begin{pmatrix} i_1 \\ \vdots \\ \vdots \\ i_n \end{pmatrix} = \frac{1}{D} \begin{pmatrix} q_1 & b^2\ c_{12} & b^2\ c_{13} & \cdots & b^2\ c_{1n} \\ b^2\ c_{12} & q_2 & b^2\ c_{23} & & \vdots \\ b^2\ c_{13} & b^2\ c_{23} & \ddots & \ddots & \\ \vdots & & \ddots & q_{n-1} & b^2\ c_{1n-1} \\ b^2\ c_{1n} & \cdots & & b^2\ c_{1n-1} & q_n \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ \vdots \\ S_n \end{pmatrix} \tag{29}$$

where

$$D = n(R_{int}+R_{ref})\prod_{k=1}^{n}(R(w_k)+R_{ref}) - b^2\sum_{k=1}^{n}\prod_{j\neq k,j>0}(R(w_k)+R_{ref}) \tag{30}$$

$$c_{ij} = \prod_{k=1,k\neq i,j}^{n}(R(w_j)+R_{ref}) \tag{31}$$

$$q_i = n(R_{int}+R_{ref})\prod_{k=1,k\neq i}^{n}(R(w_k)+R_{ref}) - b^2\sum_{k=1,k\neq i}^{n}\prod_{j\neq k,i,j>0}(R(w_j)+R_{ref}) \tag{32}$$

$$b = R_{ref} \tag{33}$$

We are in particular interested in the inverse of the submatrix which acts only on the memristor currents. This can be easily obtained, and is given by $-\frac{R_{ref}^2}{n(R_{int}+R_{ref})}J + (R(\vec{w})+R_{ref})I$, where J is the matrix made of ones and $R(\vec{w})$ is the diagonal matrix with the resistances of each memristor. This implies the following dynamics:

$$\frac{d}{dt}\vec{w} = \alpha\vec{w} - \frac{R_{on}}{\gamma}\left(\text{diag}\left(R_{ref}+R(w)\right) - \frac{R_{ref}^2}{n(R_{int}+R_{ref})}J\right)^{-1}\vec{S}. \tag{34}$$

here $\text{diag}(\vec{x}) = \delta_{ij}x_j$, and which is the equation we find in the paper.

Formal mean field dynamics solution and perturbative expansion

In the mean field approximation, eqn. (34) can be solved. If we use the Sherman-Morrison formula assuming that all memristors are equal, in such a case the equation becomes:

$$\frac{as}{bw(t)+c} + w'(t) - \alpha w(t) = 0 \tag{35}$$

with $a = \frac{R_{on}}{\gamma}$, $s = \langle\vec{S}\rangle = \frac{1}{n}\sum_{i=1}^{n} S_i$, $c = R_{ref} + R_{on} - \frac{R_{ref}^2}{R_{int}+R_{ref}} = R_{on} + \frac{R_{ref}R_{int}}{R_{ref}+R_{int}}$ and $b = R_{off} - R_{on}$, and where we are assuming that $S_i = \langle S_i\rangle$.

An analytical solution for such an equation can be found in terms of an inverse. Let us define:

$$Q(t) = \frac{c\ \text{ArcTan}\left(\frac{\sqrt{\alpha}(2(c_1+t)b+c)}{\sqrt{-4abs-\alpha c^2}}\right)}{\sqrt{\alpha}\sqrt{-4abs-\alpha c^2}} + \frac{\log(as-(c_1+t)\alpha\left((c_1+t)b+c\right))}{2\alpha} \tag{36}$$

for an arbitrary integration constant c_1 due to time invariance symmetry. Then, the solution of eqn. (35) is given by the inverse function of $Q(t)$:

$$w(t) = Q^{-1}(t). \tag{37}$$

which is not analytical. In order to solve this equation, we use the a perturbative method in $\epsilon = c/b$, assuming that $R_{int} \gg R_{on} \gg 1$. In fact, $\frac{1}{2} < \frac{R_{ref}R_{int}}{R_{ref}+R_{int}} < 1$ for positive resistances. In this case, the differential equation becomes:

$$\frac{as}{bw(t)} + w'(t) - \alpha w(t) = \epsilon\frac{as}{bw(t)^2} + O(\epsilon^2) \tag{38}$$

We thus search for perturbative solutions $w(t) = w_0(t) + \frac{c}{b}\,w_1(t) + \cdots$ up to the first order. We have the two differential equations up to the first perturbative order in ϵ, which are:

$$\begin{aligned} O(\epsilon^0)\ &:\ \frac{as}{bw_0(t)} + w_0'(t) - \alpha w_0(t) = 0 \\ O(\epsilon^1)\ &:\ \frac{as}{bw_0^2(t)} + w_1'(t) - \left(\alpha + \frac{as}{bw_0^2(t)}\right)w_1(t) = 0 \end{aligned} \tag{39}$$

The zeroth perturbative order equation has solutions:

$$w_0(t) = \pm\frac{\sqrt{as + e^{2\alpha(bz_0+t)}}}{\sqrt{\alpha}\sqrt{b}} \tag{40}$$

with z_0 associated with the initial condition, and of which we take the positive sign only. For $t > \frac{1}{2\alpha}$ we have an exponential function of the form:

$$w_0(t \gg 2\alpha) = \frac{e^{\alpha t}}{\sqrt{\alpha b}} + \frac{ase^{-2\alpha t}}{2\sqrt{\alpha b}} + O((as)^2) \tag{41}$$

The first perturbative order is of the form:

$$x'(t) = h(t)x(t) + g(t) \tag{42}$$

and has a general solution of the form:

$$x(t) = \left(e^{-\int^t h(t')dt'}\right)\left(z_1 + \int^t g(t')e^{\int^{t'} h(t'')dt''}dt'\right) \tag{43}$$

where we identify $h(t) = \alpha + \frac{as}{bw_0^2(t)}$ and $g(t) = \frac{a}{bw_0^2(t)}$. For large times $h(t \gg 2\alpha) \approx \alpha$ and $g(t \gg 2\alpha) \approx \frac{a}{b}\sqrt{\alpha b}^2 e^{-2\alpha t} = a\alpha e^{-2\alpha t}$. Thus we have:

$$w_1(t \gg 2\alpha) \approx e^{-\alpha t}(z_1 + \frac{a}{b}e^{-\alpha t}) \tag{44}$$

We thus obtain the dynamic susceptibility for long times, which is given by:

$$\partial_s w(t \gg 2\alpha) \approx \frac{a}{2\sqrt{\alpha}\sqrt{b}\sqrt{as + e^{2\alpha(bc_1+t)}}} \to \frac{a}{2\sqrt{\alpha b}}e^{-\alpha t} = \frac{R_{on}}{2\gamma\sqrt{\alpha(R_{off} - R_{on})}}e^{-\alpha t} \tag{45}$$

which falls off exponentially in time.

Fixed point structure

Since the interaction matrix is always invertible, we can study the equivalent equation:

$$\left(\text{diag}\left(R_{ref} + R(w)\right) - \frac{R_{ref}^2}{n(R_{int} + R_{ref})}J\right)\frac{d}{dt}\vec{w} = \alpha\left(\text{diag}\left(R_{ref} + R(w)\right) - \frac{R_{ref}^2}{n(R_{int} + R_{ref})}J\right)\vec{w} - \frac{R_{on}}{\gamma}\vec{S} \tag{46}$$

which we note we can rewrite the fixed point equation as:

$$0 = \frac{1}{\alpha}\frac{d}{dt}\left(\chi w_i + \frac{\xi}{2}w_i^2 - \rho\langle w\rangle\right) = \chi w_i + \xi w_i^2 - \rho\langle w\rangle - \frac{S_i}{\alpha\gamma} \tag{47}$$

Fixed points then require that the following two equations are satisfied at the same time:

$$\chi w_i + \frac{\xi}{2}w_i^2 - \rho\langle w\rangle = c \tag{48}$$

$$\chi w_i + \xi w_i^2 - \rho\langle w\rangle - \frac{S_i}{\alpha\gamma} = 0 \tag{49}$$

for an arbitrary constant c and with $w_i \in [0, 1]$.

Lyapunov function

Let us now consider the function

$$H(w_i) = \frac{\rho}{2n}\sum_i w_i^2 - \frac{\rho}{n}\sum_{i,j} w_i w_j - \sum_i \frac{S_i}{\alpha\gamma}w_i + \sum_i E(w_i) \tag{50}$$

we have

$$\frac{d}{dt}H = \sum_i(\partial_{w_i}H)\frac{dw_i}{dt} = \sum_{i,j}\left(-\rho\langle w\rangle - \frac{S_i}{\alpha\gamma} + \frac{\partial E}{\partial w_i}\right)\delta_{ij}\frac{dw_j}{dt} \tag{51}$$

and using the equations of motion we obtain

$$\frac{d}{dt}H = \frac{1}{\alpha}\sum_{i,j}\frac{dw_i}{dt}\left((1 + \frac{R_{ref}}{R_{on}} + \frac{R_{off} - R_{on}}{R_{on}}w_i)\delta_{ij} - \frac{R_{ref}^2}{R_{on}(R_{ref} + R_{int})}\frac{1}{n}J_{ij}\right)\frac{dw_j}{dt} \tag{52}$$

Let us define $Q_{ij} = \Big((1+\frac{R_{ref}}{R_{on}}+\frac{R_{off}-R_{on}}{R_{on}}w_i)\delta_{ij} - \frac{R_{ref}^2}{R_{on}(R_{ref}+R_{int})}\frac{1}{n}J_{ij}\Big)$. We have:

$$\frac{d}{dt}H = \frac{1}{\alpha}\langle Q\frac{d}{dt}\vec{w}, \frac{d}{dt}\vec{w}\rangle = \frac{1}{\alpha}\langle \frac{d}{dt}\vec{w}, \frac{d}{dt}\vec{w}\rangle_Q = \frac{1}{\alpha}||\frac{d}{dt}\vec{w}||_Q^2 \tag{53}$$

This quantity is positive or negative depending on the sign of α and the eigenvalues of Q. If Q is positive definite, then the sign of $\frac{d}{dt}H$ depends only on α. We note that Q is the sum of two Hermitean matrices. Thus the minimum eigenvalue of Q satisfies the bound $\lambda_{min}(A+B) \geq \lambda_{min}(A)+\lambda_{min}(B)$ for A and B Hermitean. Thus:

$$\lambda_{min}(Q) \geq \lambda_{min}\left((1+\frac{R_{ref}}{R_{on}}+\frac{R_{off}-R_{on}}{R_{on}}w_i)\delta_{ij}\right) + \lambda_{min}\left(-\frac{R_{ref}^2}{R_{on}(R_{ref}+R_{int})}\frac{1}{n}J_{ij}\right) \tag{54}$$

Since $\frac{1}{n}J_{ij}$ has maximum eigenvalue 1, we immediately observe that, since R_{on} is positive by construction:

$$\lambda_{min}(Q) \geq 1+\frac{R_{ref}}{R_{on}} - \frac{R_{ref}^2}{R_{on}(R_{ref}+R_{int})} = 1+\frac{R_{ref}}{R_{on}}\left(1-\frac{R_{ref}}{R_{ref}+R_{int}}\right) \geq 1 \ \ \forall R_{int}, R_{ref} \geq 0 \tag{55}$$

This implies that Q is positive definite. The function H is thus a decreasing function of the dynamics when $\alpha < 0$. Since the function is a weighted negative norm of the derivative of the memristor memories, then it is also zero at the fixed point. For $\alpha > 0$, it is sufficient to define $-H$ as a Lyapunov function. In this case, the fixed points become $w = 1$ and $w = 0$.

MEAN FIELD THEORY FOR THE FULL MODEL

We are interested in the low temperature regime of this model. Its partition function can be written as:

$$Z(\beta, n, S) = \mathrm{Tr}_w e^{-\beta H(w)} \tag{56}$$

with $H(w)$ from eqn. (15), $\beta = 1/T$ and we have implicitly defined the trace:

$$\mathrm{Tr}_w\,(\cdot) \equiv \prod_{i=1}^n \int_0^1 dw_i\,(\cdot)\,.$$

We now use the Hubbard-Stratonovich identity, with $m = \frac{\sum_i \sigma_i}{n}$,

$$e^{bm^2} = \sqrt{\frac{b}{\pi}}\int_{-\infty}^{\infty} dx\ e^{-bx^2+2mbx} \tag{57}$$

with $b = \frac{n\rho\beta}{2}$. Let us define $\tilde{E}(w_i) = E(w_i) + \frac{\rho}{2n}\sum_i w_i^2$. We write:

$$\begin{aligned} Z(\beta,n,S) &= \mathrm{Tr}_w e^{-\beta\sum_{i=1}^n(\tilde{E}(w_i)-w_i\frac{S_i}{\alpha\gamma})}\sqrt{\frac{n\beta\rho}{2\pi}}\,. \\ &\quad\cdot \int_{-\infty}^{\infty} d\psi\ e^{-\frac{n\beta\rho}{2}\psi^2+mn\beta\rho\psi} \\ &= \sqrt{\frac{n\beta\rho}{2\pi}}\int_{-\infty}^{\infty} d\psi\ e^{-\frac{n\beta\rho}{2}\psi^2}Q(\beta,S,\psi)^n \\ &= \sqrt{\frac{n\beta\rho}{2\pi}}\int_{-\infty}^{\infty} d\psi\ e^{-\frac{n\beta\rho}{2}\psi^2+n\log(Q(\beta,S,\psi))} \end{aligned} \tag{58}$$

where $Q(\beta,S,\psi) = \mathrm{Tr}_w e^{\beta\left((\rho\psi+\frac{S}{\alpha\gamma})w-\tilde{E}(w)\right)}$.If we take the limits $n\to\infty$ first, for which $\lim_{n\to\infty}\tilde{E}(w_i) = E(w_i)$, which gives

$$Z \approx e^{n\beta\tilde{f}(\beta)} \tag{59}$$

with $f(\beta) = \arg\min_\psi\left(\frac{1}{2}\rho\psi^2 - \frac{1}{\beta}\log\left(Q(\psi)\right)\right)$.

In turn, $f(\beta)$ is given by ψ solution of

$$\rho\psi = \partial_\psi\left[\frac{1}{\beta}\log Q\left(\beta, w(\psi,S)\right)\right] = \frac{1}{\beta}\frac{\partial_\psi Q\left(\beta, w(\psi,S)\right)}{Q\left(\beta, w(\psi,S)\right)} \tag{60}$$

Now we have

$$\frac{1}{\beta}\frac{\partial_\psi Q\left(\beta, w(\psi,S)\right)}{Q\left(\beta, w(\psi,S)\right)} = \frac{\rho\beta}{\beta}\frac{\mathrm{Tr}_w w e^{\beta\left((\rho\psi+\frac{S}{\alpha\gamma})w-E(w)\right)}}{\mathrm{Tr}_w e^{\beta\left((\rho\psi+\frac{S}{\alpha\gamma})w-E(w)\right)}} = \rho\frac{\frac{1}{\beta}\mathrm{Tr}_w w e^{\beta\left((\rho\psi+\frac{S}{\alpha\gamma})w-E(w)\right)}}{\frac{1}{\beta}\mathrm{Tr}_w e^{\beta\left((\rho x+\frac{S}{\alpha\gamma})w-E(w)\right)}} \tag{61}$$

which, in the limit $\beta\to\infty$ is given by the following mean field equation:

$$\psi = \arg\,\sup_{w\in[0,1]}\left(\left(\rho\psi+\frac{S}{\alpha\gamma}\right)w - E(w)\right) = \sqrt{\frac{\chi^2}{4\xi^2}+\frac{\frac{S}{\alpha\gamma}+\rho\psi}{\xi}} - \frac{\chi}{2\xi} \tag{62}$$

which is the result presented in the paper.