A Letters Journal Exploring the Frontiers of Physics

September 2008

EPL, **83** (2008) 60010 www.epljournal.org
doi: 10.1209/0295-5075/83/60010

Single elimination competition

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received 12 January 2008; accepted in final form 1 August 2008
published online 15 September 2008

PACS 02.50.-r – Probability theory, stochastic processes, and statistics
PACS 01.50.Rt – Physics tournaments and contests

Abstract – We study a simple model of competition in which each player has a fixed strength: randomly selected pairs of players compete, the stronger one wins and the loser is eliminated. We show that the best indicator of future success is not the number of wins but a player's *wealth*: the accumulated wealth of all defeated players. We calculate the distributions of strength and wealth for two versions of the problem: in the first, the loser is replaced; in the second, the loser is not. The probability of attaining a given wealth is shown to be path-independent. We illustrate our model with the popular game of conkers and discuss an extension to round-robin sports competition.

Pairwise competition within a population of agents is found in nearly all branches of science: in biology, between males for the same female and between species for fixed resources; in physics, in phase ordering kinetics [1] and galaxy formation [2]; in economics, between individuals or companies in a given industry [3,4]; in sociology, in social stratification [5,6], the minority game [7] and gambling tournaments [8]; and in all kinds of organised sport [9].

Many of these systems have been modelled using techniques from statistical mechanics, in which simple pairwise interactions between agents give rise to complex global behaviour within the population.

In a model of social stratification studied by Redner and co-workers [5,6], each agent has a positive integer strength. When two agents interact, the strength of the stronger increases by one. At the same time, the fitness of all agents decreases at a fixed rate. The authors find a phase transition from a homogeneous, single-class society to a heterogeneous, multi-class society.

In a model of asset exchange [3], when two agents interact, the wealth of the richer agent increases by one and the wealth of the poorer decreases by one. In the long-time limit, the distribution of wealth approaches a Fermi-like distribution.

The scaling behaviour of tournament competition in organized sport was recently studied in [9]. When two players interact, the winner stays on and the loser is eliminated, where the stronger player wins with probability p and the weaker with probability $1-p$.

Here we exactly solve deterministic single elimination competition. Deterministic means the stronger player always wins; single elimination competition means that the loser is eliminated. Pairs of players compete one at a time, as opposed to multiple players competing simultaneously.

Summary of the paper. – In our model, competition occurs within a population of M players, each of which is assigned a fixed, unique strength. Randomly chosen pairs of players compete sequentially and the stronger player wins. At any given time, we do not know the players' strengths, only their history of wins and losses up to that point. We show that the optimal indicator of strength is not the number of wins but the *wealth*: each new player starts with unit wealth, and when two players compete the winner inherits the wealth of the loser. A player with a wealth of n is called an n-er. For example, if a 2-er and a 3-er compete, the winner becomes a 5-er.

We study two versions of the problem. In the first version, each time two players compete, the loser is

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replaced with a new player with random strength and unit wealth. Thus the number of players is always fixed at M and play continues indefinitely. The system approaches steady-state behaviour and we calculate the limiting distribution of strength, $p(s)$, and wealth, $q(n)$.

In the second version, each time two players compete the loser is not replaced, and after $t = M - 1$ competitions only the strongest player remains. Because there is no limiting behaviour, we calculate the distribution of strength $\tilde{p}(s, t, M)$ and wealth $\tilde{q}(n, t, M)$ as a function of time and system size. (The tilde (˜) symbol designates the diminishing, as opposed to fixed, version of the problem.)

For both versions of the problem, we show that the wealth of a player—which is equivalent to the number of players he is demonstrably stronger than—is the most accurate indicator of the likelihood of future success. We calculate the probability that a player with wealth i would beat one with wealth j and show that all strategies for obtaining a high wealth are equivalent—the probability of achieving a given wealth is path independent.

We illustrate our model with the popular game of conkers, in which horse-chestnuts (conkers) are swiped one at another until the weaker one breaks. A conker's score increases by the score of the defeated with each win. We calculate statistics of a conker with a given score and offer advice on strategy. We also describe an extension to round-robin sports competition, in which a group of teams all play each other once.

Fixed population. – We first consider the version of the problem in which each defeated player is replaced with a new player with random strength and unit wealth. Since the number of players is fixed and play continues indefinitely, the system approaches steady-state behaviour and we can calculate its long-term properties.

We first calculate the limiting distribution of strength, $p(s)$. Each player starts with a strength s drawn from a uniform distribution over the unit interval. (Note that it is only the rank of the strengths that matters, and the uniform distribution could be replaced by any other continuous distribution.) Then $p(s)$ can be determined as follows.

At steady state, we know that the distribution of strength of the lesser of two samples of $p(s)$ must be uniform because the player that we remove must be drawn from the same distribution as the player which we add. Our condition on $p(s)$ is

$$1 = 2p(s) \int_s^\infty p(s)\,\mathrm{d}s, \tag{1}$$

which gives rise to the differential equation

$$\frac{\mathrm{d}p(s)}{\mathrm{d}s} = 2p^3(s), \tag{2}$$

which has solution

$$p(s) = \frac{1}{2}(1-s)^{-\frac{1}{2}}. \tag{3}$$

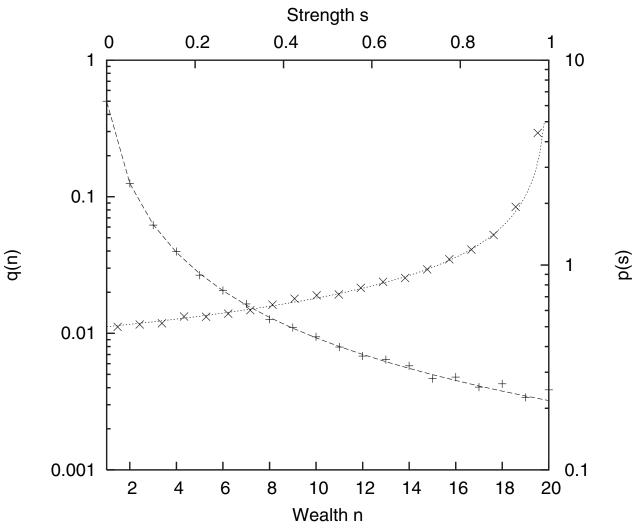

Fig. 1: The steady-state distribution of strength $p(s)$ (dotted line) and wealth $q(n)$ (dashed line) for a fixed population, on a logarithmic scale. (In a fixed population, every time a player is defeated, a new one is added with random strength and unit wealth.) STRENGTH (top and right axes): our theoretical prediction (3) closely matches the result of simulation ($\times$) for $M = 10^5$ players after 10^6 competitions. The distribution is independent of system size, and unbounded as strength approaches 1. WEALTH (bottom and left axes): our prediction (5) again matches the result of simulation (+). The distribution is also independent of system size; the fraction of players with wealth 1, 2, 3, . . . , is 1/2, 1/8, 5/128, ...

This is plotted in fig. 1 and matches the distribution of strength from a simulated population of 10^5 players after 10^6 competitions (right-hand curve).

Now we calculate the limiting distribution of wealth $q(n) \equiv q_n$, which can be determined without reference to strength. The problem is equivalent to calculating the distribution of mass in a collection of randomly aggregating particles all initially of mass 1, where each collision event yields a new mass-1 particle: $i + j \to k + 1$, where $k = i + j$. Let q_n be the fraction of players with wealth n. Since every collison yields a 1-er and a non-1-er, after a long time $q_1 = \frac{1}{2}$. The only way of producing a 2-er is from two 1-ers: $q_2 = \frac{1}{2}(q_1 q_1) = \frac{1}{8}$. Likewise, $q_3 = \frac{1}{2}(q_1 q_2 + q_2 q_1) = \frac{1}{16}$ and, in general,

$$q_{n+1} = \frac{1}{2} \sum_{i=0}^{n-1} q_i\, q_{n-i}. \tag{4}$$

The solution to this difference equation is the steady state distribution of player wealth, namely

$$q_{n+1} = \frac{C_n}{2^{(2n+1)}}, \tag{5}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The values of q_n are plotted in fig. 1 and match the distribution of wealth from a simulated population (left-hand curve).

By Stirling's approximation, the distribution of wealth n scales as

$$q_n \sim \frac{1}{\sqrt{\pi}} n^{-3/2}. \tag{6}$$

Note that (5) becomes more and more valid in the tail (high wealth n) as time increases. New wealth is added to the population in the form of 1-ers, which then flows to the right, and for this reason the average wealth $\sum_{n=1}^{\infty} n q_n = (M+t)/M$ diverges, even though half the players are of unit wealth (the median is finite).

Diminishing population. – Here we consider competition between a fixed number of players without replacement. Unlike the previous version of the problem, this one has no limiting behaviour. We solve it exactly as a function of time t and number of players M.

There is a finite number of competition histories (who plays whom when) that M labelled players can realize. The number of histories grows with M as

$$H = \frac{1}{2^{M-1}} M!\,(M-1)!. \tag{7}$$

It is convenient to visualize the histories as trees on M labelled nodes, in which two branches merge when two players compete. No two branches can merge simultaneously; the merger events are ordered. Statistical properties of the system of players can be determined by averaging over all relevant trees.

We first calculate the analogue to (3), the distribution of strength $\tilde{p}(s,t,M)$. For convenience, we first relabel the M strengths by their rank order, that is, $1, 2, \ldots, M$. The quantity $\tilde{p}$ is the probability that a player with strength rank s will end up in the last $M-t$ players. It is given by

$$\tilde{p}(s,t,M) = \frac{1}{M-t} \sum_{\mathbf{b}\in B_{t-1}} \bar{C}(s,t,m,1) \times \prod_{k=1}^{t-1} C(s,k,M,2)^{b_k} \big(\bar{C}(s,k,M,1) - C(s,k,M,2)\big)^{1-b_k}, \tag{8}$$

where

$$C(s,k,M,z) = \frac{\binom{M-s-\sum_{l}^{k-1} b_l}{z}}{\binom{M-k+1}{2}} \tag{9}$$

and $\bar{C} = 1 - C$ and $\mathbf{b}$ is a $(t-1)$-dimensional binary vector that in the sum runs over all possible binary vectors B.

Let $\tilde{q}(n,t,M)$ be the distribution of wealth for a diminishing population, analogous to q_n in (5) for a fixed population. At time $t=0$, the distribution is entirely peaked at 1; at $t = M-2$, the distribution is uniform over all n. The exact form of $\tilde{q}$ is

$$\tilde{q}(n,t,M) = \frac{\binom{M-n-1}{t-n+1}}{\binom{M-1}{t}}, \tag{10}$$

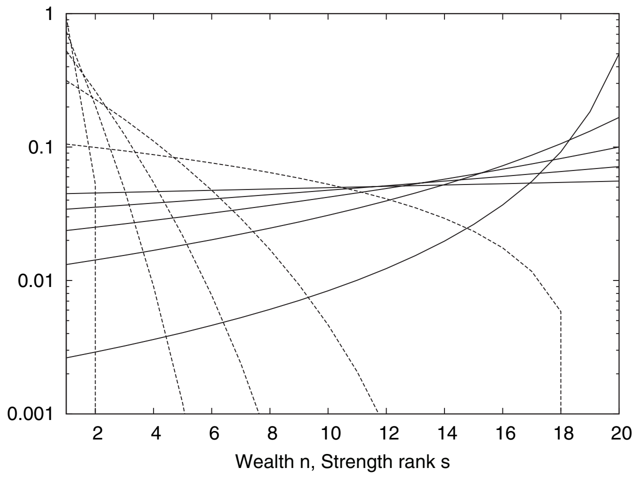

Fig. 2: The distributions of strength $\tilde{p}(s,t,M)$ (solid line) and wealth $\tilde{q}(n,t,M)$ (dotted line) for a diminishing population. (In a diminishing population, every time a player is defeated it is not replaced.) STRENGTH (solid lines): we show our prediction (8) for $M=20$, at various times t. WEALTH (dotted lines): we show our prediction (10) for $M=20$ and various t. For $M \leqslant 7$, exact enumeration perfectly matches our results.

which can alternatively be written

$$\tilde{q}(n,t,M) = \frac{M(M-t-1)}{(M-n)(t-n+1)} \prod_{i=0}^{n-1} \frac{t-i}{M-i}, \tag{11}$$

keeping in mind that $t \leqslant M-1$ and $n \leqslant t+1$.

Equations (8) and (10) are plotted in fig. 2 for $M=20$. For smaller, enumerable values of M ($M \leqslant 7$), we find that (8) and (10) perfectly match exact enumeration.

Strategy. – Assume all players compete randomly apart from one, which is free to choose which players it plays. What is the optimal strategy in order to maximise the probability of achieving some score? For example, what is the best way for a 2-er to become a 6-er: play 1 4-er, or 2 2-ers, or 4 1-ers?

Our plan for answering this is as follows. We first calculate the distribution of strength for a player with wealth n (an n-er). From this we can write down the probability that an i-er beats a j-er, for arbitrary i and j. We then show that the form of this quantity ensures path independence of wealth attainment.

After a competition between players with wealths i and j, the strength of the survivor, with wealth $k = i+j$, has density distribution

$$f_k(s) = f_i(s) \int_{-\infty}^{s} f_j(s)\,\mathrm{d}s + f_j(s) \int_{-\infty}^{s} f_i(s)\,\mathrm{d}s. \tag{12}$$

The solution is

$$f_k(s) = \partial_s (F_i(s) F_j(s)), \tag{13}$$

where $F_i(s)$ is the cumulative density distribution

$$F_i(s) = \int_{-\infty}^{s} f_i(s)\,\mathrm{d}s. \tag{14}$$

Since a 2-er can only be produced by a collision between two 1-ers, we have $f_2(s) = \partial_s(F_1^2(s))$ and, by induction,

$$f_k(s) = \partial_s(F_1^k(s)). \quad (15)$$

If we take $f_1(s)$ to be uniform, then (15) becomes

$$f_k(s) = ks^{k-1}. \quad (16)$$

With the density distribution for strength in terms of wealth, we can find the probability of an i-er beating a j-er. It is

$$P(s_i > s_j) = \int_{s_i=-\infty}^{\infty} \int_{s_j=-\infty}^{s_i} f_i(s_i) f_j(s_j)\, \mathrm{d}s_i\, \mathrm{d}s_j. \quad (17)$$

Substituting (15) into (17), we find

$$P(s_i > s_j) = \frac{i}{i+j}. \quad (18)$$

We now see that the probability of a 2-er becoming a 6-er is $\frac{1}{3}$ for all three strategies above: $\frac{2}{6} = \frac{2}{4}\frac{4}{6} = \frac{2}{3}\frac{3}{4}\frac{4}{5}\frac{5}{6}$. In general, for $a < b < c$,

$$P(a \to c) = P(a \to b) P(b \to c) = \frac{a}{c}, \quad (19)$$

and we see that the probability of gaining some score is path independent—all strategies are equivalent.

Discussion. – Why is wealth the correct predicter of a player's strength, and why is the probability of attaining a high wealth path independent? The key reason for both is the absence of loops in the competition histories—no two players can have defeated the same player twice. This means that the wealth of a player is equivalent to the number of players that he is demonstrably better than and, crucially, the sets of inferior players for each extant winner are disjoint. Both the fixed and diminishing population versions of the problem ensure that there are no loops present in the dynamics, and therefore eqs. (12)–(19) are valid for both versions.

We can calculate two other properties of players applicable to both versions of the problem: the typical strength in terms of wealth, and the typical wealth at time of death in terms of present wealth (life expectancy).

We first calculate strength s in terms of wealth n. Because the distribution of s for a given n is highly skewed for moderate n, we consider the median instead of the mean. The median strength s_n^{med} also has a natural interpretation: it is the probability of beating a new player with random strength. By definition, the median strength satisfies $\int_{-\infty}^{s_n^{\mathrm{med}}} f_n(s)\, \mathrm{d}s = 1/2$, and thus by (16)

$$s_n^{\mathrm{med}} = \sqrt[n]{\tfrac{1}{2}}. \quad (20)$$

Table 1: Statistics for an n-er (a player with wealth n). From left: wealth; fraction of players with a given wealth (in the fixed population model); median strength; wealth at time of death.

Wealth (Score)	Fraction q_n	Med. strength s_n^{med}	Life expect. n^{exp}
1-er	50%	0.500	2-er
2-er	12.5%	0.707	4-er
3-er	6.25%	0.794	6-er
4-er	3.906%	0.841	8-er
5-er	2.734%	0.871	10-er
6-er	2.051%	0.891	12-er
7-er	1.611%	0.906	14-er
8-er	1.309%	0.917	16-er
9-er	1.091%	0.926	18-er
10-er	0.927%	0.933	20-er
20-er	0.321%	0.966	40-er
30-er	0.173%	0.977	60-er
40-er	0.113%	0.983	80-er
50-er	0.080%	0.986	100-er

Second, we calculate the life expectancy n_{exp}: the expected wealth before defeat. The mean life expectancy is $n + \sum_{i=1}^{\infty} \frac{i\, n}{i+n}$, which is infinite. The median life expectancy, on the other hand, follows from (18):

$$n_{\mathrm{exp}}^{\mathrm{med}} = 2n. \quad (21)$$

The probability of an n-er attaining wealth $2n$ is precisely 1/2, independent of strategy. Values of the median strength and life expectancy are listed in table 1.

Conkers. – One popular example of competition in which wealth is known but strength is not is conkers [10,11], a game played with the nuts of the common horse-chestnut tree (*Aesculus hippocastanum*). A hole is drilled through the centre of the nut and a string or shoelace is threaded through the hole with a stopping knot tied at one end to retain the nut. Pairs of players take turns swiping each other's conker with their own until one conker is sufficiently damaged to fall off the string. Each conker is assigned a score as follows. All new conkers start with a score of 1. Each time a conker beats another conker, it adds to its score the score of the defeated[1]. We assume that the stronger conker wins.

Apart from closed tournaments, ordinary competition between conkers on the playground is best modelled by our fixed population model, since a defeated conker is likely to be replaced by a newly fallen nut. Then the distribution of strength and score (wealth) are given by the curves in fig. 1. Regardless of the number of conkers in play, typically 1/2 of all conkers will be 1-ers, 1/8 will be 2-ers, 5/128 3-ers, and so on.

[1]The most popular way of scoring conkers is defined differently: all conkers start with a score of 0, and when one conker beats another, it adds to its score the score of the defeated plus one for winning. This scoring system can be translated to the one described in the text by adding 1 to each score.

The scoring method used in conkers turns out to be extremely well chosen. No other scoring system better reflects a conker's strength and likelihood of future success. Moreover, there is no optimal strategy for maximising a conker's score—playing a few high-score conkers is just as sensible as playing many low-score ones. However, not all strategies for getting a high score are equally *fast*. If you have a number of conkers, the quickest way to achieve a high score is to play high score conkers. Chances are high that the conker will lose —the probability an i-er beats a j-er is $\frac{i}{i+j}$— in which case you simply try again with another conker. For large n, the typical number of 1-ers necessary to beat an n-er is $n \ln 2$.

If all conkers' strength is uniformly distributed between 0 and 1, the median strength of an n-er is $\sqrt[n]{\frac{1}{2}}$. Unlike in nature, where an organism's expected remaining lifespan decreases (or at best remains constant [12]) with age, a conker's life expectancy (typical score at time of death) increases linearly with score. In table 1 we list some statistics for conkers with a score of n.

Round-robin competition. – The concept of wealth can be extended to round-robin competition in organized sport [13]. In round-robin competition, each of M teams plays all the other teams, making $\binom{M}{2}$ games in total. Sports events which are wholly or party organized in this way include: the FIFA World Cup (football, called soccer in the US); the UEFA Cup (football); some American football college conferences; the Cricket World Cup; and the Super 14 (rugby union). Typically, the $\binom{M}{2}$ games are divided into $M-1$ different stages, where in each stage the M teams play $M/2$ games.

Again, we assume that each team has a fixed strength and when any two teams play the stronger one wins. This time, however, the loser is not eliminated. As we showed earlier, the wealth of a team is the optimal indicator of its likelihood of future success in the absence of loops. However, it is also a good approximation when loops are allowed, and is exactly valid as t approaches $t_{\max}$, where time $t \in [0, \binom{M}{2}]$ is the number of games that have been played.

As a round-robin competition proceeds, construct the following directed graph. Let M labelled nodes represent the M teams, and every time some team A beats another team B, draw a directed edge from B to A. In round-robin competition, we define the wealth of a team A to be the basin of attraction of node A, that is, the number of points which eventually flow to A. This is the number of teams that A is demonstrably stronger than.

At any time t, we can infer the relative strength of the M teams by ordering them (or partially ordering them if there are ties) by their wealth. At time $t = \binom{M}{2}$, the wealth perfectly corresponds to the relative strength of the teams. Our investigations show that, at time $t < \binom{M}{2}$, the wealth of a team remains a good indicator of strength, significantly better than the number of wins.

Conclusion. – We have studied one of the simplest and most common forms of competition, in which pairs of players compete sequentially and the stronger player wins. We showed that the best indicator of future success is not the number of past wins but rather the player's wealth: one plus the wealths of all defeated players, where all new players begin with wealth one. We calculated the distributions of wealth and strength when the loser is replaced and when the loser is not replaced, and showed that the probability of attaining a high wealth is path independent. This is likely to modify the way we order players in real-world competition.

Apart from competition, our model may also be used to describe systems of random aggregation. In this case the notion of strength is abandoned, there is no winner or loser, and we are only interested in the distribution and attainment of wealth. Examples of this are family inheritances, which aggregate through marriage, and corporate assets which are combined through mergers and aquisitions.

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This work was funded in part by the Defense Advanced Research Projects Agency (DARPA) Fundamental Laws of Biology program (FunBio) grant HR 0011-05-1-0057. JBC is supported by FunBio. SEA is supported by a Leverhulme Trust Fellowship.

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