Network Valuation in Financial Systems

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Abstract

We introduce a network valuation model (hereafter NEVA) for the ex-ante valuation of claims among financial institutions connected in a network of liabilities. Similar to previous work, the new framework allows to endogenously determine the recovery rate on all claims upon the default of some institutions. In addition, it also allows to account for ex-ante uncertainty on the asset values, in particular the one arising when the valuation is carried out at some time before the maturity of the claims. The framework encompasses as special cases both the ex-post approaches of Eisenberg and Noe and its previous extensions, as well as the ex-ante approaches, in the sense that each of these models can be recovered exactly for special values of the parameters. We characterize the existence and uniqueness of the solutions of the valuation problem under general conditions on how the value of each claim depends on the equity of the counterparty. Further, we define an algorithm to carry out the network valuation and we provide sufficient conditions for convergence to the maximal solution.

Keywords: Interbank Claim Valuation; Network Valuation; Financial Network; Systemic Risk; Credit risk; Default; Contagion

1 Introduction

Uncertainty and interdependence are two fundamental features of financial systems. While uncertainty over the future value of assets is traditionally very central in the financial lit-

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erature (Merton, 1974), interdependence of financial claims' values, especially in interconnected banking systems, has been investigated only more recently (Allen and Gale, 2001; Eisenberg and Noe, 2001; Elsinger et al., 2006; Freixas et al., 2000; Rochet and Tirole, 1996), taking center stage mostly after the recent financial crisis (Acemoglu et al., 2015; Battiston et al., 2016b; Elliott et al., 2014; Stiglitz, 2010). When the two features are considered together, the valuation of interdependent claims at a given time with arbitrary maturity is, in general, a non-trivial problem (Eisenberg and Noe, 2001) with crucial policy implications. The seminal work of Eisenberg and Noe (2001) (EN), which has been very influential in the literature on interbank contagion and systemic risk, has developed a framework to deal with the problem of interdependence in the absence of *ex-ante* uncertainty and bankruptcy costs. Their main result is that, in the case in which contracts are interbank debt securities, mild conditions on the network topology, and a simple maturity structure, it is possible to uniquely determine the so-called clearing payment vector, i.e. how much each bank is required to pay to the other banks in order to maximise the total payments in the system. A more recent work (Rogers and Veraart, 2013) has extended the EN model by showing that, despite the fact that bankruptcy costs may imply multiple solutions, one can still uniquely determine the solution that is preferable for all banks.

An important limitation of the EN framework is that the valuation of claims is carried out *ex-post*, i.e. at the maturity of the contracts, once the amount of external funds of each bank is known. One only needs to compute how losses, if any, should be redistributed among the surviving financial institutions. While EN emphasized the importance of moving towards an *ex-ante* valuation accounting for uncertainty (deriving, e.g., from possible shocks or cashflows between the valuation date and the maturity date), and despite the great interest in financial contagion spurred by the 2008 crisis, most works on systemic risk and stress-testing of interbank networks have focused on the original ex-post framework in which valuation and maturity times coincide (Acemoglu et al., 2015; Cifuentes et al., 2005; Elsinger et al., 2006; Glasserman and Young, 2015; Rogers and Veraart, 2013).

Therefore, a gap has emerged between this growing body of works on interconnectedness with ex-post valuation and the vast literature on the ex-ante valuation of corporate obligations. The latter, building on the classic Merton approach (Merton, 1974), deals with the problem of ex-ante valuation in the presence of uncertainty but does not encompass interdependence between claims, with a few exceptions (Cossin and Schellhorn, 2007; Fischer, 2014; Hain and Fischer, 2015; Suzuki, 2002).

The framework introduced in Suzuki (2002) applies also to the case in which cross-ownerships exist and Fischer (2014) further generalizes it to the case of obligations with different seniorities and derivatives. However, such results rely on two crucial assumptions. First, the payments made by each firm are a *continuous* function of the payments made by all firms. The assumption of continuity rules out the presence of mechanisms such as costs of default and abrupt loss of values in assets due to fire sales. Second, for *any* given level of seniority of the cross-holdings of debt *each* firm must also hold external liabilities with the same seniority.

Moreover, in the approach of Suzuki (2002) and Fischer (2014), in order to carry out an ex-ante valuation of the claims on a given institution (under the assumption of no arbitrage opportunities and market completeness), one needs to compute all possible trajectories of the stochastic processes followed by the assets of all institutions involved. In other words, this approach can be thought of as *global valuation mechanism*, since it implicitly assumes that the computation can be carried out by an entity with full knowledge of all the parameters of the financial system including all interbank exposures.

In contrast, as we show, the computation of the EN solution requires banks to know only local information about their counterparties at each step, although this holds at the expenses of not accounting for uncertainty. While such a global valuation approach is of great theoretical interest, it remains unclear whether the computation could be decentralized and therefore how feasible it would be its application as a valuation mechanism.

Another gap in the literature has also emerged between the theoretical insights from the stress-testing exercises based on the EN approach and the experience of practitioners and policy makers. Indeed, according to the BIS, the largest part of losses suffered by financial institutions during the financial crisis was not due to actual counterparties' defaults, but to the mark-to-market revaluation of obligations following the deterioration of counterparties' creditworthiness. This approach is called Credit Valuation Adjustment (CVA)[1]. This means that, while in the EN framework the default of a bank is the only event that matters for triggering losses down the chain of lending contracts, in practice also the deterioration of a bank's book matters for triggering those losses. Currently, this mechanism is not taken into account by most models of systemic risk (Furfine, 2003; Gai and Kapadia, 2010; Upper, 2011), and in particular by all those based on the EN model, although its importance has been increasingly acknowledged (Glasserman and Young, 2015). While the framework of DebtRank (Batiz-Zuk et al., 2015; Battiston et al., 2012) is one of the few models building on the idea of distress propagating even in the absence of defaults, its current formulation does not provide a consistent endogenous treatment of the recovery rate.

In light of the above considerations, in this paper we develop a novel general framework, which we refer to as *Network Valuation Model* (NEVA), for the valuation of claims among institutions interconnected within a network of liabilities, with the following characteristics. Similarly to EN, it is possible to endogenously determine a set of consistent values for the claims, following the default of some institutions. Differently from EN, we account for ex-ante uncertainty on the values of external assets, arising when the valuation is carried out at some time before the maturity of the claims, as commonly done in practice. From this point of view, while our approach is perfectly compatible with a global valuation mechanism (as in Fischer (2014); Suzuki (2002)), it also provides a *decentralized valuation mechanism* in which banks perform an ex-ante valuation of their claims in a decentralized

[1] The Basel Committee on Banking Supervision states that "roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults." Basel Committee on Banking Supervision (2011).

fashion. In other words, the financial system as a whole performs the valuation recursively via a distributed mechanism through which each bank only needs information about the valuation of the claims of its own counterparties at each step of the calculation. Finally, in contrast with Suzuki (2002) and Fischer (2014) we avoid specific assumptions about the continuity of the valuation performed by banks, therefore allowing for the possibility to account for costs of default and fire sales, e.g. as in Rogers and Veraart (2013).

More in detail, the timing of the framework can be best conceived of in four steps. At time zero all contracts are set, resulting from various possible investment allocation strategies, which are not explicitly modelled here and are not relevant to our results. At time one there is a shock on one or more of the external assets of the banks. At time $t \geq 1$ the valuation is carried out, while contracts mature at time $T \geq t$. The framework also allows for contracts with multiple maturities. Between t and T possible changes in the value of external assets are accounted for. Remarkably, on the one hand, we obtain the ex-post approach of Eisenberg and Noe, as well as the Rogers and Veraart extension as limit cases of the NEVA model when the time of the valuation is assumed to be the same as the time of the maturity and there is no uncertainty on the value of the external assets held by banks. On the other hand, the classic ex-ante valuation Merton approach can also be obtained as a limit case in which there are no interbank claims and external assets follow a geometric Brownian motion up to maturity. More interestingly, it is possible to extend the EN decentralized computation of consistent valuations to the case in which valuations are consistent expected values of the claims under local knowledge of the shock distribution. In particular, we show that the DebtRank model (Bardoscia et al., 2015a; Battiston et al., 2016a, 2012) is obtained as a limit case in which shocks on external assets before the maturity follow a uniform distribution. We characterize the existence and uniqueness of the solutions of the valuation problem under general conditions on the functional form of the valuation function, i.e. on how the value of each claim depends on the equity of the counterparty. Further, we define an algorithm to carry out the network valuation and provide sufficient conditions for convergence in finite time to the greatest solution (in the sense that the equity of each banks is greater or equal than in the other solutions) with a given precision. Finally, under additional assumptions, we show that the solutions of parametric families of ex-ante NEVAs models (i.e. before the maturity) smoothly converge to the solutions of the corresponding ex-post NEVA models (i.e. at the maturity).

2 Framework and Definitions

We consider a financial system consisting of n institutions (for brevity "banks" in the following) engaging in credit contracts with some others. Our goal is to set out a general framework (NEVA) in which banks can evaluate their own interbank claims on other banks (and thus their own equity) in a network of liabilities, by taking into account simultaneously the claims of all the banks in the network. Credit contracts are established at time 0 and are taken as given, with L_{ij} denoting the book value of the debt of bank i towards bank j, and A_{ji} denoting the book value of the corresponding asset of bank j, with $A_{ji} = L_{ij}$. Banks also have assets and liabilities external to the interbank system, which we denote respectively with A_i^e and L_i^e. The external assets of banks are subject to stochastic shocks, and this is the only source of stochasticity in the model.

We denote by T_{ij} the maturity of the contract between i and j (with respect to a reference time zero) and by t the time at which the evaluation of the financial claims is carried out. In the special case in which $t = T_{ij}$, for all i, j, the evaluation takes place *at* maturity. This is precisely the case considered in the ex-post clearing procedure in most works based on Eisenberg and Noe (2001). However, more in general, it is of practical interest the case in which the evaluation takes place before the maturity, hereafter referred to as *ex-ante* evaluation. In this case, we want to determine at $t < T$ the value of banks' liabilities knowing the underlying distribution of shocks that could affect banks external assets between t and T.

We denote the book value of the equity of bank i, i.e. the difference between its total assets and liabilities taken at their book (face) value as M_i:

$$M_i = A_i^e - L_i^e + \sum_{j=1}^{n} A_{ij} - \sum_{j=1}^{n} L_{ij} \, . \tag{1}$$

However, a proper valuation of equity of bank i, denoted here with E_i, will depend on how much bank i values its own assets. Such valuation can markedly differ from the face value, and will certainly depend on several parameters associated with specific contracts. Most importantly, it will depend on other banks' equities, or more precisely on their valuation of their equities. For instance, one can expect the value of an asset corresponding to a loan extended from bank i to bank j to depend on both E_i and E_j and, reasonably, larger values of equities will imply larger valuations of the assets. Such intuition is formalised in the following definition:

Definition 1. *Given an integer $q \leq n$, a function $\mathbb{V} : \mathbb{R}^q \to [0,1]$ is called* feasible valuation function *if and only if:*

1. *it is nondecreasing:* $\mathbf{E} \leq \mathbf{E}' \Rightarrow \mathbb{V}(\mathbf{E}) \leq \mathbb{V}(\mathbf{E}'), \forall \mathbf{E}, \mathbf{E}' \in \mathbb{R}^q$
2. *it is continuous from above.*

The general idea behind the definition of feasible valuation function is that one can write the value of any asset as the product of its face value times a valuation function, such that it ranges from the face value of the asset to zero. In general, the valuation performed by bank i of its claim against bank j, under the assumption that such valuation depends only on banks' equities and other parameters associated with their specific contract, is:

$$A_{ij}\mathbb{V}_{ij}(\mathbf{E}|\alpha) \, , \tag{2}$$

where $\mathbb{V}_{ij} : \mathbb{R}^n \to [0,1]$ is a valuation function and α is a set of parameters. Analogously, we can write the value of external assets of bank i assuming that it is function of equities (and additional parameters):

$$A_i^e \mathbb{V}_i^e(E_i|\alpha)\,, \tag{3}$$

where $\mathbb{V}_i^e : \mathbb{R} \to [0,1]$ is a valuation function. In the remainder of this paper we mostly focus on examples in which the valuation function of external assets has the form (3) and in which the valuation function of interbank assets depends on the equity of the borrower, i.e. $\mathbb{V}_{ij}(\mathbf{E}|\ldots) = \mathbb{V}_{ij}(E_j|\ldots)$. However, we point out that all the results that we derive still hold in the more general case in which the valuation functions depend on all the equities.

Each bank will assess the value of its equity at time t as the difference between the *valuation* of its assets minus the value of its liabilities:

$$E_i(t) = A_i^e \mathbb{V}_i^e(\mathbf{E}(t)|\ldots) - L_i^e + \sum_{j=1}^{n} A_{ij} \mathbb{V}_{ij}(\mathbf{E}(t)|\ldots) - \sum_{j=1}^{n} L_{ij} \quad \forall i\,, \tag{4}$$

where, as customary, we consider all liabilities to be fixed at their book value. Since all valuation functions take values in the interval $[0,1]$, equities $E_i(t)$ are bounded both from below and from above:

$$m_i \equiv -L_i^e - \sum_{j=1}^{n} L_{ij} \leq E_i(t) \leq M_i \quad \forall i\,. \tag{5}$$

By introducing the following map:

$$\Phi \;:\; \bigtimes_{i=1}^{n} [m_i, M_i] \to \bigtimes_{i=1}^{n} [m_i, M_i] \tag{6a}$$

$$\Phi_i(\mathbf{E}(t)) = A_i^e \mathbb{V}_i^e(\mathbf{E}(t)|\ldots) - L_i^e + \sum_{j=1}^{n} A_{ij} \mathbb{V}_{ij}(\mathbf{E}(t)|\ldots) - \sum_{j=1}^{n} L_{ij} \quad \forall i\,, \tag{6b}$$

the set of equations (4) can be rewritten in compact form:

$$\mathbf{E}(t) = \Phi(\mathbf{E}(t))\,. \tag{7}$$

The map Φ allows each bank to compute its own equity given the equities of all the banks in the network. Such valuations are self-consistent only for the equity vectors $\mathbf{E}(t)$ that satisfy relation (7). In order to implement a consistent network-based valuation of interbank claims it is essential to prove the existence of solutions of (7). For the sake of readability in the following we will drop the explicit dependence of equities on the time t at which the valuation is performed.

3 Main results

We now outline the most general results, which apply to generic feasible valuation functions.

Theorem 1 (Existence)**.** *The set of solutions of* (7) *is a complete lattice.*

This implies in particular that the set of solutions is non-empty and that there exist a least $\mathbf{E}^-$ and greatest solution $\mathbf{E}^+$ such that for any solution $\mathbf{E}^*$, $E_i^- \leq E_i^* \leq E_i^+$, for all i. Within the set of solutions, the greatest solution is the most desirable outcome for all banks, as it simultaneously minimizes losses for all of them. Understanding how to compute such solution is therefore of paramount importance. Let us explicitly note that every solution $\mathbf{E}^*$ of (7) corresponds to a fixed point of the iterative map

$$\mathbf{E}^{(k+1)} = \Phi(\mathbf{E}^{(k)}) \,, \tag{8}$$

and viceversa. Eq. (8) defines the usual Picard iteration algorithm (called "fictitious sequential default algorithm" in Eisenberg and Noe (2001)) and in principle provides a method to compute the solutions with arbitrary precision, as we will show in the following. Iterating the map starting from an arbitrary $\mathbf{E}^{(0)}$ does not guarantee that the solutions $\mathbf{E}^+$ and $\mathbf{E}^-$ can be attained. In fact different solutions of (7) can be found depending on the chosen starting point. Moreover, some solutions might be *unstable*, in the sense that, while still satisfying (7), choosing a starting point for Picard iteration algorithm arbitrary close to (but not equal to) such solutions, will result in the iterative map converging to another solution of (7). The problem of finding the least and greatest solution this problem is solved by the following theorems:

Theorem 2 (Convergence to the greatest solution)**.** *If* $\mathbf{E}^{(0)} = \mathbf{M}$*:*

1. *the sequence* $\{\mathbf{E}^{(k)}\}$ *is monotonic non-increasing:* $\forall k \geq 0$, $\mathbf{E}^{(k+1)} \leq \mathbf{E}^{(k)}$,
2. *the sequence* $\{\mathbf{E}^{(k)}\}$ *is convergent:* $\lim_{k\to\infty} \mathbf{E}^{(k)} = \mathbf{E}^{\infty}$,
3. $\mathbf{E}^{\infty}$ *is a solution of* (7) *and furthermore* $\mathbf{E}^{\infty} = \mathbf{E}^+$.

Theorem 2 shows that, if the starting point of the iteration is $\mathbf{E}^{(0)} = \mathbf{M}$, which corresponds to taking all assets at their face value, the iterative map (8) converges to the greatest solution $\mathbf{E}^+$. Theorem 2 guarantees that for all $\epsilon > 0$, there exists $K(\epsilon)$ such that for all $k > K(\epsilon)$ we have that $||\mathbf{E}^{(k)} - \mathbf{E}^+|| < \epsilon$. In other words, once a precision ϵ has been chosen, starting from the face values of equities $\mathbf{M}$, and after a finite number of iterations, the Picard algorithm provides equities (8) that are undistinguishable from the greatest solution, within precision ϵ.

Mutatis mutandis, it is possible to prove that:

Theorem 3 (Convergence to the least solution)**.** *If* $\mathbf{E}^{(0)} = \mathbf{m}$ *and the valuation functions in* Φ *are continuous from below, then:*

1. *the sequence* $\{\mathbf{E}^{(k)}\}$ *is monotonic non-decreasing:* $\forall k \geq 0$, $\mathbf{E}^{(k+1)} \geq \mathbf{E}^{(k)}$,
2. *the sequence* $\{\mathbf{E}^{(k)}\}$ *is convergent:* $\lim_{k\to\infty} \mathbf{E}^{(k)} = \mathbf{E}^{\infty}$,
3. $\mathbf{E}^{\infty}$ *is a solution of* (7) *and furthermore* $\mathbf{E}^{\infty} = \mathbf{E}^{-}$.

Analogous results to the ones proved after Theorem 2 also hold in this case.

Therefore, Theorems 2 and 3, provide a simple algorithmic way to check whether the solution of (7) is unique within numerical precision:

Corollary 1 (Uniqueness)**.** *If* $\mathbf{E}^{+} = \mathbf{E}^{-}$, *the solution of* (7) *is unique.*

Let us now put these results in the context of the existing literature. In order to prove the existence of a solution, Suzuki (2002) and Fischer (2014) exploit the Brouwer-Schauder fixed point theorem, which requires payments made by each firm to be a *continuous* function of the payments made by all firms. The assumption of continuity does not allow to account for default costs. However, in Suzuki (2002) and Fischer (2014) the iterative map is not required to be monotonic, allowing to model some derivatives having a specific functional form. Since the Brouwer-Schauder fixed point theorem does not give any information about the structure of the solution space (e.g. the existence of a greatest and a least solution) it is important to have a unique solution. In order to prove uniqueness Suzuki (2002) and Fischer (2014) resort to the additional hypothesis that the ownership matrix (the analogous of our matrix A_{ij}) is strictly left substochastic, meaning that for *any* given level of seniority of the cross-holdings of debt *each* firm must also hold external liabilities with the same seniority. Here we use instead the Knaster-Tarski fixed point theorem, which requires valuation function to be monotonic – preventing a straightforward modeling of derivatives – and not necessarily continuous. As a consequence, default costs and analogous mechanisms can be easily accommodated in our framework (see Sec. 4). Through the Knaster-Tarski fixed point theorem we prove, not only the existence of a solution, but also the existence of a greatest and a least solution. Remarkably, Theorem 2 shows that the greatest solution is attained if the starting point of the valuation is the face value of claims, providing a clear prescription to perform the valuation even when multiple solutions exist.

3.1 Results on Directed Acyclic Graphs (DAGs)

Proof of the existence of an algorithm that ensures the convergence to a solution in a finite time are usually based on assumptions on the form of the valuation function (Hain and Fischer, 2015). In contrast, here we show that such result holds for a specific topology of the network of interbank liabilities, namely a DAG (Directed Acyclic Graph), regardless of the functional form of the interbank valuation functions.

Proposition 1 (DAG)**.** *If the matrix defined by interbank assets A_{ij} is the adjacency matrix of a DAG and $\mathbb{V}^e_i(E) = 1$, $\forall i$:*

1. *the map (8) converges in a finite number of iterations,*

2. *the solution of (7) is unique.*

Proof. We define source banks as those banks that do not hold interbank assets, i.e. $S_0 = \{i : A_{ij} = 0, \forall j\}$, which is a non-empty set if the matrix of interbank exposures is a DAG. We then partition banks based on the maximum graph distance from the set of source banks S_0, the partition being $\{S_d\}_{d=0}^{d_{\max}}$. Starting from the initial condition $\mathbf{M}$, banks in S_0 converge in zero iterations to their face value as their equity does not depend on the equity of any other bank (neither their own). Banks in S_1 converge in one iteration as their equity only depends on the equities of banks in S_0. By induction, banks in $S_{d_{\max}}$ converge in $d_{\max}$ iterations. Starting from the initial condition $\mathbf{m}$ banks in S_0 converge in one iteration to their face value as the Picard iteration algorithm corrects the value of their equities exactly in one iteration. Consequently, $\Phi^{(d_{\max})}(\mathbf{M}) = \Phi^{(d_{\max}+1)}(\mathbf{m})$, and therefore all banks converge to $\mathbf{E}^- = \mathbf{E}^+$ in (at most) $d_{\max} + 1$ iterations. □

4 Examples

We now highlight the generality of the NEVA outlined in Section 2 by presenting a few relevant examples. More specifically, we show that four different models well known in the literature about systemic risk can be recovered as limit cases.

Proposition 2 (Eisenberg and Noe)**.** *If:*

1. $\mathbb{V}^e_i(E_i) = 1$, $\forall i$,

2. $\mathbb{V}_{ij}(E_j) = \mathbb{1}_{E_j \geq 0} + \left(\frac{E_j + \bar{p}_j}{\bar{p}_j}\right)^+ \mathbb{1}_{E_j < 0}$, $\forall i,\, j$,

there is a one-to-one correspondance between the solutions of (7) and the solutions of the map Φ introduced in Eisenberg and Noe (2001).

Proof. As already noted, since in EN the evaluation happens *at* maturity, $t = T_{ij}$ for all i, j. Under the assumptions of (i) limited liabilities, (ii) priority of debt over equity, (iii) proportional repayments, EN aims at computing a clearing payment vector $\mathbf{p}^*$ whose component p^*_i is the total payment made by bank i to its counterparties. To conform to their notation, we also introduce the obligation vector $\bar{\mathbf{p}}$, defined as $\bar{p}_i = \sum_j L_{ij}$, which is the total interbank liability that bank i needs to settle. Eisenberg and Noe (2001) show that:

$$p^*_i = \min\left[e_i + \sum_j L_{ji}\frac{p^*_j}{\bar{p}_j}, \bar{p}_i\right] , \tag{9}$$

where $e_i = A_i^e - L_i^e$, and external liabilities are also due at the same maturity of interbank liabilities. Eq. (9) can be equivalently rewritten as:

$$p_i^* = \bar{p}_i \mathbb{1}_{E_i \geq 0} + \left(E_i(\mathbf{p}^*) + \bar{p}_i\right)^+ \mathbb{1}_{E_i < 0}\,, \tag{10a}$$

with

$$E_i(\mathbf{p}) = A_i^e - L_i^e + \sum_j A_{ij} \frac{p_j}{\bar{p}_j} - \sum_j L_{ij}\,, \tag{10b}$$

where $\mathbb{1}_{x>0}$ is the indicator function relative to the set defined by the condition $x > 0$ and $(x)^+ = (x + |x|)/2$. The above equations are equivalent to (4) by choosing the valuation functions as in the hypotheses of the Proposition 2. In fact, when $E_j > 0$, the cash inflow of bank j is enough to cover its due payments, and therefore $\bar{\mathbf{p}} = \mathbf{p}^*$. In contrast, when $E_j < 0$, bank j employs its residual assets $(E_j + \bar{p}_j)^+$ to repay its creditors proportionally as much as it can. □

Proposition 3 (Rogers and Veraart)**.** *If:*

1. $\mathbb{V}_i^e(E_i) = \mathbb{1}_{E_i \geq 0} + \alpha \mathbb{1}_{E_i < 0}$, $\forall i$,
2. $\mathbb{V}_{ij}(E_i, E_j) = [\mathbb{1}_{E_i \geq 0} + \beta \mathbb{1}_{E_i < 0}] \left[\mathbb{1}_{E_j \geq 0} + \left(\frac{E_j + \bar{p}_j}{\bar{p}_j}\right)^+ \mathbb{1}_{E_j < 0}\right]$, $\forall i,\ j$,

there is a one-to-one correspondance between the solutions of (7) *and the solutions of the map* Φ *introduced in Rogers and Veraart (2013).*

Proof. The proof is entirely analogous to the proof of Proposition 2. □

Let us note that, if $\alpha < 1$ ($\beta < 1$), then $\mathbb{V}_i^e$ ($\mathbb{V}_{ij}$) is not a continuous function. In particular, a value of α (β) strictly smaller than one means that when a bank defaults its external (interbank) assets will suddenly experience a relative loss of $\alpha - 1$ ($\beta - 1$), due e.g. to the necessity to liquidate them in a fire sale.

Proposition 4 (Furfine)**.** *If:*

1. $\mathbb{V}_i^e(E_i) = 1$, $\forall i$,
2. $\mathbb{V}_{ij}(E_j) = \mathbb{1}_{E_j \geq 0} + R \mathbb{1}_{E_j < 0}$, $\forall i,\ j$,

there is a one-to-one correspondence between the solutions of (7) *and the solutions of the map* Φ *introduced in Furfine (2003).*

Proof. According to the Furfine algorithm a counterparty with non-negative equity is always able to fully repay its liabilities, while, if its equity is negative, it will only repay a fraction R of them. This is exactly what the valuation function in Proposition 4 accounts for. □

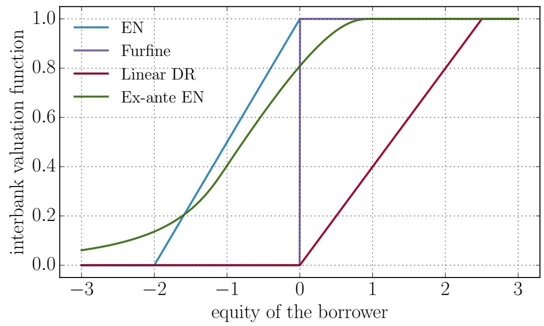

Figure 1: Interbank valuation functions as a function of the equity of the borrower. Parameters as follows. EN: $\bar{p}=2$, Furfine: $R=1$, Linear DebtRank: $M=2.5$, Ex-ante EN: $A^e=1$, $\bar{p}=2$, $\beta=1$, $\sigma=1$.

Proposition 5 (Linear DebtRank)**.** *If:*

1. $\mathbb{V}_i^e(E_i)=1$, $\forall i$,

2. $\mathbb{V}_{ij}(E_j)=\frac{E_j^+}{M_j}$, $\forall i, j$,

there is a one-to-one correspondance between the solutions of (7) and the solution of recursive map (linear DebtRank) introduced in Bardoscia et al. (2015a).

Proof. The easiest way to prove the correspondence is to compute the incremental variation of the iterative map (8), which in this case reads: $E_i^{(k+1)}-E_i^{(k)}=\sum_j A_{ij}\frac{\left(E_j^{(k)}\right)^+-\left(E_j^{(k-1)}\right)^+}{M_j}$, for all i. Starting the Picard iteration algorithm from $\mathbf{M}$ we recover (7) in Bardoscia et al. (2015a), in which $\mathbf{M}$ has been denoted with $\mathbf{E}(0)$. As soon as the equity of bank j becomes equal to zero in the iterative map in Bardoscia et al. (2015a), it will not change anymore, which is consistent with the incremental variation derived above. □

DebtRank has been introduced in Battiston et al. (2012) as an effective model to propagate shocks in the interbank network. Subsequently, a generalization of the model (which we call linear DebtRank here) has been derived from the balance sheet identity and from simplified assumptions about the propoagation of distress in the interbank market in Bardoscia et al. (2015a). The model has been further extended in Bardoscia et al. (2016, 2015b).

In Fig. 1 we plot several interbank valuation functions: EN (see Proposition 2), Furfine (see Proposition 4), Linear DebtRank (see Proposition 5), and ex-ante EN, which will be introduced in Sec. 5.

5 An application: Eisenberg-Noe with ex-ante valuation

On one hand, as already remarked, EN allows to compute the payments banks have to make to their counterparties. From Proposition 2 it is clear that to compute such payments one needs to know the values of all equities, thus implying that such computation should happen at the maturity. On the other hand, the evaluation of corporate debt before the maturity done by means of the Merton model (Merton, 1974) does not account for the recursive evaluations that are needed when creditors and debtors form a complex interconnected network. The aim of this section is to bridge this gap by introducing a set of valuation functions that allows to perform the ex-ante valuation of interbank claims, therefore accounting for the additional source of uncertainty deriving by the impossibility to have an unerring estimate of the assets values and equities before the maturity.

In the context of Asset Pricing Theory (APT), Fischer (2014) and Suzuki (2002) show that, assuming no arbitrage and market completeness, the ex-ante valuation at time $t < T$ of the random variable $E_i(T)$ at maturity can be performed by computing its conditional expectation with respect to the (unique) Equivalent Martingale Measure (EMM) $\mathbb{Q}$: $E_i(t) = \mathbb{E}_{\mathbb{Q}}[E_i(T)|\mathcal{F}(t)]$, where $\mathcal{F}(t)$ is the filtration at time t associated with E_i and, for simplicity, the returns of the riskless bond have been set to zero. In order to compute such conditional expectation, one needs to find the fixed point of (7), for any realization of the underlying stochastic processes. Such computation can be only performed by a central authority with full knowledge of all the parameters of the system.

Although the global ex-ante valuation can be also performed within our framework, here we focus instead on *local* ex-ante valuation, in the sense that, while the valuation is performed collectively by all banks, each bank only has information about its own counterparties. This happens precisely because, (i) the aforementioned uncertainty is entirely incorporated into the valuation functions and (ii), as it can be seen from (4), in order to compute an iteration of the Picard iteration algorithm (8), each bank only needs information about its own counterparties. Moreover, such set of valuation functions will naturally extend EN.

As regards the valuation functions of external assets, we simply take them as in Proposition 2. For what concerns interbank assets, we note that if the valuation occurred at maturity banks could use the interbank valuation functions in Proposition 2, in which the equity would be the equity at the maturity T. This information, however, is not available to banks before T. We now assume that, from the time t at which the valuation happens until maturity T, (i) the only uncertainty that banks must consider during the evaluation process is due to external assets, and (ii) external assets follow a stochastic process $\mathbf{A}^e(t)$. Hence, the variation of equity between t and T is equal to the variation of the external assets, i.e. $\mathbf{E}(T) = \mathbf{E} + \Delta\mathbf{A}^e$. From a technical point of view, the difference between global and local ex-ante valuations is that in the first case one computes the expectation of the fixed point of (7), while in the second case one computes the fixed point of (7) in which the expectations of the valuation functions appear:

$$\begin{aligned} E_i(t) = \mathbb{E}_{\mathbb{Q}}[E_i(T)|\mathcal{F}(t)] &= \mathbb{E}_{\mathbb{Q}}[A_i^e(T)|\mathcal{F}(t)] - L_i^e + \sum_{j=1}^{n} A_{ij}\mathbb{E}_{\mathbb{Q}}[\mathbb{V}_{ij}^{(\mathrm{EN})}(E_j(T))|\mathcal{F}(t)] - \sum_{j=1}^{n} L_{ij} \\ &= A_i^e(t) - L_i^e + \sum_{j=1}^{n} A_{ij}\mathbb{E}_{\mathbb{Q}}[\mathbb{V}_{ij}^{(\mathrm{EN})}(E_j(T))|\mathcal{F}(t)] - \sum_{j=1}^{n} L_{ij}\,, \end{aligned} \tag{11}$$

where $\mathbb{V}_{ij}^{(\mathrm{EN})}$ are the valuation functions in Proposition 2. Once the stochastic process followed by external assets is known, we can identify the valuation functions for interbank assets with their expectation conditioned on the observation at time t:

$$\begin{aligned} \mathbb{V}_{ij}(E_j) \equiv \mathbb{E}_{\mathbb{Q}}[\mathbb{V}_{ij}^{(\mathrm{EN})}(E_j)|\mathcal{F}(t)] &= \mathbb{E}\left[\mathbb{1}_{E_j(T)\geq 0} + \beta\left(\frac{E_j(T)+\bar{p}_j}{\bar{p}_j}\right)^{+}\mathbb{1}_{E_j(T)<0}\middle| A_j^e(t)\right] \\ &= \mathbb{E}\left[\mathbb{1}_{E_j(T)\geq 0}|A_j^e(t)\right] + \beta\mathbb{E}\left[\left(\frac{E_j(T)+\bar{p}_j}{\bar{p}_j}\right)^{+}\mathbb{1}_{E_j(T)<0}\middle| A_j^e(t)\right] \\ &= 1 - p_j^D(E_j) + \beta\rho_j(E_j)\,, \end{aligned} \tag{12}$$

where we have defined the probability of default:

$$\begin{aligned} p_j^D(E_j) &= \mathbb{E}\left[\mathbb{1}_{E_j(T)<0}|A_j^e(t)\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\Delta A_j^e < -E_j}\right] \end{aligned} \tag{13a}$$

and the endogenous recovery:

$$\begin{aligned} \rho_j(E_j) &= \mathbb{E}\left[\left(\frac{E_j(T)+\bar{p}_j}{\bar{p}_j}\right)^{+}\mathbb{1}_{E_j(T)<0}\middle| A_j^e(t)\right] \\ &= \mathbb{E}\left[\left(\frac{E_j(T)+\bar{p}_j}{\bar{p}_j}\right)\mathbb{1}_{-\bar{p}_j - E_j \leq \Delta A_j^e < -E_j}\right]\,. \end{aligned} \tag{13b}$$

Strictly speaking, β appearing in (12) would be equal to one in EN. Here, we include it to account for additional default costs. Moreover, its presence will be relevant in the context of Proposition 7. From (12) we can see that the valuation function can be thought of as the expectation over a two-valued probability distribution: If the borrower j does not default at maturity, bank i will recover the full amount A_{ij}, while if bank j defaults, bank i will in general recover a smaller amount. From this point of view, β can be thought of as an additional exogenous recovery rate on top of the endogenous recovery rate ρ_j. Finally, we note that (12) defines feasible valuation functions.

Proposition 6. *In the limit in which the maturity is approached, i.e. $t \to T$, the interbank valuation function (12) converges to the interbank valuation function of EN (Proposition 2).*

Proof. First we notice that, as $t \to T$ the variation in external assets goes to zero with probability approaching one, and therefore from (13) we have that $p_j^D(E) \to \mathbb{1}_{E_j<0}$ and that $\rho_j(E) \to \left(\frac{E_j+\bar{p}_j}{\bar{p}_j}\right)^+ \mathbb{1}_{E_j<0}$, from which the proposition easily follows. $\square$

5.1 Ex-ante valuation with geometric Brownian motion

In the spirit of the Merton model we will compute the expected value of assets at maturity (here, for the sake of convenience, maturity T is common to all interbank claims) given our observation of the value of external assets before the maturity (at time t), assuming that external assets follow independent geometric brownian motions:

$$dA_i^e(s) = \sigma A_i^e(s) dW_i(s) \qquad \forall i\, s\,, \tag{14}$$

where, for simplicity, we consider the drift to be equal to zero. The PDF of ΔA_i^e is:

$$p(\Delta A_i^e) = \frac{1}{\sqrt{2\pi(T-t)}\sigma(\Delta A_i^e + A_i^e)} e^{\frac{-\left[\log\left(1+\frac{\Delta A_i^e}{A_i^e}\right)+\frac{1}{2}\sigma^2(T-t)\right]^2}{2\sigma^2(T-t)}}\,. \tag{15}$$

From (13) we then have:

$$p_j^D(E) = \frac{1}{2}\left[1+\mathrm{erf}\left[\frac{\log(1-E/A_j^e)+\sigma^2(T-t)/2}{\sqrt{2(T-t)}\sigma}\right]\right]\mathbb{1}_{E<A_j^e} \tag{16a}$$

$$\rho_j(E) = \left(1+\frac{E}{\bar{p}}\right)\left(p_j^D(E) - p_j^D(E+\bar{p})\right) + \frac{1}{2\bar{p}}c_j(E) \tag{16b}$$

with

$$\begin{aligned} c_j(E) = &-\mathrm{erf}\left[\frac{\sigma^2(T-t)/2 - \log\left(1-E/A_j^e\right)}{\sqrt{2(T-t)}\sigma}\right]\mathbb{1}_{E<A_j^e} \\ &-\mathrm{erf}\left[\frac{\sigma^2(T-t)/2 + \log\left(1-E/A_j^e\right)}{\sqrt{2(T-t)}\sigma}\right]\mathbb{1}_{E<A_j^e} \\ &+\mathrm{erf}\left[\frac{\sigma^2(T-t)/2 + \log\left(1-(E+\bar{p})/A_j^e\right)}{\sqrt{2(T-t)}\sigma}\right]\mathbb{1}_{E<A_j^e-\bar{p}} \\ &+\mathrm{erf}\left[\frac{\sigma^2(T-t)/2 - \log\left(1-(E+\bar{p})/A_j^e\right)}{\sqrt{2(T-t)}\sigma}\right]\mathbb{1}_{E<A_j^e-\bar{p}}\,. \end{aligned}$$

Theorems 1 and 2 ensure that there exists a greatest solution (therefore optimal for all banks) and that such solution can be computed with arbitrary precision using the Picard iteration algorithm (8).

5.2 Comparison between valuation functions

For illustrative purposes here we perform a stress test on a small financial system composed by three banks, A, B, C. We choose a simple ring topology, $A \to B \to C \to A$ with the following parameters:

$$A^e = \begin{pmatrix} 10 \\ 5 \\ 3 \end{pmatrix} \qquad L^e = \begin{pmatrix} 9 \\ 4 \\ 3 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \\ 0.5 & 0 & 0 \end{pmatrix}, \tag{17}$$

so that all three banks have a book value of their equity equal to one. Total leverages, defined as the ratio between total assets and book values of equity, range from 10.5 to 3.5. Our stress test consists in applying an exogenous shock to the external assets of all banks, resulting in a relative devaluation α, i.e. $A_i^e \to (1-\alpha)A_i^e$. The variation in external assets of bank i, measured as the difference between its external assets before the shock and its external assets after the shock is $\Delta A_i^e = \alpha A_i^e$. Using (4) we can readily compute the corresponding variation in equity, again measured as the difference between the equity before the shock (i.e. its book value) and the equity after the shock: $\Delta E_i = \alpha A_i^e + \sum_j A_{ij}(1 - \mathbb{V}_{ij}(E_j^*))$. Network effects can be quantified as the total losses in the system minus the losses directly caused by the exogenous shock: $\sum_i \Delta E_i - \Delta A_i^e = \sum_{ij} A_{ij}(1 - \mathbb{V}_{ij}(E_j^*))$, which can be conveniently normalised by its maximum, $\sum_{ij} A_{ij}$:

$$\frac{\sum_i \Delta E_i - \Delta A_i^e}{\sum_{ij} A_{ij}} = \frac{\sum_{ij} A_{ij}\left[1 - \mathbb{V}_{ij}(E_j^*)\right]}{\sum_{ij} A_{ij}}. \tag{18}$$

In the left panel of Fig. 2 we show the behaviour of the quantity (18) as a function of the exogenous shock on external assets, for several valuation functions. For Furfine we use $R = 0$, while for ex-ante EN (NEVA in the legend) we use $\beta = 1$ and (14) with $\sigma = 0.1$, for all banks. Interestingly, we can see that for smaller values of the exogenous shock network effects are larger for the ex-ante EN than for EN at maturity, while the situation is reversed for larger values of the exogenous shock. This is consistent with the fact that uncertainty deriving from being before the maturity can both lead to lower and higher valuations of interbank claims. Lower valuations correspond to potential sizeable losses that can happen even in the presence of smaller shocks, while higher valuations correspond to positive fluctuations in the value of external assets that can lead to a recovery in the presence of larger shocks.

Another way to assess the extent of network effects is the following. Let us imagine that each bank wants to valuate the interbank assets of its counterparty using the standard

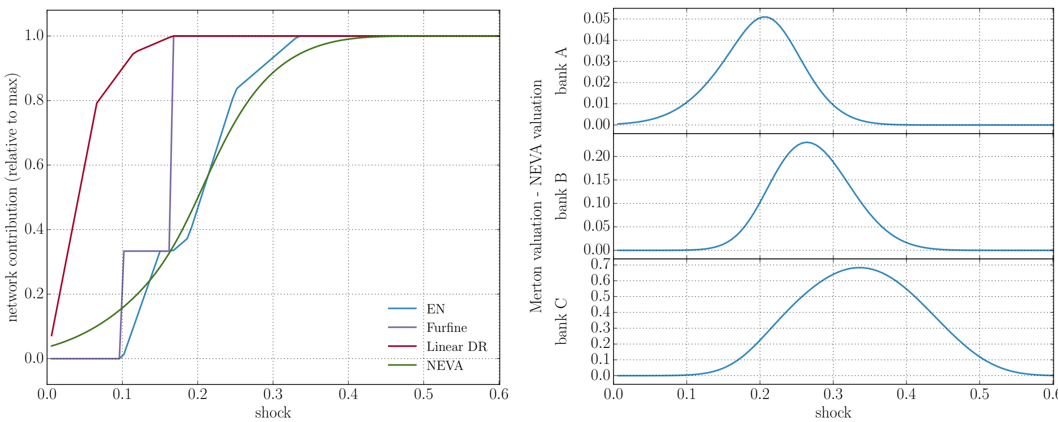

Figure 2: Stress test consisting in applying an exogenous shock to external assets of all banks and by re-evaluating interbank claims. Left panel: network contribution (normalised to its maximum value) as a function of the exogenous shock, for several valuation functions. Right panel: difference between the discount factor of the valuation of an interbank claim performed with the standard Merton approach and discount factor of the valuation of an interbank claim performed with ex-ante EN (NEVA).

Merton approach. This amounts to using the valuation function (12) and evaluating it in the book value of the equity of the counterparty. Hence, the lender i discounts its interbank assets A_{ij} towards the borrower j by a factor $\mathbb{V}_{ij}(M_j)$. If the same valuation is performed using NEVA, more specifically using ex-ante EN, the discount factor equals to $\mathbb{V}_{ij}(E_j^*)$. In the right panel of Fig. 2 we show the difference between such discount factors, i.e. between the discount factor of the valuation of an interbank claim performed with the standard Merton approach and the valuation of an interbank claim performed with ex-ante EN (NEVA in the legend) valuation functions with $\beta = 1$ and (14) $\sigma = 0.1$, for all banks. The difference is maximal for intermediate values of the exogenous shock. In fact, for small values of the shock network effects are small, while for large values of the shock the valuation of interbank claims becomes less and less important, as most losses will be direct losses due to the exogenous shock.

6 Limit behavior of solutions

We now introduce a sequence $\{\mathbb{V}_{ij}^l\}$ of valuation functions. For example, different values of l could correspond to different values of a parameter in the NEVA. For each value of l we have a different equation (7) of the form $\mathbf{E} = \Phi_l(\mathbf{E})$. We will denote the k-th iteration of the corresponding map in (8) with $\Phi_l^{(k)}$. In order to clarify the rationale behind the introduction of sequences of valuation functions, we will consider the following example.

Let l be an index associated with the distance to maturity, so that maturity is approached as l increases. For a given l we can compute the solution $\mathbf{E}_l$ corresponding to the given maturity. The question now arises: in the limit $l \to \infty$ (that is, as maturity approaches) will the solutions of the ex-ante valuations corresponding to the equations $\mathbf{E} = \Phi_l(\mathbf{E})$ approach the solution of the ex-post valuation (that is, at maturity) corresponding to the equation $\mathbf{E} = \Phi_\infty(\mathbf{E})$? Solving this problem essentially boils down to identifying sufficient conditions under which an interchange of the two limits (one on the model parameters and the other on the iterations of the Picard algorithm) is legit. A positive answer to this question will allow us to relate ex-ante and ex-post valuation models, correctly identifying ex-ante models as genuine generalizations of ex-post ones to the case of arbitrary maturity. The following theorem provides an affirmative answer for non-decreasing valuation functions.

Theorem 4. *If:*

1. *the sequences* $\{\mathbb{V}^l_{ij}\}$ *are monotonic non-decreasing:* $\mathbb{V}^l_{ij}(E) \leq \mathbb{V}^{l+1}_{ij}(E)$, $\forall E, i, j, l$,
2. *the sequences* $\{\mathbb{V}^l_{ij}\}$ *are pointwise convergent:* $\lim_{l\to\infty} \mathbb{V}^l_{ij}(E) = \mathbb{V}^\infty_{ij}(E)$, $\forall E, i, j$,
3. *there exists a unique solution for* $\mathbf{E}^* = \Phi_\infty(\mathbf{E}^*)$, *where* Φ_∞ *is the map corresponding to the valuation functions* $\mathbb{V}^\infty_{ij}(E)$,
4. $\mathbb{V}^\infty_{ij}(E)$ *is a continuous function,* $\forall i, j$.

then $\lim_{l\to\infty} \mathbf{E}_l = \mathbf{E}^*$, *where* $\mathbf{E}_l = \lim_{k\to\infty} \Phi^{(k)}_l(\mathbf{M})$.

For non-increasing valuation functions the requirement that Φ_∞ has a unique solution can be relaxed.

Theorem 5. *If:*

1. *the sequences* $\{\mathbb{V}^l_{ij}\}$ *are monotonic non-increasing:* $\mathbb{V}^l_{ij}(E) \geq \mathbb{V}^{l+1}_{ij}(E)$, $\forall E, i, j, l$,
2. *the sequences* $\{\mathbb{V}^l_{ij}\}$ *are pointwise convergent:* $\lim_{l\to\infty} \mathbb{V}^l_{ij}(E) = \mathbb{V}^\infty_{ij}(E)$, $\forall E, i, j$,

then $\lim_{l\to\infty} \mathbf{E}_l = \mathbf{E}^*$, *where* $\mathbf{E}_l = \lim_{k\to\infty} \Phi^{(k)}_l(\mathbf{M})$.

6.1 Limit to linear DebtRank

As a first application of the limit theorems we now consider the following sequence of valuation functions:

$$\mathbb{V}^l_{ij}(E_j) = 1 - p^D_j(E_j) + \beta_l \rho_j(E_j) \qquad l = 1, 2, \dots, \tag{19}$$

where $\{\beta_l\}$ is a monotone non-increasing sequence of real parameters such that $\lim_{l\to\infty}\beta_l = 0$ and ΔA_j^e has the uniform distribution in the interval $[-M_j,\, 0]$, $\forall j$. Then:

$$p_j^D(E) = 1 - \frac{(E)^+}{M_j} \tag{20a}$$

$$\rho_j(E_j) = \left[\frac{E_j + \bar{p}_j}{\bar{p}_j M_j}(b-a) + \frac{b^2 - a^2}{2\bar{p}_j M_j}\right]\mathbb{1}_{b>a}\,, \tag{20b}$$

where $b(E) = -(E)^+$ and $a(E) = \max(-\bar{p}_j - E, -M_j)$. The valuation functions $\mathbb{V}_{ij}^l$ are monotonic non-increasing in l and converge to the valuation functions, $\mathbb{V}_{ij}^\infty(E_j) = (E_j)^+/M_j$.

Proposition 7. *If:*

1. *the sequence* $\{\beta_l\}$ *is such that* $\lim_{l\to\infty}\beta_l = 0$*,*
2. *the sequences* $\{\mathbb{V}_{ij}^l\}$ *are chosen as in* (19)*,* $\forall i,\, j$*,*
3. *the probability of default and the endogenous recovery* (13) *are computed with* ΔA_j^e *having a uniform distribution in the interval* $[-M_j,\, 0]$*,* $\forall j$*,*

then the solution of the corresponding equation (7) *converges to the solution* $\mathbf{E}^*$ *of the linear DebtRank.*

Proof. The proof follows immediately from Theorem 5. By using the previous proposition one can effectively re-interpret linear DebtRank as an ex-ante EN model in which shocks are negative and the exogeneous recovery rate β is equal to zero. □

6.2 The case of geometric Brownian motion

We have already discussed the case in which external assets follow a geometric brownian motion in Section 5.1. Here we consider a sequence $\{\mathbb{V}_{ij}^l\}$ of valuation functions with corresponding maturities $\{T_l\}$, with $\lim_{l\to\infty} T_l = t$, i.e. the limit $l \to \infty$ corresponds to the limit in which the distance to maturity goes to zero. In this limit the valuation functions $\mathbb{V}_{ij}^l$ converge pointwisely to the valuation functions of EN (see Proposition 6). If we could apply either of Theorems 4 and 5, the solutions $\mathbf{E}_l$ of NEVA with interbank valuation functions in (12) (the ex-ante EN) would converge to the solution $\mathbf{E}^*$ of the (ex-post) EN. Unfortunately, $\mathbb{V}_{ij}^l$ are neither non-increasing nor non-decreasing and the theorems cannot be applied. To see why this is the case, let us note that the interbank valuation functions of EN are equal to zero for $E < -\bar{p}$ and equal to one for $E > 0$ (see Proposition 2). From (15) we can see that there is always a non-zero probability that the variation of external assets ΔA^e is either positive or negative. Hence, the interbank valuation functions obtained by plugging (16) into (12) take values in the open interval

$(0, 1)$. Therefore, for $E < -\bar{p}$ ($E > 0$) they are larger (smaller) than interbank valuation functions of EN.

Nevertheless, in the Appendix we present three numerical examples in which the convergence of the solutions of the NEVA with interbank valuation functions in (12) (the ex-ante EN) with external assets following a geometric Brownian motion to the solutions of EN holds. This provides a sound background to conjecture that the convergence might be proven to hold under more general hypotheses.

7 Conclusions

In this paper, we introduce a general framework that allows financial institutions to perform an ex-ante network-adjusted valuation of interbank claims in a decentralized fashion. On the one hand, our framework encompasses some of the most widely used models of financial contagion (Bardoscia et al., 2015a; Eisenberg and Noe, 2001; Furfine, 2003; Rogers and Veraart, 2013), in the precise sense that the model is equivalent to those models for specific choices of the valuation functions and the parameters. On the other hand, our framework relates also to the stream of literature (Fischer, 2014; Suzuki, 2002) carrying out the valuation of claims à la Merton when cross-holdings of debt exist between different firms. An important contribution of our approach is that the valuation is decentralized, meaning that it does not assume the existence of an entity with perfect information on the parameters of the financial system.

Our main result is that, under mild assumptions about valuation functions, the valuation problem admits a greatest solution, i.e. a solution in which the losses of all banks are minimal. Moreover, we provide a simple iterative algorithm to compute such solution. Furthermore, we derive a set of conditions under which the solution of the valuation problem *at* the maturity time T is equal to the limit of the sequence of solutions obtained for the valuation problems at $t < T$ as the maturity is approached (i.e. $t \to T$). In other words, the solution of the problem at the maturity coincides with the limit for the valuation time approaching the maturity of the solutions of problems at a given valuation time.

A natural application of our framework is in devising *stress-tests* to assess losses on banks' portfolios in a network of liabilities, conditional to shocks on their external assets in order to determine capital requirements and value at risk. Indeed, to any given shock on the external assets of the banks it corresponds a different valuation of banks' equities. Therefore, by assuming a known distribution of shocks, one can derive a corresponding distribution of equity losses. Finally, such distribution can be taken as the input of any axiomatic risk measure (Biagini et al., 2015; Chen et al., 2013).

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Appendix: Proofs of Theorems

Theorem 1: Existence of a greatest solution

To prove it we just need to show that: (a) the function Φ maps a complete lattice into itself, $\Phi : T \to T$, (b) the function Φ is an order-preserving function. To prove (a) we notice that if valuation functions are feasible then:

$$\forall \mathbf{E} \in \mathbb{R}^n \quad m_i = -L_i^e - \sum_j L_{ij} \leq \Phi_i(\mathbf{E}) \leq A_i^e - L_i^e + \sum_j A_{ij} - \sum_j L_{ij} = M_i$$

and consequently $T = \bigtimes_{i=1}^n [m_i, M_i]$ is a complete lattice such that $\Phi : T \to T$, that proves (a). Since Φ is a linear combination of monotonic non-decreasing functions in $\mathbf{E}$, then $\forall \mathbf{E}, \mathbf{E}'$ if $\mathbf{E} < \mathbf{E}'$, follows $\Phi(\mathbf{E}) \leq \Phi(\mathbf{E}')$, where the partial ordering relation in T is component-wise, i.e. $\mathbf{x} \leq \mathbf{y}$ iff $\forall i \; x_i \leq y_i$. So both conditions (a) and (b) hold and the Knaster-Tarski theorem applies. The set of solutions S of (7) is then a complete lattice, therefore it is non-empty (the empty set cannot contain its own supremum) and, more importantly, it admits a supremum solution, $\mathbf{E}^+$, and an infimum solution, $\mathbf{E}^-$, such that $\forall \mathbf{E}^* \in S$, $\mathbf{E}^- \leq \mathbf{E}^* \leq \mathbf{E}^+$.

Theorem 2: Convergence to the greatest solution

Convergence will be proved by induction. For $n = 0$ we have

$$\mathbf{E}^{(1)} = \Phi(\mathbf{E}^{(0)}) \leq \mathbf{M} = \mathbf{E}^{(0)}$$

Assume now that the claim is true for all $0 \leq m \leq n$, then

$$\mathbf{E}^{(n+1)} = \Phi(\mathbf{E}^{(n)}) \leq \Phi(\mathbf{E}^{(n-1)}) = \mathbf{E}^{(n)}$$

where we have used the fact that Φ is monotonic non-decreasing and $\mathbf{E}^{(n)} \leq \mathbf{E}^{(n-1)}$ by hypothesis, We know that $\{\mathbf{E}^{(n)}\}$ is bounded below and monotonic non-increasing, by the Monotone Convergence Theorem we have that $\mathbf{E}^* = \lim_{n\to\infty} \mathbf{E}^{(n)} = \inf_n \{\mathbf{E}^{(n)}\}$ exists and is finite. By hypothesis Φ is continuous from above (because under assumptions of Theorem (1) we know that the valuation functions are feasible), hence

$$\Phi(\mathbf{E}^*) = \Phi(\lim_n \mathbf{E}^{(n)}) = \lim_n \Phi(\mathbf{E}^{(n)}) = \lim_n \mathbf{E}^{(n+1)} = \mathbf{E}^*$$

So that $\mathbf{E}^* \in S$. We will now prove it must be that $\mathbf{E}^* = \mathbf{E}^+$. First we need to establish a preliminary result, namely that $\mathbf{E}^{(n)} \geq \mathbf{E}^+, \forall n$. Reasoning by induction, it is trivially true for the initial point that $\mathbf{E}^{(0)} \geq \mathbf{E}^+$. Suppose now that it is true up to a given $\bar{n}$, $\mathbf{E}^{(\bar{n})} \geq \mathbf{E}^+$ then, since Φ is order-preserving,

$$\mathbf{E}^{(\bar{n}+1)} = \Phi(\mathbf{E}^{(\bar{n})}) \geq \Phi(\mathbf{E}^+) = \mathbf{E}^+$$

Now, knowing that $\mathbf{E}^{(n)} \geq \mathbf{E}^+, \forall n$ we have that $\mathbf{E}^* = \inf_n\{\mathbf{E}^{(n)}\} \geq \mathbf{E}^+$. But $\mathbf{E}^* \in S$, hence $\mathbf{E}^* = \mathbf{E}^+$.

Theorem 3: Convergence to the least solution

Convergence will be proved by induction. For $n = 0$ we have

$$\mathbf{E}^{(1)} = \Phi(\mathbf{E}^{(0)}) \geq \mathbf{m} = \mathbf{E}^{(0)}$$

Assume now that the claim is true for all $0 \leq m \leq n$, then

$$\mathbf{E}^{(n+1)} = \Phi(\mathbf{E}^{(n)}) \geq \Phi(\mathbf{E}^{(n-1)}) = \mathbf{E}^{(n)}$$

where we have used the fact that Φ is monotonic non-decreasing and $\mathbf{E}^{(n)} \geq \mathbf{E}^{(n-1)}$ by hypothesis. We know that $\{\mathbf{E}^{(n)}\}$ is bounded above and monotonic non-decreasing, by the Monotone Convergence Theorem we have that $\mathbf{E}^* = \lim_n \mathbf{E}^{(n)} = \sup_n\{\mathbf{E}^{(n)}\}$ exists and is finite. By hypothesis Φ is continuous from below, hence

$$\Phi(\mathbf{E}^*) = \Phi(\lim_n \mathbf{E}^{(n)}) = \lim_{n\to\infty} \Phi(\mathbf{E}^{(n)}) = \lim_{n\to\infty} \mathbf{E}^{(n+1)} = \mathbf{E}^*$$

So that $\mathbf{E}^* \in S$. We will now prove it must be that $\mathbf{E}^* = \mathbf{E}^-$. First we need to establish a preliminary result, namely that $\mathbf{E}^{(n)} \leq \mathbf{E}^-, \forall n$. Reasoning by induction, it is trivially true for the initial point that $\mathbf{E}^{(0)} \leq \mathbf{E}^-$. Suppose now that it is true up to a given $\bar{n}$, $\mathbf{E}^{(\bar{n})} \leq \mathbf{E}^-$ then, since Φ is order-preserving,

$$\mathbf{E}^{(\bar{n}+1)} = \Phi(\mathbf{E}^{(\bar{n})}) \leq \Phi(\mathbf{E}^-) = \mathbf{E}^-$$

Now, knowing that $\mathbf{E}^{(n)} \leq \mathbf{E}^-, \forall n$ we have that $\mathbf{E}^* = \sup_n\{\mathbf{E}^{(n)}\} \leq \mathbf{E}^-$. But $\mathbf{E}^* \in S$, hence $\mathbf{E}^* = \mathbf{E}^-$.

Theorem 4: Limit behavior from non-decreasing valuation functions

Let us consider the sequences $\Phi_l^n \equiv \Phi_l^{(n)}(\mathbf{M})$, where the index n denotes composition of Φ with itself n times. Since the valuation functions are monotonically non-decreasing in l then $\forall \mathbf{E}\, \Phi_l(\mathbf{E}) \leq \Phi_{l+1}(\mathbf{E})$ that implies $\Phi_l^n \leq \Phi_{l+1}^n$. Since $\mathbb{V}_{ij}^l(E)$ are all feasible valuation functions we also have that $\Phi_l^n \geq \Phi_l^{n+1}$. From this follows, and boundedness of the sequences in both indices, it follows that $\lim_l \lim_n \Phi_l^n = \tilde{\mathbf{E}}$ exists. Monotonicity, punctual convergence and continuity of the limit valuation function imply, by Dini's Theorem, uniform convergence of $\Phi_l(\mathbf{E})$ to $\Phi_\infty(\mathbf{E})$. Uniform convergence and continuity of Φ_∞ imply that $\mathbf{E}^* = \Phi_\infty(\mathbf{E}^*)$. Since, by assumption, the solution is unique then we must have that $\tilde{\mathbf{E}} = \mathbf{E}^*$. Thus $\lim_l \lim_n \Phi_l^n = \lim_l \lim_n \Phi_l^n$ that is equivalent to the thesis.

Theorem 5: Limit behavior from non-increasing valuation functions

Let us consider the sequences $\Phi_l^n \equiv \Phi_l^{(n)}(\mathbf{M})$. Since the valuation function is monotonic non-increasing in l then $\forall \mathbf{E}\, \Phi_l(\mathbf{E}) \geq \Phi_{l+1}(\mathbf{E})$ that implies $\Phi_l^n \geq \Phi_{l+1}^n$. Since $\mathbb{V}_{ij}^l(E)$ are all feasible valuation functions we also have that $\Phi_l^n \geq \Phi_l^{n+1}$. Hence, the sequences Φ_l^n are all non-increasing in both indices and bounded from below, from this follows that $\lim_l \lim_n \Phi_l^n = \lim_l \lim_n \Phi_l^n$ that is equivalent to the thesis.

Appendix: Numerical examples

In our examples, the financial system is composed of three banks that use the interbank valuation functions (12) and in which external assets follow the geometric brownian motion (14). For each network topology we vary the time to maturity T_l and we compute equities via (7). We also compute equities for the (ex-post) EN, also via (7). We set $\beta = 1.0$, i.e. we do not include an additional exogeneous recovery. External liabilities are equal to zero for all banks in all cases.

Open chain

$A \to B \to C$

$$A^e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \\ 0 & 0 & 0 \end{pmatrix}$$

Tree

$B \leftarrow A \to C$

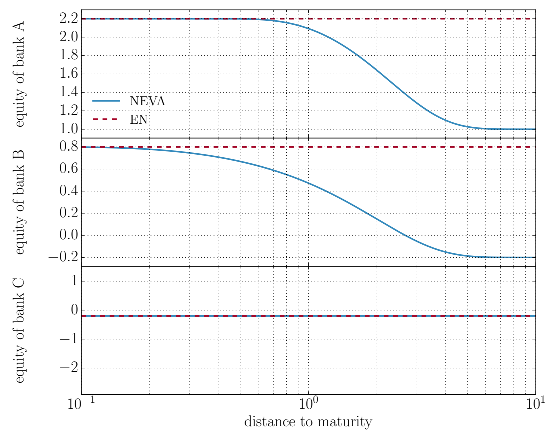

Figure 3: Open chain. Solid blue: equities $E^*_{j,l}$ as a function of the distance to maturity T_l for the NEVA with interbank valuation functions in (12) (ex-ante Eisenberg and Noe model). Dashed red: equities E^*_j for the (ex-post) Eisenberg and Noe model. Panels from the top to the bottom refer to banks A, B, and C.

$$A^e = \begin{pmatrix} 1 \\ 0.1 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

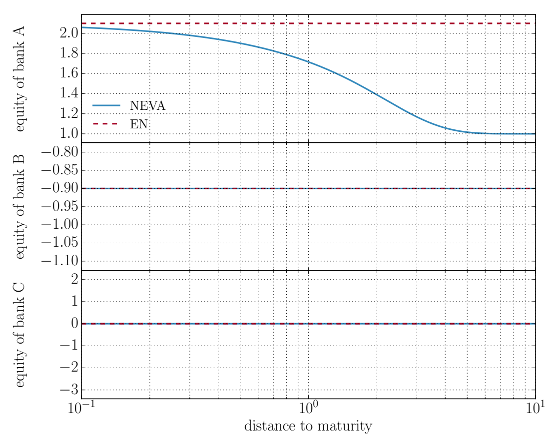

Figure 4: Analogous of Fig. 3 for a tree.

Closed chain

$A \to B \to C \to A$

$$A^e = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1.1 & 0 \\ 0 & 0 & 1.2 \\ 1.5 & 0 & 0 \end{pmatrix}$$

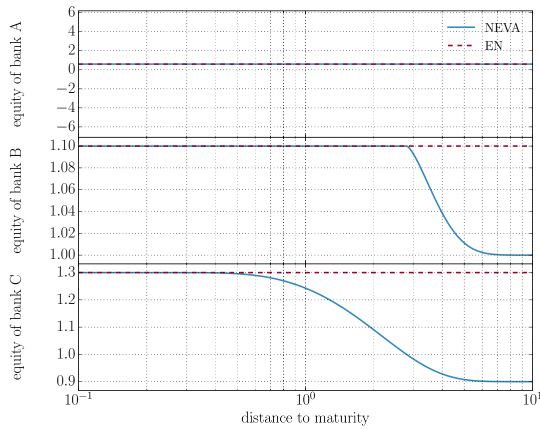

Figure 5: Analogous of Fig. 3 for a closed chain.

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